Dynamical Aperture Control in Accelerator Lattices with Multipole Potentials

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Abstract

We develop tools for symbolic representation of a non-linear accelerator model and analytical methods for description of non-linear dynamics. Information relevant to the dynamic aperture (DA) is then obtained from this model and can be used for indirect DA control or as a complement to direct numerical optimization. We apply two analytical methods and use multipole magnets to satisfy derived analytical constraints. The accelerator model is represented as a product of unperturbed and perturbed exponential operators with the exponent of the perturbed operator given as a power series in the perturbation parameter. Normal forms can be applied to this representation and the lattice parameters are used to control the normal form Hamiltonian and normal form transformation. Hamiltonian control is used to compute a control term or controlled operator. Lattice parameters are then fitted to satisfy the imposed control constraints. Theoretical results, as well as illustrative examples, are presented.

Keywords

Non-linear dynamics; dynamic aperture; analytical methods; optimization.

1 Introduction

The dynamic aperture (DA) is an important parameter for circular accelerators; it can be defined as a stable area in phase space or, more practically, as an area stable for a fixed number of turns in co-ordinate space. Reduction of DA is mainly caused by non-linear effects, which manifest themselves through frequency dependence on the amplitude, appearance of resonances, and chaos. For DA optimization, usually sophisticated numerical algorithms are used [1]. In this case, the DA is computed directly via particle tracking. In this study, we try to increase the DA by constructing symbolic constraints with analytical tools, such as normal forms [2–4] and Hamiltonian control [5, 6]. Thus, analytical results can be used as a complement to direct DA optimization by providing good initial values for system parameters and additional constraints; it can also be used for indirect optimization. We use a simple FODO cell as an example for which a symbolic model is computed and analytical methods are applied (Fig. 1).

2 Non-linear accelerator model

To study DA, one first needs to build a non-linear model of an accelerator lattice. Here, we use the framework of single-particle Hamiltonian (or symplectic) dynamics. Relevant information about DA can be extracted from the accelerator one-turn map:

$$x_2 = f(x_1)\,, \tag{1}$$

where f is a one-turn map that describes propagation of a particle for one turn in an accelerator, x_1 are the initial co-ordinates, and x_2 are the co-ordinates after one turn. It is well known [2,3,7] that single-particle dynamics can be described in terms of compositional operators $(\mathcal{M}_f g)(x) := (g \circ f)(x) = g(f(x))$; then the lattice operator is given by

$$x_2 = (\mathcal{M}\mathcal{I})(x_1)\,, \tag{2}$$
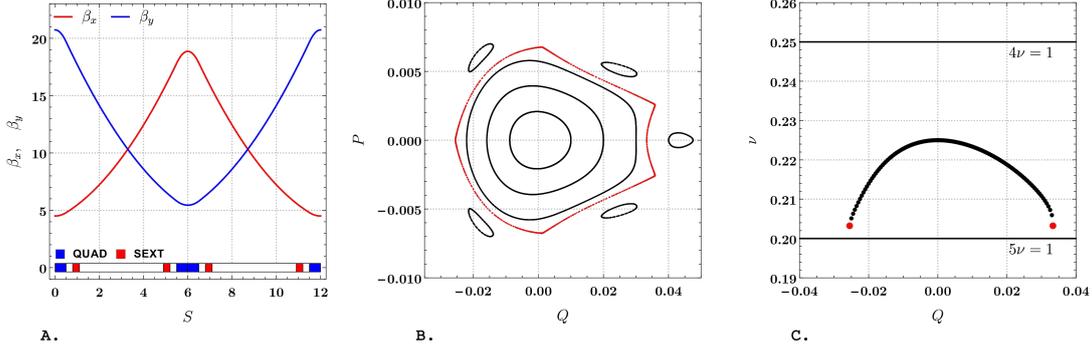
Fig. 1: FODO lattice with sextupole perturbation: (A) layout and lattice functions; (B) phase space trajectories (black) and dynamic aperture (red); (C) frequency dependence on amplitude.

where $\mathcal{I}$ is an identity function and the reference to the associated one-turn map f is dropped from now on.

For analytical (symbolic) calculations, we want to factorize the lattice operator $\mathcal{M}$. Our goal is to express $\mathcal{M}$ as a product of unperturbed and perturbed parts and to obtain their exponential representations:

$$\mathcal{M} = \mathcal{M}_L \mathcal{M}_N = \exp([F_L]) \exp([F_N]), \tag{3}$$

where $\mathcal{M}_L$ is the unperturbed part, $\mathcal{M}_N$ is the perturbed part, $[F_L]$ and $[F_N]$ are Lie operators, and the Poisson bracket operator is defined as $[f]g := [f, g] = \partial_q f \partial_p g - \partial_p f \partial_q g$. The non-linear perturbation generator $F_N = F_N^{(1)} \varepsilon + F_N^{(2)} \varepsilon^2 + \dots$ is computed up to some order of the formal perturbation parameter ε.

Often, the unperturbed part is associated with linear motion and the perturbed part with non-linear motion. For multipole perturbations, it is common to associate perturbation with homogeneous polynomials, e.g., for 2D phase space with co-ordinates (q, p), one has

$$F_N^{(k-2)} = F^{(k,0)} q^k + F^{(k-1,1)} q^{k-1} p + \dots + F^{(0,k)} p^k.$$

In the case of accelerators, it is convenient to express the lattice operator as a product of functionally independent parts, e.g., each part describes propagation through a certain accelerator element:

$$\mathcal{M} = \mathcal{M}_1 \mathcal{M}_2 \dots \mathcal{M}_n, \tag{4}$$

where $\mathcal{M}_\alpha = \mathcal{M}_{\alpha,L} \mathcal{M}_{\alpha,N} = \exp([F_{\alpha,L}]) \exp([F_{\alpha,N}])$ is an element operator that satisfies [2]

$$\mathcal{M}_\alpha{}' = \mathcal{M}_\alpha[-H_\alpha] = \mathcal{M}_\alpha[-H_{\alpha,L} - H_{\alpha,N}], \tag{5}$$

where H_α is the element's Hamiltonian function, and $H_{\alpha,L}$ and $H_{\alpha,N}$ are its linear and non-linear parts, respectively. The operator $\mathcal{M}$ also satisfies this equation for the global Hamiltonian function H, but it is far more complex than individual Hamiltonian functions and thus is not practical to use. Hence, we first need to factorize the element's operators; for the autonomous case, single-exponent representation is straightforward:

$$\mathcal{M}_\alpha = \exp([-s_\alpha(H_{\alpha,L} + H_{\alpha,N})]) = \exp([F_{\alpha,L}]) \exp([F_{\alpha,N}]), \tag{6}$$

where s_α is the element length. It is not the required form, but the linear part is simple to identify:

$$F_{\alpha,L} = -s_\alpha H_{\alpha,L}. \tag{7}$$

Decomposition of Eq. (7) can be motivated by symplectic integrators [8]. In this case, the element is first split into slices and then factorized with BCH formulae [9] and operator identities. Another possibility is to use BCH directly for $\exp([F_{\alpha,N}]) := \exp([-F_{\alpha,L}])\exp([F_{\alpha,L}] + [F_{\alpha,N}])$. This can be done for each slice or an exact expression can be obtained in both cases for pure multipole magnets, i.e., when $F_{\alpha,L} = F_{\alpha,L}(p)$.

These decomposition methods can be used to obtain approximate factorization or exact factorization in the special case. In the generic case, Magnus expansion [10, 11] can be used, which was originally introduced to obtain an exponential solution $Y(t) = \exp(\Omega(t))Y(0)$ of the matrix-differential equation:

$$Y' = A(t)Y. \tag{8}$$

The exponent $\Omega(t)$ satisfies the following differential equation with $\Omega(0) = 0$:

$$\Omega' = \sum_{k=0}^{\infty} \frac{1}{k!} B_k \{\Omega\}^k A,$$

where $\{X\}Y := \{X, Y\} = XY - YX$ denotes the commutator operator and B_k are Bernoulli numbers. This equation can be solved by iteration, $\Omega(t) = \Omega_1 + \Omega_2 + \ldots$, and one then has the following recursion:

$$\Omega_1' = A, \qquad \Omega_n' = \sum_{k=1}^{n-1} \frac{1}{k!} B_k S_n^{(k)}, \qquad n \geq 2,$$

$$S_n^{(1)} = \{\Omega_{n-1}\}A, \qquad S_n^{(n-1)} = \{\Omega_1\}^{n-1} A, \qquad S_n^{(k)} = \sum_{m=1}^{n-k} \{\Omega_m\} S_{n-m}^{(k-1)}, \qquad 2 \leq k \leq n-1.$$

Explicit solutions can be obtained as multidimensional integrals:

$$\Omega_1(t) = \int_0^t A(t_1)dt_1, \quad \Omega_2(t) = \frac{1}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 \ \{A(t_1), A(t_2)\}.$$

For the Hamiltonian case, $\mathcal{M}' = \mathcal{M}[-H]$ with $\mathcal{M} = \exp([F])$ and $F = F_1 + F_2 + \ldots$ one only needs to replace A with $-H$ and the commutator brackets with Poisson brackets, and change signs $F_n = (-1)^{n+1}\Omega_n$. Since the Hamiltonian has an unperturbed part, an additional step is required before applying the Magnus expansion:

$$\mathcal{M}' = \mathcal{M}[-H] = \mathcal{M}[-H_L - H_N], \qquad\qquad \mathcal{M}_L' = \mathcal{M}_L[-H_L], \qquad \mathcal{M}_L = \exp([F_L]),$$

$$\mathcal{M}_M := \mathcal{M}_L \mathcal{M}_N \mathcal{M}_L^{-1}, \qquad\qquad \mathcal{M}_M' = \mathcal{M}_M[-\mathcal{M}_L H_N] = \mathcal{M}_M[-H_M],$$

$$\mathcal{M}_M = \exp([F_M]) = \exp([F_{M,1} + F_{M,2} + \ldots])a, \qquad F_N = \mathcal{M}_L^{-1} F_M = \exp([-F_L])F_M.$$

Here, the operator $\mathcal{M}_M$ is defined, for which the Hamiltonian H_M is first-order in perturbation. Thus, an approximation of F_M can be obtained with the Magnus expansion and F_N can then be computed from it. This formally solves the factorization problem for the element operator.

As an example, we give the factorization of a thick sextupole and a thick sextupole with a quadrupole component. In the first case, the Hamiltonian is given by

$$H = \frac{1}{2}p^2 + \frac{1}{3}k_S q^3,$$

where k_S is the sextupole amplitude. The linear part is $F_L = -\frac{L}{2}p^2$ and the non-linear part F_N can be computed up to some order with a Magnus expansion, e.g. first and second orders are given by

$$F_N^{(1)} = \frac{1}{12}k_S L^4 p^3 - \frac{1}{3}k_S L^3 p^2 q + \frac{1}{2}k_S L^2 pq^2 - \frac{1}{3}k_S L q^3,$$

$$F_N^{(2)} = \frac{1}{168}k_S^2 L^7 p^4 - \frac{1}{24}k_S^2 L^6 p^3 q + \frac{1}{8}k_S^2 L^5 p^2 q^2 - \frac{1}{6}k_S^2 L^4 pq^3 + \frac{1}{12}k_S^2 L^3 q^4 \,,$$

where L is the element length, $F_N^{(1)}$ is the first-order perturbation given by a degree-three homogeneous polynomial with coefficients that depend on sextupole parameters, and $F_N^{(2)}$ is a degree-four polynomial.

For a thick sextupole with a quadrupole component, the Hamiltonian is $H = \frac{1}{2}\left(p^2 + k_Q q^2\right) + \frac{1}{3}k_S q^3$, where k_S and k_Q are the sextupole and quadrupole amplitudes, respectively. $F_L = -\frac{L}{2}\left(p^2 + k_Q q^2\right)$ is the linear part and the first-order non-linear part $F_N^{(1)} = F_N^{(3,0)}q^3 + F_N^{(2,1)}q^2 p + F_N^{(1,2)}qp^2 + F_N^{(0,3)}p^3$ is a degree-three polynomial with coefficients that now depend on the quadrupole amplitude:

$$F_N^{(3,0)} = k_S \left(-\frac{\sin\left(L\sqrt{k_Q}\right)}{4\sqrt{k_Q}} - \frac{\sin\left(3L\sqrt{k_Q}\right)}{36\sqrt{k_Q}} \right) ,$$

$$F_N^{(2,1)} = k_S \left(-\frac{\cos\left(L\sqrt{k_Q}\right)}{4k_Q} - \frac{\cos\left(3L\sqrt{k_Q}\right)}{12k_Q} + \frac{1}{3k_Q} \right) ,$$

$$F_N^{(1,2)} = k_S \left(\frac{\sin\left(3L\sqrt{k_Q}\right)}{12k_Q^{3/2}} - \frac{\sin\left(L\sqrt{k_Q}\right)}{4k_Q^{3/2}} \right) ,$$

$$F_N^{(0,3)} = k_S \left(-\frac{\cos\left(L\sqrt{k_Q}\right)}{4k_Q^2} + \frac{\cos\left(3L\sqrt{k_Q}\right)}{36k_Q^2} + \frac{2}{9k_Q^2} \right) .$$

Having obtained factorization of individual elements, we can go back to Eq. (4) and, with a slight change of notation $\mathcal{M}_\alpha = \mathcal{M}_{\alpha,L}\mathcal{M}_{\alpha,N} = \exp([F_{\alpha,L}])\exp([F_{\alpha,N}]) = \mathcal{M}[\alpha-1,\alpha]\mathcal{M}[\alpha]$, the full lattice operator can be written as

$$\mathcal{M} = \mathcal{M}[0,n]\hat{\mathcal{M}}[1]\ldots\hat{\mathcal{M}}[n]\,, \tag{9}$$

where $\mathcal{M}[0,n] = \mathcal{M}[0,1]\ldots\mathcal{M}[n-1,n]$ is the linear part, the non-linear part is given by $\hat{\mathcal{M}}[1]\ldots\hat{\mathcal{M}}[n]$, with transformed perturbation, $\hat{\mathcal{M}}[\alpha] = \exp([\mathcal{M}^{-1}[\alpha,n]F_{\alpha,N}])$, and the operator identity $\exp([f])\exp([g])\exp([-f]) = \exp([\exp(g)f])$ was used. The product of linear operators is known and the product of non-linear operators can be computed using BCH formulae [9]. Finally, the factorized lattice operator is given by

$$\mathcal{M} = \mathcal{M}_L\mathcal{M}_N = \mathcal{M}[0,n]\exp([F_N^{(1)}\varepsilon + F_N^{(2)}\varepsilon^2 + \ldots])\,. \tag{10}$$

The factorized operator can be used for the normal form and Hamiltonian control computations with generic polynomial perturbation or can be used for geometric indirect optimization (see Section 5). A comparison of element-by-element tracking and analytical model implicit tracking for a FODO example is shown in Fig. 2. We also note that non-linear perturbation computed analytically matches that obtained with COSY INFINITY [12].

3 Normal form computation

In Section 2, it was shown that a circular accelerator can be viewed as a discrete dynamic system specified by a lattice operator Eq. (10). This operator (or the one-turn map associated with it) contains all the relevant information about single-particle non-linear dynamics and thus can be used to study non-linear effects that influence the DA. In this section, for a system defined by $x_2 = f(x_1)$, we want to obtain a simpler representation $y_2 = g(y_1)$, where f is a one-turn map of original system with x_1 and x_2 the initial and final co-ordinates, and g is a normal form of f with y_1 and y_2 the initial and final normal form
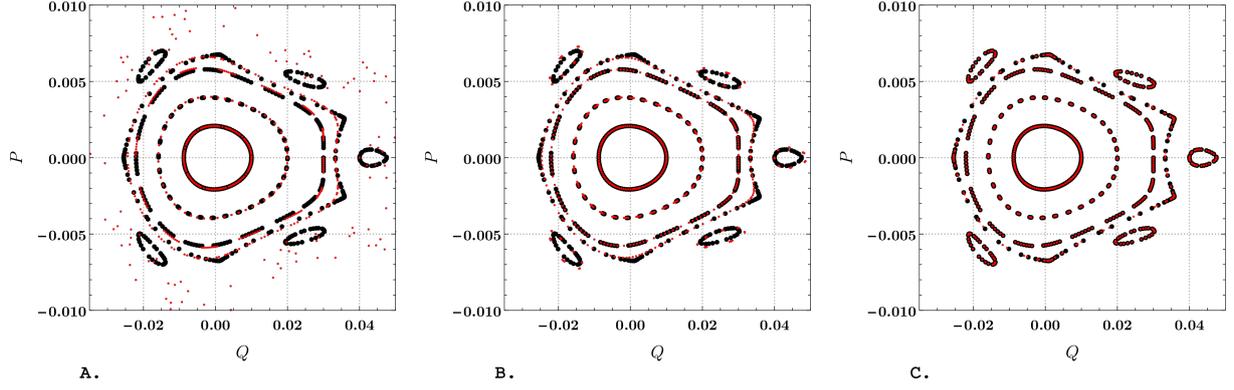
Fig. 2: Comparison of element-by-element tracking (black dots) and analytical model implicit tracking (red dots): (A) $\mathcal{M}_L \exp([F_N^{(1)}\varepsilon])$; (B) $\mathcal{M}_L \exp([F_N^{(1)}\varepsilon + F_N^{(2)}\varepsilon^2])$; (C) $\mathcal{M}_L \exp([F_N^{(1)}\varepsilon + \cdots + F_N^{(6)}\varepsilon^6])$.

co-ordinates related to the original co-ordinates x by symplectic transformation $x = h(y)$. From these definitions we have

$$x_2 = f(x_1) = f(h(y_1)) = (f \circ h)(y_1),$$
$$x_2 = h(y_2) = h(g(y_1)) = (h \circ g)(y_1),$$

and since y_1 is arbitrary, we obtain the functional relation $h \circ g = f \circ h$ or $g = h^{-1} \circ f \circ h$. The goal of normal form computation is to find a normal form g and a transformation h for a given one-turn map f. This functional relation can be written in operator form:

$$\mathcal{M}_g = \mathcal{M}_h \mathcal{M}_f \mathcal{M}_h^{-1}, \tag{11}$$

where the definition of the compositional operator $\mathcal{M}_f g := g \circ f$ was used and all operators can be represented by exponential operators or by the product of such operators, since the symplectic case is assumed.

For a given lattice operator $\mathcal{M}_f \equiv \mathcal{M} = \mathcal{M}_L \mathcal{M}_N$, we first want to normalize the linear part $\mathcal{M}_L$; to do so, we assume that linear motion is stable (a generic case can be found in Refs. [2, 3]). In this case, $\mathcal{M}_L$ is conjugate to rotation $\mathcal{R} = \mathcal{A} \mathcal{M}_L \mathcal{A}^{-1}$ with transformation $\mathcal{A}$. The normal form is given by

$$\mathcal{R} = \exp([-2\pi\nu I]), \tag{12}$$

where ν denotes the frequency, $I = 1/2\left(q^2 + p^2\right)$ is the action, and q and p are normal form co-ordinates. For the linear case, the normal form relation Eq. (11) can be written in terms of matrices $R = A^{-1}M_L A$. If we use Courant–Snyder parametrization, then

$$M_L = \begin{bmatrix} \cos(2\pi\nu) + \alpha\sin(2\pi\nu) & \beta\sin(2\pi\nu) \\ -\gamma\sin(2\pi\nu) & \cos(2\pi\nu) - \alpha\sin(2\pi\nu) \end{bmatrix}, \qquad A = \begin{bmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{bmatrix},$$

$$R = A^{-1}M_L A = \begin{bmatrix} \cos(2\pi\nu) & \sin(2\pi\nu) \\ -\sin(2\pi\nu) & \cos(2\pi\nu) \end{bmatrix},$$

where α, β, and $\gamma = 1/\beta\left(1 + \alpha^2\right)$ are lattice functions and matrix R is a rotation. Operator $\mathcal{R}$ also corresponds to rotation, for example:

$$\mathcal{R}q = \exp([-2\pi\nu I])q = (1 - [2\pi\nu I] + \frac{1}{2}[2\pi\nu I]^2 - \frac{1}{6}[2\pi\nu I]^3 + \dots)q$$

$$= q\left(1 - \frac{1}{2}(2\pi\nu)^2 + \frac{1}{24}(2\pi\nu)^4 + \dots\right) + p\left(2\pi\nu - \frac{1}{6}(2\pi\nu)^3 + \frac{1}{120}(2\pi\nu)^5 + \dots\right)$$

$$= q\cos(2\pi\nu) + p\sin(2\pi\nu)\,.$$

Having normalized the linear part, we can proceed with the non-linear part $\mathcal{M}_N = \exp([F_N])$ normalization. The effect of linear normalization on operator $\mathcal{M}$ is to normalize its linear part and to transform the non-linear part:

$$\mathcal{M}_F := \mathcal{A}\mathcal{M}_L\mathcal{M}_N\mathcal{A}^{-1} = \left(\mathcal{A}\mathcal{M}_L\mathcal{A}^{-1}\right)\left(\mathcal{A}\mathcal{M}_N\mathcal{A}^{-1}\right) = \mathcal{R}\mathcal{A}\exp([F_N])\mathcal{A}^{-1}$$
$$= \mathcal{R}\exp(\mathcal{A}[F_N]\mathcal{A}^{-1}) = \mathcal{R}\exp([\mathcal{A}F_N]) =: \mathcal{R}\exp([F_{F,N}])\,.$$

Thus, a new system to normalize is

$$\mathcal{M}_F = \mathcal{R}\exp([F_{F,N}]) = \mathcal{R}\exp([\varepsilon F_{F,N}^{(1)} + \varepsilon^2 F_{F,N}^{(2)} + \ldots])\,, \tag{13}$$

where the unperturbed part is given by operator $\mathcal{R}$ defined by Eq. (12), the perturbation $F_{F,N}$ is given in linear normal form co-ordinates, and each order is associated with homogeneous polynomials, for example:

$$F_{F,N}^{(1)} = F_{F,N}^{(3,0)}q^3 + F_{F,N}^{(2,1)}q^2p + F_{F,N}^{(1,2)}qp^2 + F_{F,N}^{(0,3)}p^3\,.$$

In general, coefficients of these polynomials depend on lattice parameters, but we can keep this dependence implicit and substitute actual coefficients in the final answer.

According to Eq. (11), the non-linear normal form can be written as

$$\mathcal{N} = \mathcal{T}\mathcal{M}_F\mathcal{T}^{-1}\,, \tag{14}$$

where $\mathcal{N} = \mathcal{R}\mathcal{K}$ is a normal form of $\mathcal{M}_F$, $\mathcal{K} = \exp([K]) = \exp([\varepsilon K^{(1)} + \varepsilon^2 K^{(2)} + \ldots])$ is a non-linear part of $\mathcal{N}$, and the normal form transformation is $\mathcal{T} = \exp([T]) = \exp([\varepsilon T^{(1)} + \varepsilon^2 T^{(2)} + \ldots])$. Then, the non-linear part is

$$e^{[K]} = \mathcal{R}^{-1}e^{[T]}\mathcal{R}e^{[F_{F,N}]}e^{[-T]} = e^{[\mathcal{R}^{-1}T]}e^{[F_{F,N}]}e^{[-T]} =: e^{[X]}e^{[Y]}e^{[Z]}\,.$$

The non-linear part can be computed order-by-order, using BCH formulae [9] and the relation between commutator and Poisson brackets $\{[f],[g]\} = [[f,g]]$. The order k equation is then

$$K^{(k)} = X^{(k)} + Y^{(k)} + Z^{(k)} + R^{(k)}\,,$$

where $R^{(k)}$ is known and depends on previous orders. To solve this equation, we expand all functions in resonance basis with basis functions $Q^{(n,m)}$:

$$F^{(k)} = \sum_{n,m} F^{(n,m)}Q^{(n,m)}\,, \qquad Q^{(n,m)} = \prod_q I^{\frac{1}{2}n_q}e^{im_q\varphi_q}\,.$$

And since auxiliary functions $X^{(k)}$ and $Z^{(k)}$ are related to transformation $T^{(k)}$:

$$X^{(k)} = \sum_{n,m} T^{(n,m)}e^{-i2\pi m\nu}Q^{(n,m)}\,, \qquad Z^{(k)} = -\sum_{n,m} T^{(n,m)}Q^{(n,m)}\,,$$

the transformation coefficients are found to be

$$T^{(n,m)} = \frac{1}{2}\left(1 - i\cot(\pi m\nu)\right)\left(R^{(n,m)} - K^{(n,m)}\right)\,. \tag{15}$$

This expression is singular when the resonance condition is satisfied, i.e., $m\nu = p \in \mathbb{Z}$, or when $m = 0$. These terms must be absorbed into normal form by setting $K^{(n,m)} = R^{(n,m)}$ to resolve singularity. For the case when $m\nu \neq p \in \mathbb{Z}$, only terms with $m = 0$ contribute to the normal form. Then K depends
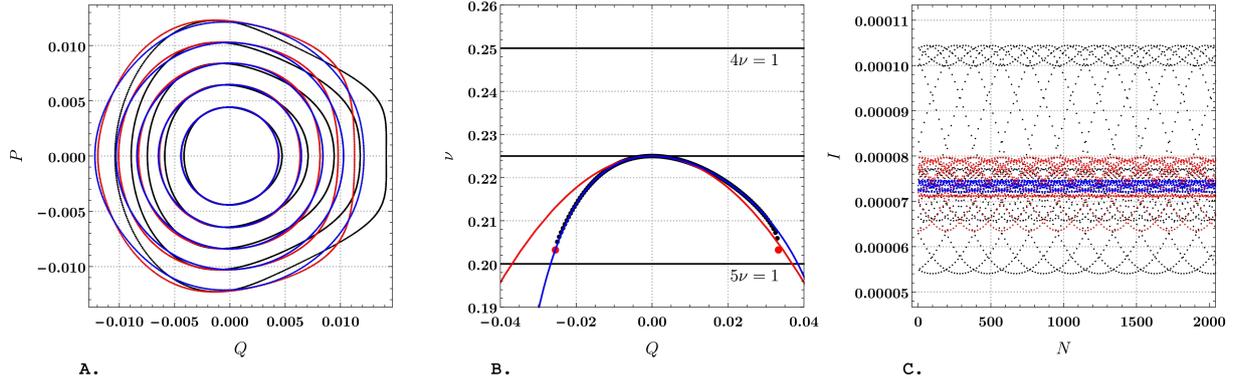
Fig. 3: Full normal form computation for FODO example: (A) effect of normal form transformation on trajectories; (B) frequency dependence on the amplitude; (C) invariant conservation for first order (black), second order (red) and sixth order (blue).

only on actions and thus commutes with the linear part $\mathcal{R}$. Such a full normal form allows one to obtain dependence of frequencies on the amplitude as well as invariants (Fig. 3).

As an example, we perform full normalization up to second order for the generic case and then substitute parameters for thin and thick single sextupole perturbation. The normal form procedure is also applied to the FODO example (Fig. 3). Generic perturbation in resonance basis up to the second order is given by

$$Y = \varepsilon Y^{(1)} + \varepsilon^2 Y^{(2)} \,,$$
$$Y^{(1)} = Y^{(-3,3)}Q^{(-3,3)} + Y^{(-1,3)}Q^{(-1,3)} + Y^{(1,3)}Q^{(1,3)} + Y^{(3,3)}Q^{(3,3)} \,,$$
$$Y^{(2)} = Y^{(-4,4)}Q^{(-4,4)} + Y^{(-2,4)}Q^{(-2,4)} + Y^{(0,4)}Q^{(0,4)} + Y^{(2,4)}Q^{(2,4)} + Y^{(4,4)}Q^{(4,4)} \,.$$

The first-order normal form $K^{(1)} = 0$ and the second-order expression is

$$K^{(2)} = \left(Y^{(0,4)} + \frac{3}{2}Y^{(-1,3)}Y^{(1,3)}\cot(\pi\nu) + \frac{9}{2}Y^{(-3,3)}Y^{(-3,3)}\cot(3\pi\nu)\right)Q^{(0,4)} \,. \qquad (16)$$

This expression is not valid when resonance conditions $\nu = p$ or $3\nu = p \in \mathbb{Z}$ are satisfied. For a thin sextupole, the perturbation in original co-ordinates is

$$F_N^{(1)} = -\frac{1}{3}k_S L q^3 \,, \qquad F_N^{(2)} = 0 \,,$$

where L is the sextupole effective length and k_S is the amplitude. The transformed perturbation is given by

$$F_{F,N}^{(1)} \equiv Y^{(1)} = -\frac{1}{3}\beta^{3/2}L q^3 k_S = -\frac{1}{6\sqrt{2}}\beta^{3/2}L k_S \left(Q^{(-3,3)} + 3Q^{(-1,3)} + 3Q^{(1,3)} + Q^{(3,3)}\right) \,,$$

and the second-order normal form is obtained by substituting corresponding coefficients into Eq. (16):

$$K^{(2)} = \frac{1}{16}\beta^3 k_S^2 L^2 \left(3\cot(\pi\nu) + \cot(3\pi\nu)\right)Q^{(0,4)} \,.$$

The thick sextupole perturbation can be computed with a Magnus expansion (see Section 2 for details). The normal form in this case is

$$K^{(2)} = \frac{1}{16}\beta^3 k_S^2 L^2 \left(C_1 + 3\left(1 + C_2\right)\cot(\pi\nu) + \left(1 + C_3\right)\cot(3\pi\nu)\right)Q^{(0,4)} \,,$$

$$C_1 = \frac{\gamma^2 L^5}{7\beta^3} + \frac{\alpha\gamma L^4}{\beta^3} + \frac{(3\alpha^2 + 1) L^3}{\beta^3} + \frac{4\alpha L^2}{\beta^2} + \frac{2L}{\beta},$$

$$C_2 = \frac{\gamma^3 L^6}{16\beta^3} + \frac{\alpha\gamma^2 L^5}{2\beta^3} + \frac{(63\alpha^2 + 13) \gamma L^4}{36\beta^3} + \frac{\alpha (21\alpha^2 + 13) L^3}{6\beta^3} + \frac{(51\alpha^2 + 11) L^2}{12\beta^2} + \frac{3\alpha L}{\beta},$$

$$C_3 = \frac{\gamma^3 L^6}{16\beta^3} + \frac{\alpha\gamma^2 L^5}{2\beta^3} + \frac{(7\alpha^2 + 1) \gamma L^4}{4\beta^3} + \frac{\alpha (7\alpha^2 + 3) L^3}{2\beta^3} + \frac{(17\alpha^2 + 1) L^2}{4\beta^2} + \frac{3\alpha L}{\beta}.$$

One can see that in the limit $L \to 0$ with fixed $k_S L$ this expression is reduced to the thin sextupole case.

For DA optimization, we can perform full normalization and then fit lattice parameters to reduce the frequency dependence on the amplitude and thus reduce the tune footprint size. However, this procedure can lead to increase in resonance strengths and such resonances (that can be selected based on FMA [13]) should be reduced as well. Another option is to control the shapes of the frequency curves to avoid crossing of strong resonances. The shape of invariants can be modified to obtain curves that resemble circles. All these options do not provide direct control of DA, but can be used in indirect optimization or as a complement to a numerical one. Several examples of indirect optimization are given in Section 5.

4 Hamiltonian control theory

The goal of Hamiltonian control theory is to modify the perturbed system $\mathcal{M}_F$ by adding a control operator $\mathcal{C} = \exp([C])$, which is second order in the perturbation parameter, i.e., $C = C^{(2)}\varepsilon^2 + C^{(3)}\varepsilon^3 + \ldots$, such that the controlled system $\mathcal{M}_C$,

$$\mathcal{M}_C = \mathcal{M}_F\mathcal{C} = \mathcal{R}\exp([F_{F,N}])\exp([C]), \tag{17}$$

is conjugate to a system that is close to the unperturbed one:

$$e^{[T]}\mathcal{M}_C e^{[-T]} = \mathcal{R}e^{[\mathcal{G}_R F_{F,N}]}.$$

This is the case when the control operator is defined as

$$e^{[C]} := e^{[-F_{F,N}]} e^{[(\mathcal{G}_N - \mathcal{G}) F_{F,N}]} e^{[\mathcal{G}_R F_{F,N}]} e^{[\mathcal{G} F_{F,N}]}, \tag{18}$$

where $\mathcal{R}$ is an unperturbed part of uncontrolled operator $\mathcal{M}_F$ is given by Eq. (13), $\mathcal{G} := \mathcal{G}(1 - \mathcal{R}^{-1})\mathcal{G}$ is a pseudo-inverse operator of $(1 - \mathcal{R}^{-1})$, $\mathcal{G}_N := (1 - \mathcal{R}^{-1})\mathcal{G}$ is the non-resonant operator, $\mathcal{G}_R := 1 - \mathcal{G}_N$ is the resonant operator, and $T := \mathcal{G} F_{F,N}$ is the transformation. The controlled operator is then given by

$$\mathcal{M}_C = \mathcal{R}e^{[-\mathcal{R}^{-1}\mathcal{G} F_{F,N}]} e^{[\mathcal{G}_R F_{F,N}]} e^{[\mathcal{G} F_{F,N}]} =: \mathcal{R}e^{[F_C]}. \tag{19}$$

The perturbation $F_{F,N}$ is transformed into resonance basis and the action of the above operators on the basis function is given by the following expressions:

$$\mathcal{G} Q^{(n,m)} = \frac{1}{2}(1 - i\cot(\pi m\nu))\Delta(m\nu \notin \mathbb{Z})Q^{(n,m)},$$

$$\mathcal{R}^{-1}\mathcal{G} Q^{(n,m)} = -\frac{1}{2}(1 + i\cot(\pi m\nu))\Delta(m\nu \notin \mathbb{Z})Q^{(n,m)},$$

$$\mathcal{G}_N Q^{(n,m)} = \Delta(m\nu \notin \mathbb{Z})Q^{(n,m)},$$

$$\mathcal{G}_R Q^{(n,m)} = \Delta(m\nu \in \mathbb{Z})Q^{(n,m)},$$

where the function Δ is defined as $\Delta(\top) := 1$ and $\Delta(\bot) := 0$.

In general, the closed form of the controlled operator (or the control term) cannot be obtained, but it can be approximated up to some order in the perturbation parameter $F_C = F_C^{(1)}\varepsilon + F_C^{(2)}\varepsilon^2 + \ldots$
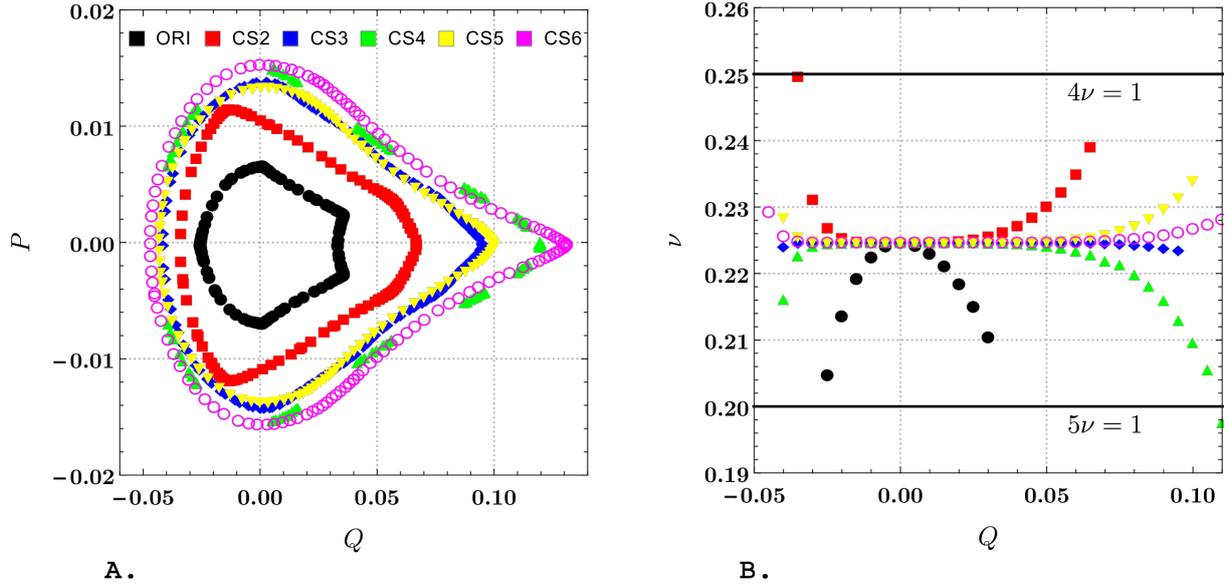
Fig. 4: Formal controlled system for FODO example: (A) DA curves for original system and controlled systems up to order six; (B) corresponding frequency dependence on the amplitude.

with BCH formulae [9]. Control does not change the first-order properties of a system, i.e., $F_C^{(1)} \equiv F_{F,N}^{(1)}$; thus, to construct the controlled operator one can choose only first-order perturbation $F_{F,N} = F_{F,N}^{(1)}\varepsilon$ and, since $F_{F,N}^{(1)}$ is associated with a homogeneous polynomial of degree three, $\mathcal{G}_R F_{F,N}^{(1)} = 0$. Hamiltonian control theory is very flexible, since one can define both the target system and the perturbation.

As an example, we compute the first term of the control operator for thin sextupole perturbation. Like normal form computation (see Section 3 for details), the perturbation is first transformed by linear normalization and then expanded into resonance basis. The first term of the control operator can then be computed:

$$C^{(2)} = -\frac{1}{16}\beta^3 L^2 k_S^2 (\cot(\pi\nu) + \cot(3\pi\nu))q^4 - \frac{1}{16}\beta^3 L^2 k_S^2 (3\cot(\pi\nu) - \cot(3\pi\nu))p^2 q^2 \, .$$

The effect of formal control for a FODO example is shown in Fig. 4. In this case, the target system is $\mathcal{R}$ and $F_{F,N}^{(1)}$ is a perturbation. It can be seen that formal control provides significant improvement of the DA. The frequency curves become flatter as the order of computation is increased. This is the case because the target system is a rotation. If full normal form computation is performed for a controlled system then the normal form has no tune shifts and the transformation is zero for orders greater than one.

Here, DA optimization is not direct but motivated by the fact that the controlled system is conjugate to a good one. Control can be realized as a special element or a controlled system can be obtained with suitable distribution of multipoles. In the following section, examples of fitting octupole and decapole distributions to realize a controlled system are shown (Fig. 5).

5   Examples of indirect optimization

As stated in Section 1, direct DA optimization requires particle tracking. An accelerator model for direct optimization can include a number of effects that are difficult and impractical to include in the analytical model of Eq. (10). The analytical model itself can be used for particle tracking (see Fig. 2) and can potentially speed up direct optimization.

Indirect optimization, on the contrary, does not use particle tracking and thus can be much faster. In this case, lattice parameters should be fitted to satisfy constraints derived from theory. Realization
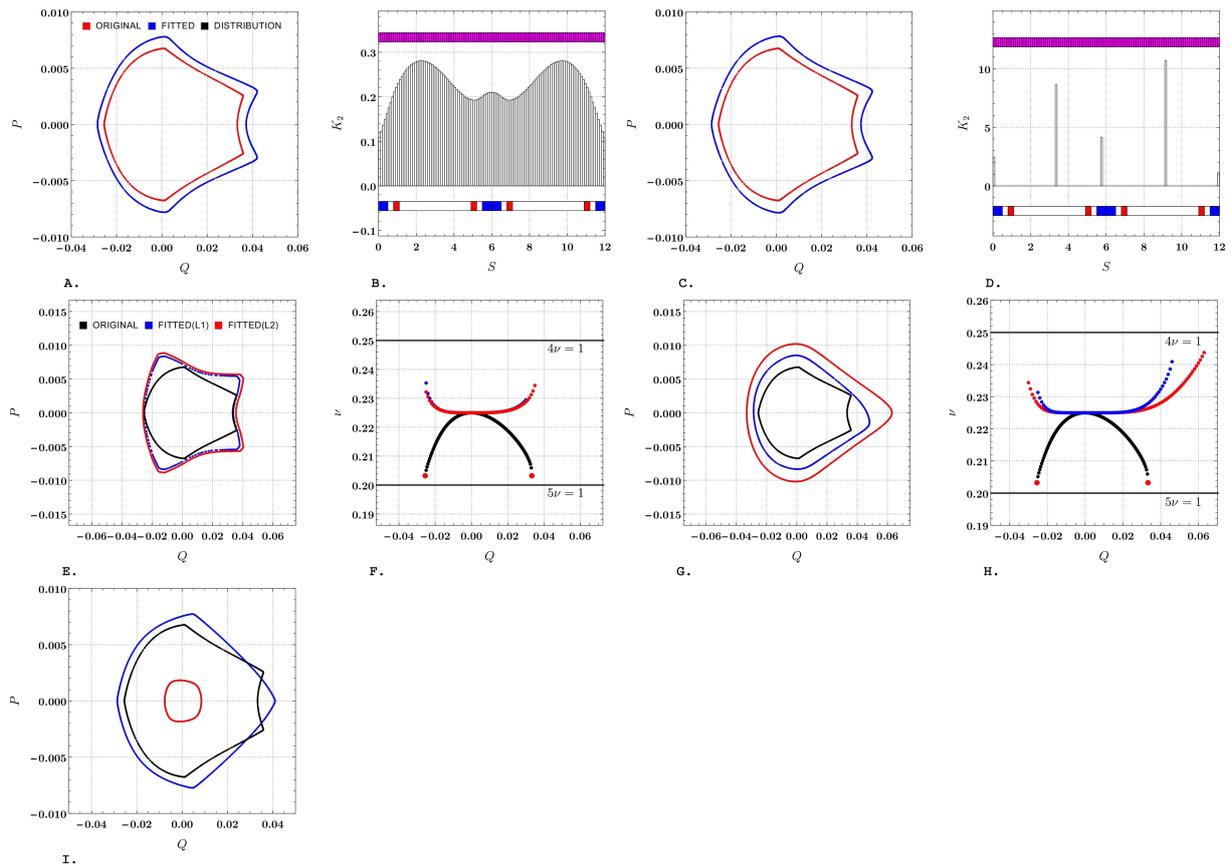
Fig. 5: Examples of linear L1 and L2 indirect optimization. Original lattice is replaced by 120 multipole magnets. (A) L2 geometric optimization, second-order perturbation removed; original DA (red) and L2 DA (blue). (B) Octupole strength distribution for (A). (C) L1 geometric optimization original, second-order perturbation removed; DA (red) and L1 DA (blue). (D) Octupole strength distribution for (C). (E) Normal form L2 and L1 optimizations, second-order tune shift removed; original DA (black), L1 DA (blue), L2 DA (red). (F) Frequency dependence on the amplitude for (E). (G) Normal form L2 and L1 optimizations, second- and fourth-order tune shifts removed and $4\nu = 1$ and $5\nu = 1$ resonances reduced; original DA (black), L1 DA (blue), L2 DA (red). (H) Frequency dependence on amplitude for (G). (I) L2 controlled system realization; original DA (black), second order (blue), third order (red).

of normal form and Hamiltonian control objectives can be reduced to a minimization problem, since in both cases one needs to fit system parameters so that the coefficients of some polynomials have desired values. In principle, one can obtain full symbolic representation of the accelerator model, including linear and non-linear parameters. Usually, one needs to find some lattice parameters (dipole and quadrupole positions and amplitudes, as well as chromatic sextupoles) and suitable distributions of multipoles. If the positions of multipoles are allowed to vary, the minimization problem is intrinsically non-linear. However, for fixed multipole positions, the problem can be partially reduced to linear minimization (here, we use L1 [14] and L2 linear minimization). This is possible, since leading-order dependence of the kth coefficients on kth-order multipole amplitudes is linear. Moreover, polynomial coefficients are linear with respect to kth-order multipole amplitudes up to order $2k - 1$, e.g., second- and third-order coefficients are linear with respect to octupole amplitudes; third, fourth, and fifth orders are linear with respect to decapole amplitudes. Thus, one can reduce indirect optimization to several linear minimization problems.

Geometric optimization can be performed directly on the model, i.e., terms in the non-linear part of the model can be removed. This procedure is motivated by the fact that stable linear systems have infinite

DA. One can then expect an increase in DA if higher-order terms can be neglected. Another option is to use normal forms. In this case, one can reduce the shape frequency dependence on the amplitude and avoid crossing of dangerous resonances. Resonance driving terms can be reduced and invariant shapes can be modified. For Hamiltonian control, DA optimization is motivated by the fact that controlled system is conjugate to a good one. The controlled system can be realized by a suitable distribution of multipole magnets. It should be noted that since fitting is performed up to some order, the effect of higher orders is assumed to be negligible; this might not be a good assumption. Examples of indirect optimization for a FODO example are shown in Fig. 5.

6   Summary

An analytical non-linear accelerator model is important for DA aperture optimization. We have implemented the procedure described in Section 2 in a symbolic manipulator. Thus, semi-realistic symbolic models of accelerator lattices can be computed. Models can include multipole magnets (possibly inside quadrupole magnets), non-linear kinematic effects, and simple fringe field effects. We also plan to include chromatic effects as our next step.

Several methods for non-linear systems analysis were implemented. The full normal form, control term, and controlled system were precomputed for generic polynomial perturbations; one simply needs to substitute corresponding polynomial coefficients to obtain results for a given model.

Various analytical constraints can be generated for a given accelerator model based on non-linear analysis methods. These constraints can be used as a complement to direct DA optimization, or indirect optimization can be performed. We also provided several examples of indirect optimization for a simple FODO cell (Fig. 5). It can be seen that indirect optimization can be used to increase DA, but uncontrolled higher-order terms can lead to DA reduction.

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