

Quantum Field Theory and the Electroweak Standard Model

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Abstract

The Standard Model is one of the main intellectual achievements for about the last 50 years, a result of many theoretical and experimental studies. In this lecture a brief introduction to the electroweak part of the Standard Model is given. Since the Standard Model is a quantum field theory, some aspects for understanding of quantization of abelian and non-abelian gauge theories are also briefly discussed. It is demonstrated how well the electroweak Standard Model works in describing a large variety of precise experimental measurements at lepton and hadron colliders.

1 Introduction

The Standard Model (SM) of strong and electroweak (EW) interactions is the basis for understanding of nature at extremely small distances. In high-energy physics usually the relativistic system of units is used in which the Planck constant \hbar and the speed of light c are equal to unity, $\hbar = c = 1$. Taking into account well-known values for $\hbar = 1.055 \cdot 10^{27}$ erg s, $c = 3 \cdot 10^{10}$ cm/s and the positron electric charge $e = 1.6 \cdot 10^{-19}$ C and using the relation between the electronvolt and erg ($1 \text{ eV} = e \cdot 1 \text{ V} = 1 \text{ V} \cdot 1.6 \cdot 10^{-19} \text{ C} = 1.6 \cdot 10^{-12}$ erg), one easily gets the following very useful relation between length and energy units: $1/\text{GeV} = 2 \cdot 10^{-14}$ cm. Due to the Heisenberg uncertainty principle, $\Delta x \Delta p \geq 1/2$, the above relation allows us to understand which energies (momentum transfers) are needed approximately to probe certain distances:

$$100 \text{ GeV} \rightarrow 10^{-16} \text{ cm},$$

$$1 \text{ TeV} \rightarrow 10^{-17} \text{ cm},$$

$$10 \text{ TeV} \rightarrow 10^{-18} \text{ cm}.$$

Therefore, at the LHC one can study the structure of matter at distances of 10^{-18} – 10^{-17} cm. For small distances of the order of 10^{-16} cm or correspondingly 100 GeV energies the SM works very well, as follows from many studies and measurements.

The SM is a quantum field theory; it is based on a few principles and requirements:

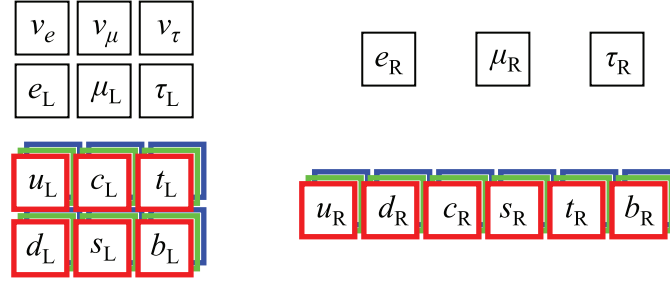
- gauge invariance with lowest dimension (dimension four) operators; SM gauge group: $SU(3)_C \times SU(2)_L \times U(1)_Y$;
- correct electromagnetic neutral currents and correct charge currents with (V–A) structure as follows from four fermion Fermi interactions (1)

$$\frac{G_F}{\sqrt{2}} [\bar{\nu}_\mu \cdot \gamma_\alpha (1 - \gamma_5) \cdot \mu] [\bar{e} \cdot \gamma_\alpha (1 - \gamma_5) \cdot \nu_e] + \text{h.c.}; \quad (1)$$

- three generations without chiral anomalies;
- Higgs mechanism of spontaneous symmetry breaking.

Fermions are combined into three generations forming left doublets and right singlets with respect to the weak isospin (see Fig.1).

$$f_{L,R} = \frac{1}{2}(1 \mp \gamma_5)f,$$

**Fig. 1:** Fermion generation

$$I_f^{3L,3R} = \pm \frac{1}{2}, 0 : \begin{aligned} L_1 &= \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, e_{R1} = e_R^-, Q_1 = \begin{pmatrix} u \\ d \end{pmatrix}_L, u_{R1} = u_R, d_{R1} = d_R, \\ L_2 &= \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, e_{R2} = \mu_R^-, Q_2 = \begin{pmatrix} c \\ s \end{pmatrix}_L, u_{R1} = c_R, d_{R1} = s_R, \\ L_3 &= \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, e_{R3} = \tau_R^-, Q_3 = \begin{pmatrix} t \\ b \end{pmatrix}_L, u_{R1} = t_R, d_{R1} = b_R. \end{aligned}$$

The SM Lagrangian written in accord with the mentioned requirements looks very simple:

$$\begin{aligned} L &= -\frac{1}{4}W_{\mu\nu}^i(W^{\mu\nu})^i - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}G_{\mu\nu}^a(G^{\mu\nu})^a \\ &+ \sum_{f=\ell,q} \bar{\Psi}_L^f (iD_\mu^L \gamma^\mu) \Psi_L^\dagger + \sum_{f=\ell,q} \bar{\Psi}_R^f (iD_\mu^R \gamma^\mu) \Psi_R^\dagger + L_H, \\ L_H &= L_\Phi + L_{\text{Yukawa}}, \\ L_\Phi &= D_\mu \Phi^\dagger D^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^4, \\ L_{\text{Yukawa}} &= -\Gamma_d^{ij} \bar{Q}_L^i \Phi d_R^j + \text{h.c.} - \Gamma_u^{ij} \bar{Q}_L^i \Phi^C u_R^j + \text{h.c.} - \Gamma_e^{ij} \bar{L}_L^i \Phi e_R^j + \text{h.c.} \end{aligned}$$

The field strength tensors and covariant derivatives have very familiar forms:

$$\begin{aligned} W_{\mu\nu}^i &= \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g_2 \varepsilon^{ijk} W_\mu^j W_\nu^k, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ G_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_S f^{abc} A_\mu^b A_\nu^c, \\ D_\mu^L &= \partial_\mu - ig_2 W_\mu^i \tau^i - ig_1 B_\mu \left(\frac{Y_L^f}{2} \right) - ig_S A_\mu^a t^a, \\ D_\mu^R &= \partial_\mu - ig_1 B_\mu \left(\frac{Y_R^f}{2} \right) - ig_S A_\mu^a t^a, \end{aligned}$$

where $i = 1, 2, 3$, $a = 1, \dots, 8$; W_μ^i are gauge fields for the weak isospin group, B_μ are gauge fields for the weak hypercharge group and A_μ^a are gluon gauge fields for the strong $SU_C(3)$ colour group.

$$Y_f = 2Q_f - 2I_f^3 \Rightarrow Y_{L_i} = -1, Y_{e_{R_i}} = -2, Y_{Q_i} = \frac{1}{3}, Y_{u_{R_i}} = \frac{4}{3}, Y_{d_{R_i}} = -\frac{2}{3}.$$

The Lagrangian is so compact that its main part can be presented on the CERN T-shirt (see Fig. 2).

It is hard to imagine that such a simple Lagrangian allows one to describe basically all the phenomena of the microworld. But the SM Lagrangian, being expressed in terms of physics components, is not that simple, leading after quantization to many interaction vertices between particles or quanta of corresponding quantum fields.

This lecture is organized as follows. In the next section some aspects of quantum field theory are briefly discussed. After a motivation as to why do we need a quantum field theory, we consider scalar

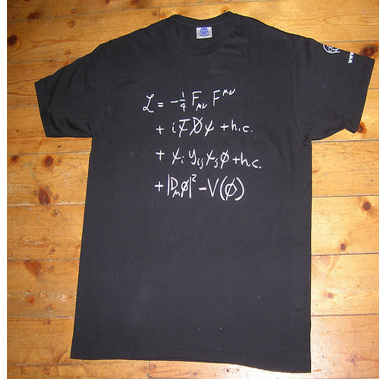


Fig. 2: CERN T-shirt with SM Lagrangian

fields and introduce the Feynman propagator and functional integral approach as a quantization method. The functional integral given in the holomorphic representation allows us to clarify boundary conditions and show the connection between the Green functions and S-matrix elements. Feynman diagrams are introduced. The formalism is extended to the fermion and gauge fields stressing peculiarities in the quantization procedure and Feynman rule derivation. In the next section a construction of the EW SM Lagrangian is presented. We discuss experimental facts and theory principles based on which the EW part of the SM Lagrangian for fermion and gauge fields is constructed. We show explicitly which conditions on weak hypercharges allow us to get correctly electromagnetic and charge current (CC) interactions and predict additional neutral currents (NCs). We demonstrate how potentially dangerous chiral anomalies cancelled out. Then spontaneous symmetry breaking, the Goldstone theorem and the appearance of Nambu–Goldstone bosons are briefly discussed. The Brout–Englert–Higgs–Hagen–Guralnik–Kibble mechanism of spontaneous symmetry breaking is introduced leading to non-zero masses of the gauge fields and appearance of the Higgs boson. Very briefly we discuss in addition to the unitary gauge the covariant gauge, propagators of Goldstone bosons and ghosts. At the end of the section it is shown how the spontaneous symmetry breaking mechanism leads to non-zero masses for the fermions in the SM and how very naturally the Cabibbo–Kobayashi–Maskawa mixing matrix appears. In the next section we concentrate on some phenomenological aspects of the EW SM such as connections between the Fermi constant G_F , the Higgs vacuum expectation value v , consistency of low-energy measurements and W, Z mass measurements, W-, Z-boson decay widths and branching ratios, number of light neutrinos, two-fermion processes in e^+e^- collisions, tests of the gauge boson self-interactions, top-quark decays and the EW top production (single top). Briefly we discuss the EW SM beyond the leading order, renormalization and running coupling in quantum electrodynamics (QED), as a simplest example, running masses and running parameters in the SM, precision EW data and global parameter fits. Concluding remarks are given in the next section. The quantum chromodynamics (QCD) part of the SM and the phenomenology of the Higgs boson are not discussed in these lectures as they are addressed in other lectures of the School.

For a deeper understanding of the topics discussed, one can recommend a number of very good textbooks and reviews [1–9] and lectures given at previous schools and specialized reviews [10, 12–15], which have been used in preparation of this lecture.

2 Introductory words to quantum theory

In classical mechanics a system evolution follows from the principle of least action:

$$\delta S = \delta \int_{t_i}^{t_f} dt L(q(t), \dot{q}(t)) = 0; \quad \int_{t_i}^{t_f} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta(\dot{q}) \right] = 0; \quad \delta(\dot{q}) = \frac{d}{dt} \delta q.$$

For an arbitrarily small variation δq , one gets the well-known Lagrange equation of motion

$$\frac{\partial L}{\partial q} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right). \quad (2)$$

For a non-relativistic system described by the Lagrangian $L = \frac{m\dot{q}^2}{2} - V(q)$, the second Newton law follows from Eq. (2),

$$m\ddot{q} = -\frac{\partial V}{\partial q} = F.$$

The Hamiltonian of the system is related to the Lagrangian in the following well-known way:

$$H(p, q) = p\dot{q} - L(q, \dot{q}),$$

where \dot{q} is a solution of the equation $p = \frac{\partial L}{\partial \dot{q}}$.

In quantum mechanics the coordinate and momentum are replaced by corresponding operators $p, q \rightarrow \hat{p}, \hat{q}$ with postulated commutator relation $[\hat{p}(0), \hat{q}(0)] = -i\hbar$. In the Heisenberg picture the system evolution is described by the Heisenberg equation with time-dependent operators; for example, the equation for the coordinate operator has the following form:

$$\frac{\partial \hat{q}}{\partial t} = \frac{i}{\hbar} [\hat{H}, \hat{q}] \quad (3)$$

with a formal solution

$$\hat{q}(t) = e^{\frac{i}{\hbar}\hat{H}t} \hat{q}(0) e^{-\frac{i}{\hbar}\hat{H}t}.$$

This easily follows from the equalities

$$\frac{\partial \hat{q}}{\partial t} = \frac{i}{\hbar} \hat{H} e^{\frac{i}{\hbar}\hat{H}t} \hat{q}(0) e^{-\frac{i}{\hbar}\hat{H}t} + e^{\frac{i}{\hbar}\hat{H}t} \hat{q}(0) e^{-\frac{i}{\hbar}\hat{H}t} \hat{H} = \frac{i}{\hbar} \hat{H} \hat{q}(t) - \frac{i}{\hbar} \hat{q}(t) \hat{H} = \frac{i}{\hbar} [\hat{H}, \hat{q}].$$

For the coordinate and momentum operators, one can prove the following inequality:

$$\Delta q \cdot \Delta p \geq 1/2,$$

which is called the Heisenberg uncertainty principle.

Let us recall a simple proof of the Heisenberg uncertainty principle. The mid value of any operator \hat{A} and its dispersion are given by the relations

$$\langle \psi | \hat{A} | \psi \rangle = \bar{A}, \quad \langle \psi | (\hat{A} - \bar{A})^2 | \psi \rangle.$$

Let us take the following operator constructed from the momentum and coordinate operators ($[\hat{p}, \hat{q}] = -i\hbar$) with arbitrary constant γ :

$$\hat{A} = \hat{p} + i\gamma\hat{q} - (\bar{p} + i\gamma\bar{q}).$$

Then the conjugated operator has the form

$$\hat{A}^\dagger = \hat{p} - i\gamma\hat{q} - (\bar{p} - i\gamma\bar{q}).$$

For any state $|\psi\rangle$ we have

$$\langle \psi | \hat{A}^\dagger \hat{A} | \psi \rangle \geq 0$$

,

$$\begin{aligned} & \langle \psi | [(\hat{p} - \bar{p}) - i\gamma(\hat{q} - \bar{q})] [(\hat{p} - \bar{p}) + i\gamma(\hat{q} - \bar{q})] | \psi \rangle \\ & = (\Delta p)^2 + \gamma^2 \Delta q^2 - i\gamma(\hat{q}\hat{p} - \hat{p}\hat{q}) = (\Delta p)^2 + \gamma^2 \Delta q^2 + \gamma\hbar \geq 0. \end{aligned}$$

This is true for any value of γ and therefore the determinant is not positive:

$$\frac{\hbar^2}{4} - \Delta q^2 \Delta p^2 \leq 0.$$

Thus, we immediately arrive at the uncertainty principle:

$$\Delta q \Delta p \geq \frac{1}{2} \hbar.$$

Let us consider the simplest system, the harmonic oscillator, described by the Hamiltonian (we put $\hbar = 1$)

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2),$$

and construct two operators \hat{a} and \hat{a}^\dagger which are called the annihilation and creation operators:

$$\begin{aligned} \hat{q} &= \frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger); & \hat{p} &= -i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger). \\ \implies \hat{a} &= \sqrt{\frac{\omega}{2}}\hat{q} + i\frac{1}{\sqrt{2\omega}}\hat{p}; & \hat{a}^\dagger &= \sqrt{\frac{\omega}{2}}\hat{q} - i\frac{1}{\sqrt{2\omega}}\hat{p}. \end{aligned} \quad (4)$$

From the equation $[\hat{p}, \hat{q}] = -i$, one gets $[\hat{a}, \hat{a}^\dagger] = 1$ and the Hamiltonian takes the form

$$\hat{H} = \frac{\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}).$$

It is easy to show the following commutation relations with the Hamiltonian:

$$[\hat{H}, \hat{a}] = -\omega\hat{a} \quad \text{and} \quad [\hat{H}, \hat{a}^\dagger] = \omega\hat{a}^\dagger.$$

From (3),

$$\frac{d\hat{a}}{dt} = i[\hat{H}\hat{a}] = -i\omega\hat{a} \implies \hat{a}(t) = \hat{a}(0)e^{-i\omega t}; \quad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}.$$

Let us consider states with definite energy:

$$\hat{H}|E\rangle = E|E\rangle.$$

Then the state $\hat{a}|E\rangle$ ($\hat{a}^\dagger|E\rangle$) corresponds to the energy $(E - \omega)$ ($(E + \omega)$). Indeed,

$$\hat{H}\hat{a}|E\rangle = \hat{a}\hat{H}|E\rangle - \omega\hat{a}|E\rangle = (E - \omega)\hat{a}|E\rangle,$$

$$\hat{H}\hat{a}^\dagger|E\rangle = (E + \omega)\hat{a}^\dagger|E\rangle.$$

Let us construct states (Hilbert space of states) starting from the ‘vacuum’ state $|0\rangle$:

$$\hat{a}|0\rangle = 0.$$

What is the energy of the vacuum state? This is

$$\hat{H}|0\rangle = \frac{\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})|0\rangle = \frac{\omega}{2}|0\rangle.$$

The state $|n\rangle$ we introduce as $|n\rangle = (\hat{a}^\dagger)^n|0\rangle$; its energy is given by

$$\hat{H}(\hat{a}^\dagger)^n|0\rangle = \omega(n + 1/2)|n\rangle.$$

As we know, such a construction is very successful in describing non-relativistic quantum phenomena (spectra of atoms, molecules, nuclei etc).

But there are well-known problems:

- In experiments we have not only particle creation and annihilation but also production of new particles and antiparticles.
- Relativity and causality might be in conflict with quantum principles.
If in four-dimensional areas X and Y points are separated by the space-like interval $(x - y)^2 < 0$, events in the points x and y are causally independent. But in QM we have the uncertainty principle, $\Delta p \Delta x \geq 1$, and in some regions of the order of $\Delta x \sim 1/m$ close to indistinct boundaries of the areas X and Y the causality might be violated.

Quantum field theory allows us to resolve both of these problems simultaneously.

To describe a particle and an antiparticle with mass m , momentum \vec{k} and energy $\omega_k = k^0 = \sqrt{\vec{k}^2 + m^2}$, let us consider two sets of creation and annihilation operators for each momentum point \vec{k} \hat{a}, \hat{a}^\dagger and \hat{b}, \hat{b}^\dagger . Vacuum is defined by the requirements $\hat{a}|0\rangle = \hat{b}|0\rangle = 0$. The Hamiltonian for every momentum point \vec{k} obviously has the following form:

$$H_k = \frac{\omega_k}{2} (\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}) + \hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}(\vec{k})\hat{b}^\dagger(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})),$$

where $\omega_k = k^0 = \sqrt{\vec{k}^2 + m^2}$. Please note that the mass parameter m is the same for all oscillators with different \vec{k} .

One can use a different normalization of an integral measure in order to get the commutation relations:

$$\int \overline{d\vec{k}} [\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k})] = 1; \quad \int \overline{d\vec{k}} [\hat{b}(\vec{k})\hat{b}^\dagger(\vec{k})] = 1.$$

We are using

$$\overline{d\vec{k}} = \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \Rightarrow [\hat{a}(\vec{k})\hat{a}^\dagger(\vec{k}')] = (2\pi)^2 2\omega_k \delta(\vec{k} - \vec{k}').$$

Total momentum and charge operators taken in so-called ‘normal ordering’ have the following form:

$$\begin{aligned} \hat{P}^\mu &= \int \overline{d\vec{k}} k^\mu [\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})], \\ \hat{Q} &= \int \overline{d\vec{k}} [\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) - \hat{b}^\dagger(\vec{k})\hat{b}(\vec{k})]. \end{aligned}$$

One can prove the following commutator relations, which clarify the meaning of the operators:

$$\begin{aligned} [\hat{P}^\mu, \hat{a}^\dagger(\vec{k})] &= k^\mu \hat{a}^\dagger(\vec{k}); \quad [\hat{P}^\mu, \hat{a}(\vec{k})] = -k^\mu \hat{a}(\vec{k}), \\ [\hat{Q}, \hat{a}^\dagger(\vec{k})] &= \hat{a}^\dagger(\vec{k}); \quad [\hat{Q}, \hat{b}^\dagger(\vec{k})] = -\hat{b}^\dagger(\vec{k}). \end{aligned}$$

Now one can construct the field operator:

$$\hat{\Phi}(x) = \int \overline{d\vec{k}} [e^{-ikx} \hat{a}(\vec{k}) + e^{ikx} \hat{b}^\dagger(\vec{k})].$$

The momentum operator acts on the field operator leading to a coordinate translation:

$$e^{i\hat{P}_\mu y^\mu} \hat{\Phi}(x) e^{-i\hat{P}_\mu y^\mu} = \hat{\Phi}(x + y).$$

Indeed, one may prove this in a very simple way. Let us introduce an operator $\hat{A}(\alpha)$ depending on some numerical parameter α :

$$e^{i\alpha y^\mu \hat{P}_\mu} \hat{a}(k) e^{-i\alpha y^\mu \hat{P}_\mu} = \hat{A}(\alpha).$$

The operator obeys the following equation:

$$\frac{dA}{d\alpha} = iy^\mu \widehat{P}^\mu \widehat{A} - iy^\mu \widehat{P}^\mu \widehat{A} = -iy^\mu [\widehat{P}^\mu \widehat{A}].$$

One finds a solution in the form $\widehat{A} = \widehat{a}(\vec{k})f(\alpha)$, from which one can immediately see the needed translation relation:

$$\begin{aligned} \implies \widehat{a}(k) \frac{df}{d\alpha} &= -iy^\mu f(\alpha) k^\mu \widehat{a}(k) \implies \frac{df}{d\alpha} = -iy^\mu k^\mu f(\alpha) \implies \\ e^{i\alpha y^\mu \widehat{P}^\mu} \widehat{a}(k) e^{-i\alpha y^\mu \widehat{P}^\mu} &= \widehat{a}(k) e^{-ik^\mu y_\mu} \implies \widehat{\Phi}(x) = \widehat{\Phi}(x + y). \end{aligned}$$

It is very important to note that such a field operator obeys the Klein–Gordon equation:

$$[\square^2 + m^2] \widehat{\Phi}(x) = \int \overline{dk} \left[e^{-ikx} (-k^2 + m^2) \widehat{a}(k) + e^{ikx} (-k^2 + m^2) \widehat{b}(k) \right] = 0$$

because of

$$-k^2 + m^2 = -k_0^2 + \vec{k}^2 + m^2 = -\omega_k^2 + \omega_k^2 = 0$$

What charge has the state created by the operator $\widehat{\Phi}(x)$? Let us act on the vacuum state by the $\widehat{\Phi}(x)$ operator: $\widehat{\Phi}(x)|0\rangle$. This state has the following charge:

$$\widehat{Q} \widehat{\Phi}(x)|0\rangle = -\widehat{\Phi}(x)|0\rangle.$$

In the same way, one gets

$$\widehat{Q} \widehat{\Phi}^\dagger(x)|0\rangle = \widehat{\Phi}^\dagger(x)|0\rangle.$$

This means that the field operator $\widehat{\Phi}(x)$ acting on vacuum produces the state with the negative charge (-1) and the field operator $\widehat{\Phi}^\dagger(x)$ produces the state with the positive charge $(+1)$.

Now let us consider two space–time points x_1 and x_2 :



and two-point correlation functions—products of field operators between vacuum states. If $t_1 < t_2$, the operator $\widehat{\Phi}(x_1)|0\rangle$ in the correlator

$$\langle 0 | \widehat{\Phi}^\dagger(x_2) \widehat{\Phi}(x_1) | 0 \rangle$$

creates the charge -1 at t_1 , and the operator $\widehat{\Phi}^\dagger(x_2) \widehat{\Phi}(x_1)|0\rangle$ annihilates this charge at t_2 . So, charge -1 propagates from the point x_1 to x_2 and $t_2 > t_1$. If $t_2 < t_1$, the operator $\widehat{\Phi}^\dagger(x_2)|0\rangle$ in the correlator

$$\langle 0 | \widehat{\Phi}(x_1) \widehat{\Phi}^\dagger(x_2) | 0 \rangle$$

creates the charge $+1$ at t_2 , and $\widehat{\Phi}(x_1) \widehat{\Phi}^\dagger(x_2)|0\rangle$ annihilates this charge at t_1 . So, charge $+1$ propagates from the point x_2 to x_1 and $t_2 < t_1$.

Since both these actions do not change the vacuum, we should take both correlators into account to see the causal relation of events in points x_1 and x_2 :

$$\langle 0 | T \{ \widehat{\Phi}^\dagger(x_2) \widehat{\Phi}(x_1) \} | 0 \rangle$$

$$\begin{aligned}
&= \langle 0 | \widehat{\Phi}^\dagger(x_2) \widehat{\Phi}(x_1) | 0 \rangle \Theta(t_2 - t_1) + \langle 0 | \widehat{\Phi}(x_1) \widehat{\Phi}^\dagger(x_2) | 0 \rangle \Theta(t_1 - t_2) \\
&= i \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik(x_2 - x_1)}}{k^2 - m^2 + i0} = D_c(x_2 - x_1).
\end{aligned}$$

The function

$$D_c(x) = i \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m^2 + i0} \quad (5)$$

is called the Feynman propagator. Obviously, the Feynman propagator is a Green function of the Klein–Gordon equation. One can check that all the commutators between the field operators in the points x and y separated by the space-like interval $(x - y)^2 < 0$ are equal to zero. So, the causality takes place. Also, one can construct multiparticle states by acting of the creation operators (operators have different quantum numbers corresponding to different kinds of particles) on the vacuum state $|\vec{k}_1, \dots, \vec{k}_n\rangle = \prod_{i=1}^n \hat{a}^\dagger(\vec{k}_i) |0\rangle$. The energy and momentum of the states are then obtained by acting of the operators

$$\begin{aligned}
\hat{P}^0 |\vec{k}_1, \dots, \vec{k}_n\rangle &= \hat{H} |\vec{k}_1, \dots, \vec{k}_n\rangle = \left(\sum_{i=1}^n k_i^0 \right) |\vec{k}_1, \dots, \vec{k}_n\rangle, \\
\hat{\vec{P}} |\vec{k}_1, \dots, \vec{k}_n\rangle &= \left(\sum_{i=1}^n \vec{k}_i \right) |\vec{k}_1, \dots, \vec{k}_n\rangle.
\end{aligned}$$

Of course, one can get the same results using the usual canonical quantization with the correspondence

$$\begin{aligned}
q(t), \dot{q}(t), L(q, \dot{q}), S &= \int dt L(q, \dot{q}) \iff \\
\varphi(x), \partial_\mu \varphi(x), L(\varphi, \partial_\mu \varphi), S &= \int d^4 x L(\varphi, \partial_\mu \varphi).
\end{aligned}$$

The field momentum is then

$$\pi(x) = \frac{\partial L}{\partial(\partial_0 \varphi(x))}; \quad \partial_0 \varphi(x) = \dot{\varphi}.$$

The Lagrangian of the complex scalar field has the form

$$\begin{aligned}
L &= \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi, \\
\pi(x) &= \dot{\varphi}^\dagger(x); \quad \pi^\dagger(x) = \dot{\varphi}(x).
\end{aligned}$$

In the same way, as in classical mechanics, the equation of motion comes from the principle of least action:

$$\frac{\partial L}{\partial \varphi} = \partial_\mu \frac{\partial L}{\partial \partial_\mu \varphi} \longrightarrow (\square - m^2)\varphi = 0.$$

The Feynman propagator D_c introduced above is a Green function of the equation of motion. For the field and momentum operators, one naturally assumes the equal-time commutation relation

$$[\hat{\pi}(\vec{x}, t), \hat{\varphi}(\vec{x}', t')] \Big|_{t=t'} = -i\delta(\vec{x} - \vec{x}').$$

The Lagrangian is invariant under a global phase shift:

$$\varphi(x) \rightarrow e^{i\alpha\varphi(x)}, \quad \alpha \equiv \text{Const.}$$

Such a combination, called a current, is conserved (this is a simple example of the first Noether theorem):

$$j^\mu(x) = i\partial^\mu \varphi^\dagger \varphi - i\varphi^\dagger \partial^\mu \varphi,$$

$$\partial_\mu j^\mu = i\Box\varphi^\dagger\varphi + i\partial^\mu\varphi^\dagger\partial_\mu\varphi - i\partial_\mu\varphi^\dagger\partial^\mu\varphi - i\varphi^\dagger\Box\varphi = 0.$$

Conservation of the current leads to the conservation of the charge:

$$Q = \int d^3\vec{x}j^0 = \int d\vec{x}(i\dot{\varphi}^\dagger\varphi - i\varphi^\dagger\dot{\varphi}),$$

$$\frac{dQ}{dt} = \int d\vec{x}\partial_0j^0 = \int d\vec{x}(\partial_i j^i) = \int_\Omega d\vec{n}\vec{j} = 0$$

for falling-off fields.

3 Functional integral in quantum mechanics

However, for our further consideration, the functional integral approach to quantum field theory is more useful. In particular, it allows us to quantize non-abelian gauge field theories, to clarify better boundary conditions and renormalization procedure and to get a reduction formula (connection between S-matrix elements and the Green functions).

Once more we begin with the quantum mechanics as a simple example.

$$L(q_i, \dot{q}_i) \rightarrow p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad H(q_i, p_i) = \dot{q}_i p_i - L(q_i, \dot{q}_i)|_{\dot{q}=f(t)}$$

$$[\hat{q}^i(t), \hat{p}^j(t)] = i\hbar\delta^{ij}\hat{1}.$$

Let us consider a simple system described by the non-relativistic Hamiltonian

$$H(\hat{p}, \hat{q}) = \frac{\hat{p}^2}{2m} + V(\hat{q}).$$

In the Schrödinger picture, the evolution of a quantum system follows from the Schrödinger equation

$$i\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle.$$

The formal solution of the Schrödinger equation is

$$|\Psi(t)\rangle = e^{-i\hat{H}t}|\Psi(0)\rangle.$$

One can define states:

$$|q, t\rangle : \hat{q}|q, t\rangle = q|q, t\rangle.$$

Then the wave function coordinate representation is

$$\Psi(q, t) = \langle q|\Psi(t)\rangle = \left\langle q\left|e^{-i\hat{H}t}\right|\Psi(0)\right\rangle \text{ and } \langle q|q'\rangle = \delta(q - q').$$

If we introduce a complete set of states $|q_0\rangle$ such that

$$\hat{1} = \int dq_0 |q_0\rangle\langle q_0|,$$

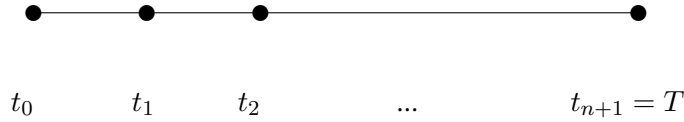
we can write

$$\Psi(q, t) = \int dq_0 \langle q|e^{-i\hat{H}t}|q_0\rangle\langle q_0|\Psi(0)\rangle = \int dq_0 K(q, q_0, t)\langle q_0|\Psi(0)\rangle,$$

where K is the so-called kernel of the Schrödinger equation.

Obviously, $[\hat{H}, \hat{H}] = 0$ and therefore

$$e^{-i\hat{H}T} = e^{-i\hat{H}(t_{n+1}-t_n)} \cdot e^{-i\hat{H}(t_n-t_{n-1})} \dots e^{-i\hat{H}(t_1-t_0)}.$$



At each time moment t_i we can introduce a unity operator

$$\hat{1} = \int dq_i |q_i\rangle \langle q_i|.$$

Then, for the kernel K , we obtain the following formula:

$$K(q, q_0, T - t_0) = \int \lim_{n \rightarrow \inf} \prod_{i=1}^n dq_i |q_{n+1-i}\rangle \langle q_i | e^{-i\hat{H}\delta t} |q_{n-i}\rangle \dots \langle q_1 | e^{-i\hat{H}\delta t} |q_0\rangle,$$

$$e^{\varepsilon(\hat{A}+\hat{B})} = e^{\varepsilon\hat{A}} + e^{\varepsilon\hat{B}}(1 + o(\varepsilon^2)),$$

$$\langle q_{i+1} | e^{-i\hat{H}\delta t} |q_i\rangle = \langle q_{i+1} | e^{-i\frac{\hat{p}^2}{2m}\delta t} \cdot e^{-i\hat{V}(q)\delta t} |q_i\rangle.$$

For very small (δt) , $e^{-i\hat{V}(q)\delta t}$ could be factorized out, and therefore

$$\langle q_{i+1} | e^{-i\hat{H}\delta t} |q_i\rangle \approx e^{-i\hat{V}(q_i)\delta t} \cdot \langle q_{i+1} | e^{-i\frac{\hat{p}^2}{2m}\delta t} |q_i\rangle.$$

The last term can be expressed in the following way:

$$\langle q_{i+1} | e^{-i\frac{\hat{p}^2}{2m}\delta t} |q_i\rangle = \int \frac{dp}{2\pi} \langle q_{i+1} | e^{-i\frac{\hat{p}^2}{2m}\delta t} |p\rangle \langle p|q_i\rangle = \int \frac{dp}{2\pi} e^{-ip(q_{i+1}-q_i)\delta t} e^{-i\frac{p^2}{2m}\delta t},$$

where $\langle q|p\rangle = e^{ipq}$, $\langle p|q\rangle = e^{-ipq}$ and $\hat{1} = \int \frac{dp}{2\pi} |p\rangle \langle p|$.

One could make the following substitution of the integration variable:

$$p' = \left[\frac{p}{\sqrt{2m}} - \sqrt{2m}(q_{i+1} - q_i) \frac{1}{2} \right] \sim e^{i\frac{m}{2}(q_{i+1}-q_i)^2} \frac{1}{\delta t}.$$

Then, for the kernel K , one gets

$$K(q, q_0; t) = N \int dq_i \prod_i e^{i \left[\frac{m}{2} \left(\frac{q_{i+1}-q_i}{\delta t} \right)^2 - V(q_i) \right] \delta t} = N \int_{Dq} e^{i \int_0^t dt \left[\frac{mv^2}{2} - V(q) \right]} = \int e^{iS} D(q).$$

For our consideration it is not needed, but if one takes the integral of dp , one gets the following representation for the functional integral measure:

$$D(q) = \lim_{n \rightarrow \inf, \delta t = \frac{t}{n} \rightarrow 0} \cdot \sqrt{\frac{m}{2\pi i \delta t}} \prod_{T=1}^n \left(\sqrt{\frac{m}{2\pi i \delta t}} dq_T \right).$$

We do not discuss mathematical aspects of how well such a construction is determined.

The formula for the kernel, being written as

$$K(q, q_0; t) = \int D(q) e^{iS} = \int D(q) e^{i \int_0^t dt L(q, \dot{q}, t)}, \quad (6)$$

can be generalized to the case of quantum field theory. For that we need to recall a few simple but important formulas for multidimensional Gaussian integrals.

4 Gaussian integrals

Let us consider n coordinates y_1, \dots, y_n as a formal vector

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Obviously, $y^T = (y_1 \dots y_n)$ with a formal definition of ‘scalar’ product

$$y^T \cdot x = \sum_i y_i x_i.$$

The well-known answer for the Gaussian integral has the form

$$Z = \int dx_1 \dots dx_n \cdot e^{-\frac{1}{2}x^T A x} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}, \quad (7)$$

where A is a positive-definite $n \times n$ matrix.

Problem. Take the integral and obtain the above formula.

Reminder: $\det A = \prod_{i=1}^n \lambda_i$, where λ_i is the eigenvalue of the matrix A .

The integral (7) is an analogue of the integral in a field theory without external sources, as we shall see in the next part. An analogue of the functional integral with a source is as follows:

$$Z[J] = \int dx_1 \dots dx_n \cdot e^{-\frac{1}{2}x^T A x + J^T x}, \quad (8)$$

where J is some vector $J^T = (J_1 \dots J_n)$. If we make a substitution of integration variable

$$x' = x - (A^{-1})J,$$

the integral (8) will take the form

$$Z[J] = e^{\frac{1}{2}J^T A^{-1} J} \cdot \int dx'_1 \dots dx'_n \cdot e^{-\frac{1}{2}x'^T A x'}.$$

So, the integral $Z[J]$ is given by

$$Z[J] = e^{\frac{1}{2}J^T A^{-1} J} \cdot Z = e^{\frac{1}{2}J^T A^{-1} J} \cdot \frac{(2\pi)^{n/2}}{\sqrt{\det A}}.$$

Generalization of the above formula for the case of complex variables of integration is straightforward.

Let $z = x + iy$ and $z^* = x - iy$. We need to compute the integral

$$Z_C = \int \prod_{k=1}^n dz_k^* dz_k e^{-z^\dagger B z},$$

where B is a Hermitian matrix and $z^\dagger = (z^*)^T$. One can diagonalize the quadratic form $z^{*\dagger} B z$ by applying a unitary transformation of variables $z' = U \cdot z$ such that the matrix UBU^\dagger becomes diagonal:

$$UBU^\dagger = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

The integral Z_C then takes the form

$$Z_C = \int \prod_{k=1}^n dz_k'^* dz_k' e^{-\lambda_k |z_k'|^2} = \int \prod_{k=1}^n dx_k dy_k e^{-\lambda_k (x_k^2 + y_k^2)} = \frac{\pi^n}{\det B}. \quad (9)$$

If we add the external complex ‘source’ J ,

$$Z_C[J] = \int \prod_{k=1}^n dz_k'^* dz_k' e^{-z^\dagger B z + J^\dagger z + z^\dagger J},$$

with the shift of variables of integration we get

$$Z_C[J] = e^{J^\dagger B^{-1} J} \cdot Z_C = e^{J^\dagger B^{-1} J} \cdot \frac{\pi^n}{\det B}. \quad (10)$$

In the field theory we have to consider interacting fields. So, we need to consider more complicated integrals involving source ‘interactions’:

$$Z_{\text{int}}[J] = \frac{1}{Z_{\text{int}}} \int \prod dx e^{-\frac{1}{2} x A x + J x - V(x)},$$

where $Z_{\text{int}} = Z_{\text{int}}[0]$.

If we expand the exponent $e^{-V(x)}$ in ‘perturbation’ theory we can easily get the following form for the integral $Z_{\text{int}}[J]$:

$$Z_{\text{int}}[J] = \frac{1}{Z[0]} e^{-V[\frac{\partial}{\partial J}]} \cdot Z[J] = e^{-V[\frac{\partial}{\partial J}]} \cdot e^{\frac{1}{2} J A^{-1} J}.$$

5 Functional integral in quantum field theory

As we have seen already, a transition from mechanics to a field theory could be done by means of a formal correspondence between the coordinate and its derivative and the field $\varphi(x)$ and its derivative:

$$q(t) \rightarrow \varphi(x); \quad \dot{q}(t) \rightarrow \partial_\mu \varphi(x).$$

With this analogy one can immediately write down the following formula for the evolution kernel in the case of quantum field theory:

$$Z[J] = \int D(\varphi) e^{i \int d^4 x L(\varphi, \partial_\mu \varphi) + i \int d^4 x J(x) \varphi(x)}, \quad (11)$$

where the measure $D(\varphi) = \prod_x d\varphi(x)$ corresponds to the integration over all possible trajectories (field configurations).

Now all the formulas we derived for Gaussian integrals in the previous section can be applied here using the functional derivative instead of the usual one. For example,

$$\frac{\delta J(y)}{\delta J(x)} = \delta^{(4)}(x - y).$$

If we consider the Lagrangian for the free scalar field

$$L = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 = -\frac{1}{2} \varphi (\square^2 + m^2) \varphi \equiv -\frac{1}{2} \varphi D_c^{-1} \varphi, \quad (12)$$

we can get for $Z[J]$,

$$Z[J] = \exp \left(\frac{1}{2} \int d^4 x d^4 y J(x) D_c(x - y) J(y) \right), \quad (13)$$

where the normalization of the measure is taken such that

$$Z[0] = 1.$$

Here the function D_c is the Green function of the equation of motion:

$$D_c^{-1} \cdot D_c = 1,$$

which more accurately means that

$$i(\square^2 + m^2)_x D_c(x - y) = \delta^{(4)}(x - y).$$

In the momentum representation, by taking the Fourier transform of both sides of this equation, one gets

$$i(-p^2 + m^2) D(p) = 1.$$

The formal solution of the equation is

$$D(p) = \frac{i}{p^2 - m^2}.$$

But we need to fix how to deal with the pole. The only possible choice is to add $+i\varepsilon$. In this case the function $D_c(p)$ has the familiar form of the Feynman propagator:

$$D_c(p) = \frac{i}{p^2 - m^2 + i\varepsilon}.$$

Indeed, such fixing of the denominator leads to the fact that in the expression for the functional integral

$$Z[J] = \int D(\varphi) \exp\left(i \int d^4x \left[\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 + J\varphi \right]\right), \quad (14)$$

m^2 gets a shift $m^2 \rightarrow m^2 - i\varepsilon$ and the term

$$\int D(\varphi) \exp\left(-\varepsilon \int dx \varphi^2(x)\right)$$

ensures the convergence of the integral.

On the other hand, as we have discussed already, such a form of the Feynman propagator leads to Feynman boundary conditions, namely if $x^0 < y^0$ the particle propagates from \vec{x} to \vec{y} , and if $x^0 > y^0$ the corresponding antiparticle propagates from \vec{x} to \vec{y} , as follows from the expression

$$D_c(x - y) = - \int \frac{d\vec{p}}{(2\pi)^3 2\omega_p} e^{i\vec{p}(\vec{x}-\vec{y})} [\Theta(t)e^{-i\omega_p t} + \Theta(-t)e^{i\omega_p t}].$$

So, the Feynman propagator is a Green function and, in other words, the inverse quadratic form in the action (12).

In the case of an interacting potential $V(\varphi)$ we get from (13) the following general expression for the generating functional:

$$Z_V[J] = \exp\left(-i \int d^4x V\left(\frac{\delta}{i\delta J(x)}\right)\right) \cdot \exp\left(\frac{1}{2} \int dy dz J(y) D_c(y - z) J(z)\right). \quad (15)$$

As will be discussed, one can get the Green functions by taking the needed number of functional derivatives with respect to the source. However, this is not enough, since we do not know how exactly the functional integral, called the generalized functional integral, and the Green functions are related to the S-matrix elements needed to compute physics observables.

6 Functional integral in holomorphic representation

We start with the harmonic oscillator as we did before:

$$H(p, q) = \frac{\hat{p}^2}{2} + \frac{\omega^2 \hat{q}^2}{2}.$$

The creation and annihilation operators have the form (4) with the commutator

$$[\hat{a}, \hat{a}^\dagger] = 1. \quad (16)$$

The Hamiltonian of the system taken in a normal form (all creation operators are on the right-hand side) is

$$H = \omega \hat{a}^\dagger \hat{a}.$$

The commutator relation (16) has a very nice representation in terms of holomorphic functions, which are introduced by means of the following scalar product:

$$\langle f_1 | f_2 \rangle = \int (f_1(a^*))^* f_2(a^*) e^{-a^* a} \frac{da^* da}{2\pi i}.$$

With such a definition of the scalar product, the set of functions $\Psi_n(a^*) = \frac{(a^*)^n}{\sqrt{n!}}$, $n \geq 0$, forms an orthonormal basis

$$\langle \Psi_n | \Psi_m \rangle = \frac{1}{\sqrt{n!m!}} \int a^n (a^*)^m e^{-a^* a} \frac{da^* da}{2\pi i} = \delta_{nm}. \quad (17)$$

One can easily prove that $\sum_n |\Psi_n\rangle \langle \Psi_n| = 1$.

Problem. Prove the relation (17).

The operators \hat{a}^\dagger and \hat{a} act according to the following rules:

$$\hat{a}^\dagger \cdot f(a^*) = a^* f(a^*), \quad \hat{a} f(a^*) = \frac{d}{da^*} f(a^*). \quad (18)$$

By direct substitution, one can prove the following relation:

$$\langle f_1 | \hat{a}^\dagger f_2 \rangle = \langle \hat{a} f_1 | f_2 \rangle, \quad (19)$$

which means that the operators \hat{a}^\dagger and \hat{a} are conjugate to each other.

Now we will show a few simple formulas for the holomorphic representation given above, which are useful for a construction of the S-matrix.

Let us take some operator \hat{A} with matrix element in our basis

$$A_{nm} = \langle \Psi_n | \hat{A} | \Psi_m \rangle. \quad (20)$$

The function

$$A(a^*, a) = \sum_{nm} A_{nm} \frac{(a^*)^n}{\sqrt{n!}} \frac{a^m}{\sqrt{m!}} = \sum_{nm} |n\rangle \langle m| \quad (21)$$

is called the kernel of the operator \hat{A} . The kernel of a product of two operators is given by the convolution of kernels:

$$A_1 A_2(a^* a) = \int A_1(a^* \alpha) A_2(\alpha^* a) e^{-\alpha^* \alpha} \frac{d\alpha^* d\alpha}{2\pi i}.$$

The operator \hat{A} can be decomposed into a formal series of normal ordered creation and annihilation operators:

$$\hat{A} = \sum_{nm} K_{nm} (\hat{a}^\dagger)^n (\hat{a})^m. \quad (22)$$

The following function is called the normal symbol of the operator \hat{A} :

$$K(a^*, a) = \sum_{nm} K_{nm} (a^*)^n a^m. \quad (23)$$

Problem. Prove the relation between the kernel and the normal symbol of the operator \hat{A} :

$$A(a^*, a) = e^{a^* a} K(a^*, a). \quad (24)$$

(Check the equality (24) for the particular case $\hat{A} = \hat{a}^{\dagger n} \hat{a}^l$.)

Now we use the relations (22), (23) and (24) to construct the functional integral in the holomorphic representation.

Let the Hamiltonian of some system be $\hat{H}(\hat{a}^{\dagger}, a)$. The evolution operator has the form

$$\hat{U} = e^{-i\hat{H} \cdot \Delta t}.$$

From (22) and (24), one can get the following formula for the kernel of the evolution operator:

$$U(a^*, a) = e^{[a^* a - i h(a^*, a)] \Delta t} \quad (25)$$

for a small time interval Δt .

In the case of a finite interval, we can split it into small pieces $t'' - t' = N \cdot \Delta t$ and using our orthonormal bases (17) we get the following form for the normal symbol of the evolution operator, which is a convolution of products of the evolution operators:

$$U(a^*, a; t'', t') = \int \exp \left([a^* \alpha_{N-1} - \alpha_{N-1}^* \alpha_{N-1} + \dots - \alpha_1^* \alpha_1 + \alpha_1^* \alpha_0] \right. \\ \left. - i \Delta t [h(a^*, \alpha_{N-1}) + \dots + h(\alpha^*, \alpha_0)] \right) \cdot \prod_{k=1}^{N-1} \frac{d\alpha_k^* d\alpha_k}{2\pi i}.$$

In the limit $\Delta t \rightarrow 0$, $N \rightarrow \infty$, $\Delta N = t'' - t'$, one gets

$$U(a^*, a; t'', t') = \int e^{a^* \alpha(t'')} \cdot \exp \left(\int_{t'}^{t''} [-\alpha^* \alpha - i h(\alpha^*, \alpha)] dt \right) \cdot \prod_t \frac{d\alpha^* d\alpha}{2\pi i}, \quad (26)$$

where the boundary conditions are $\alpha^*(t'') = a^*$, $\alpha(t') = a$.

In our case for the harmonic oscillator $h(a^*, a) = \omega a^* a$ the integral (26) can be easily computed. To do this, one should take a variation

$$\delta \left[a^* \alpha(t'') + \int_{t'}^{t''} [-\alpha^* \alpha - i h(\alpha^*, \alpha)] dt \right] \\ = a^* \delta \alpha(t'') + \int_{t'}^{t''} dt [-\delta \alpha^* \dot{\alpha} - \alpha^* \delta \dot{\alpha} - i \omega \alpha^* \delta \alpha - i \omega \delta \alpha^* \alpha] \\ = a^* \delta \alpha(t'') - \alpha^*(t'') \delta \alpha(t'') + \int_{t'}^{t''} dt [-\delta \alpha^* (\dot{\alpha} + i \omega \alpha) + \delta \alpha (\dot{\alpha}^* - i \omega \alpha^*)].$$

The extremum condition gives us the answer. Note that the first two terms cancel each other because of the boundary condition $\alpha^*(t'') = a^*$. Extremum conditions can be simply solved:

$$a(t) = e^{-i\omega(t-t')} a(t'), \quad a^*(t) = e^{-i\omega(t''-t)} a^*(t'). \quad (27)$$

Then, for the normal symbol of the evolution operator, we get

$$U(a^*, a; t'', t') = \exp(a^* a e^{-i\omega(t-t')}), \quad (28)$$

substituting the solution (27) into the exponent in (26) where a non-zero contribution comes only from the first term in the exponent.

An important consequence of this is that for the operator \hat{A} with the kernel $A(a^*, a)$ the kernel of the operator $e^{i\hat{H}_0 t''} \hat{A} e^{-i\hat{H}_0 t'}$ is given by the convolution of corresponding kernels and is equal to

$$A(a^* e^{i\omega t''}, a e^{-i\omega t'}). \quad (29)$$

This relation demonstrates the power of the holomorphic representation in which the evolution is simply reduced to the substitution of arguments:

$$a \rightarrow a e^{-i\omega t}.$$

Now we come back to the field theory. The field operator is given by

$$\hat{\Phi}(x) = \int \frac{d\vec{k}}{(2\pi)^3 2k^0} [e^{-ikx} \hat{a} + e^{ikx} \hat{a}^\dagger(k)]$$

and the corresponding 4-momentum operator is

$$\hat{P}^\mu = \int \frac{d\vec{k}}{(2\pi)^3 2k^0} k^\mu \hat{a}^\dagger(k) \hat{a}(k).$$

Vacuum $|0\rangle$ is the state $\hat{a}(k)|0\rangle = 0$ and the one-particle state is $|\vec{k}\rangle = \hat{a}^\dagger(k)|0\rangle$. The operator \hat{P}^μ acts on the one-particle state as

$$(\hat{P}^0 = \hat{H}) |\vec{k}\rangle = k^0 |\vec{k}\rangle, \quad \hat{P} |\vec{k}\rangle = \vec{k} |\vec{k}\rangle.$$

A multiparticle state is constructed as $|\vec{k}_1, \dots, \vec{k}_n\rangle = \prod_{i=1}^n \hat{a}^\dagger(\vec{k}_i) |0\rangle$. Obviously,

$$\begin{aligned} \hat{H} |\vec{k}_1, \dots, \vec{k}_n\rangle &= \left(\sum_{i=1}^n k_i^0 \right) |\vec{k}_1, \dots, \vec{k}_n\rangle, \\ \hat{P} |\vec{k}_1, \dots, \vec{k}_n\rangle &= \left(\sum_{i=1}^n \vec{k}_i \right) |\vec{k}_1, \dots, \vec{k}_n\rangle. \end{aligned}$$

If \hat{H} is the Hamiltonian of a system and \hat{H}^0 is a free Hamiltonian, then the S-matrix is determined as the following limit of the evolution operator:

$$\hat{S} = \lim_{\substack{t' \rightarrow -\infty \\ t'' \rightarrow +\infty}} e^{i\hat{H}_0 t''} \hat{U}(t'', t') e^{-i\hat{H}_0 t'}. \quad (30)$$

With such a definition, it is clear that $\hat{S} = 1$ if $\hat{H} = \hat{H}^0$.

From the formulas (26) and (29), we can obtain the following representation for the kernel of the S-matrix:

$$S(a^*, a) = \lim_{\substack{t' \rightarrow -\infty \\ t'' \rightarrow +\infty}} e^{\int \frac{d\vec{k}}{(2\pi)^3 2k^0} (\alpha^*(\vec{k}, t'') \alpha(\vec{k}, t') - \int_{t'}^{t''} [\alpha^*(\vec{k}, t) \dot{\alpha}(\vec{k}, t) + h(\alpha^*, \alpha)] dt)}. \prod_{t, \vec{k}} \frac{d\alpha^* d\alpha}{2\pi i}, \quad (31)$$

where $\alpha^*(\vec{k}, t'') = a^*(\vec{k})e^{i\omega t''}$, $\alpha(\vec{k}, t') = a(\vec{k})e^{-i\omega t'}$.

Now we consider the system with an external source $J(x)$:

$$L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + J(x)\varphi.$$

The kernel of the interaction operator $V = -J(x)\hat{\varphi}(x)$ is given by

$$V(a^*, a) = \int \frac{d\vec{k}}{(2\pi)^3 2k^0} \left[\gamma(\vec{k}, t)a^*(\vec{k}) + \gamma^*(\vec{k}, t)a(\vec{k}) \right],$$

where $\gamma(\vec{k}, t) = -\int J(\vec{x}, t)e^{-i\vec{k}\cdot\vec{x}}d\vec{x}$. In order to take the integral (31) once more, we should solve the extremum conditions

$$\begin{aligned} \dot{\alpha}(\vec{k}, t) + i\omega(\vec{k})\alpha(\vec{k}, t) + i\gamma(\vec{k}, t) &= 0, \\ \dot{\alpha}^*(\vec{k}, t) - i\omega(\vec{k})\alpha^*(\vec{k}, t) - i\gamma^*(\vec{k}, t) &= 0 \end{aligned} \quad (32)$$

with the boundary conditions

$$\alpha^*(\vec{k}, t'') = a^*(\vec{k})e^{i\omega t''}, \quad \alpha(\vec{k}, t') = a(\vec{k})e^{-i\omega t'}. \quad (33)$$

The solution of these equations has the following form:

$$\begin{aligned} \alpha(\vec{k}, t) &= a(\vec{k})e^{-i\omega t} - ie^{-i\omega t} \int_{t'}^{t''} e^{i\omega s} \gamma(\vec{k}, s) ds, \\ \alpha^*(\vec{k}, t) &= a^*(\vec{k})e^{i\omega t} + ie^{i\omega t} \int_{t'}^{t''} e^{-i\omega s} \gamma^*(\vec{k}, s) ds. \end{aligned}$$

If one substitutes the obtained solutions into the integral (31), one will get the following formula for the kernel of the S-matrix:

$$\begin{aligned} S_J(a^*, a) &= \exp \left(\int \frac{d\vec{k}}{(2\pi)^3 2k^0} \left[a^*(\vec{k})a(\vec{k}) \right. \right. \\ &+ \int dt \int d\vec{x} J(\vec{x}, t) \left(a^*(\vec{k})e^{i\omega t - i\vec{k}\cdot\vec{x}} + a(\vec{k})e^{-i\omega t + i\vec{k}\cdot\vec{x}} \right) / (2\omega) \\ &\left. \left. - \frac{1}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} ds \int d\vec{x} d\vec{y} J(\vec{x}, t) J(\vec{x}, s) / (2\omega) e^{-i\omega|t-s|} e^{i\vec{k}\cdot\vec{x} - i\vec{k}\cdot\vec{y}} \right] \right). \end{aligned} \quad (34)$$

A transition from the kernel to the normal symbol for the S-matrix corresponds to omitting the first term in the exponent.

Now let us recall the solution of the free Klein–Gordon equation

$$\varphi_0(x) = \int \frac{d\vec{k}}{(2\pi)^3 2k^0} \left[a(\vec{k})e^{i\omega t} + a^*(\vec{k})e^{-i\omega t} \right]$$

and the Green function (the propagator)

$$D_c(x) = i \int \frac{d\vec{k}}{(2\pi)^3 2\omega} e^{i\vec{k}\cdot\vec{x}} e^{-i\omega_k|t|} = i \int \frac{d^4k e^{-ikx}}{k^2 - m^2 + i0}.$$

In terms of $\varphi_0(x)$ and $D_c(x)$, one can rewrite the normal symbol for the S-matrix in the following way:

$$S_J(\varphi_0) = \exp \left(i \int d^4x J(x)\varphi_0(x) + \frac{1}{2} \int d^4x d^4y J(x)D_c(x-y)J(y) \right). \quad (35)$$

Generalization of (35) to the case of the integration potential $V(\varphi)$ is rather straightforward. We can explore the relation we already faced with

$$\exp\left(-i \int d^4x V(\varphi)\right) = \exp\left(-i \int V\left(\frac{\delta}{i\delta J}\right) d^4x\right) \exp\left(i \int \varphi J d^4y\right) \Big|_{J=0}. \quad (36)$$

Then the expression for the normal symbol of the S-matrix takes a simple and elegant form:

$$S_V(\varphi_0) = \exp\left(-i \int V\left(\frac{\delta}{i\delta J}\right) d^4x\right) \cdot \exp\left(i \int J(y)\varphi_0(y) d^4y + \frac{1}{2} \int d^4z d^4y J(y) D_c(y-z) J(z)\right) \Big|_{J=0}. \quad (37)$$

If we do not put the source $J \rightarrow 0$ after taking functional derivatives, we have the normal symbol of the S-matrix in the presence of an external source:

$$S_V(\varphi_0, J) = \exp\left(-i \int V\left(\frac{\delta}{i\delta J}\right) d^4x\right) \cdot \exp\left(i \int J(y)\varphi_0(y) d^4y + \frac{1}{2} \int d^4z d^4y J(y) D_c(y-z) J(z)\right). \quad (38)$$

One can see that if we put $\varphi_0 = 0$ we get the same formula we as already derived for the generating functional

$$Z_V(J) = \exp\left(-i \int V\left(\frac{\delta}{i\delta J}\right) d^4x\right) \cdot \exp\left(\frac{1}{2} \int d^4z d^4y J(y) D_c(y-z) J(z)\right). \quad (39)$$

This observation allows us to get a very important relation, which is called the Lehmann–Symanzik–Zimmermann reduction formula. To do this, we introduce in (38) some arbitrary external field $\varphi(x)$ instead of $\varphi_0(x)$. Then, by direct computation of functional derivatives with respect to $\varphi(x)$ and $J(x)$, one can check the following relation:

$$\int \prod_i dx_i \varphi_0(x_i) \left[\frac{1}{i} \frac{\delta}{\delta \varphi_1} \dots \frac{\delta}{\delta \varphi_n} S_V(\varphi_0, J) \Big|_{\varphi, J=0} - \frac{1}{i} \frac{\delta}{\delta \tilde{J}_1(x_1)} \dots \frac{\delta}{\delta \tilde{J}_n(x_n)} Z(\tilde{J}) \Big|_{J=0} \right] = 0, \quad (40)$$

where $\tilde{J}(x) = \int D_c(x-y) J(y)$. If we keep in mind that the expression of the S-matrix

$$S_V(\varphi_0) = \sum_n \frac{1}{n!} \int dx_1 \dots dx_n \varphi_0(x_1) \dots \varphi_0(x_n) S_n(x_1 \dots x_n)$$

gives us the coefficient functions $S_n(x_1, \dots, x_n)$ of S-matrix scattering elements, and on the other hand the expansion of the generating functional

$$Z_V(J) = \sum_n \frac{1}{n!} \int dx_1 \dots dx_n J(x_1) \dots J(x_n) \frac{\delta Z}{i\delta J(x_1) \dots i\delta J(x_n)}$$

gives us the Green functions

$$G_n(x_1 \dots x_n) = \frac{\delta Z}{i\delta J(x_1) \dots i\delta J(x_n)},$$

we observe a simple correspondence. The reduction formula (40) tells us how to compute S-matrix elements by computing corresponding Green functions:

1. one should compute the Green function;
2. multiply all legs to the inverse propagator or, in other words, apply the operator

$$\prod_{i=1}^n (\square_i + m^2)$$

to each of the legs;

3. multiply the result by the product of the corresponding free fields:

$$\frac{1}{n} \prod_i \varphi_0(x_i).$$

Schematically, the procedure is shown in Fig. 3.

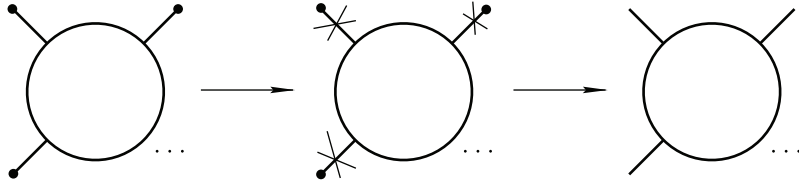


Fig. 3: Transition from the Green functions to the S-matrix elements

This rule is very general and could be applied for all types of fields, not only for the scalar fields.

7 Generating functional for Green functions and perturbation theory

Now we return to the generating functional written in the form

$$Z[J] = \int D(\varphi) \cdot \exp \left(i \int d^4x L(\varphi, \partial_\mu \varphi) + i \int d^4x J(x) \varphi(x) \right), \quad (41)$$

where $L = \frac{1}{2} \partial_\mu \varphi \cdot \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - V(\varphi)$. If we take the second derivative,

$$\left. \frac{\delta^{(2)} Z}{i \delta J(x_1) i \delta J(x_2)} \right|_{J=0} = \langle \varphi_1(x_1) \varphi_2(x_2) \rangle,$$

we obtain the Feynman propagator $D_c(x_1 - x_2)$, as can be easily seen from the form for $Z[J]$:

$$Z[J] = \exp \left(-i \int dx V \left(\frac{\delta}{\delta J} \right) \right) \cdot \exp \left(\frac{1}{2} \int dy dz J(y) D_c(y - z) J(z) \right). \quad (42)$$

We obtain the same function as we have obtained from the time-ordered product of field operators (see (5))

$$D_c(x_1 - x_2) = \langle 0 | T \{ \hat{\Phi}(x_1) \hat{\Phi}(x_2) \} | 0 \rangle = \langle \varphi_1(x_1) \varphi_2(x_2) \rangle.$$

This is always the case. Derivatives of the generating functional automatically give T -products of the corresponding field operators:

$$\frac{\delta^{(n)} Z}{i \delta J(x_1) \dots i \delta J(x_n)} \equiv \langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle = \langle 0 | T \{ \hat{\Phi}(x_1) \dots \hat{\Phi}(x_n) \} | 0 \rangle.$$

At this point let us consider as an example a theory with $V(\varphi) = \frac{\lambda}{4!}\varphi^4$. In such a theory if we take two derivatives and expand the exponent on λ , we get

$$\begin{aligned} \langle \varphi_1(x_1)\varphi_2(x_2) \rangle &= D_c(x_1 - x_2) + \lambda [D_c(x_1 - x_2)D_c^2(0) + \dots] \\ &+ \frac{1}{2} \left(\frac{\lambda}{4!}\right)^2 [72D_c^2(0)D_c^2(x_1 - x_2) + 24D_c^4(x_1 - x_2) + \dots] + \dots \end{aligned}$$

It is very useful to introduce the Feynman rules and Feynman diagrams. The closed line corresponds to the propagator

$$\frac{1}{i}D_c(x - y).$$

Each interaction vertex corresponds to

$$-i\lambda = -i \frac{d^4V(\varphi)}{d\varphi^4} \Big|_{\varphi=0}.$$

In terms of Feynman rules, the corrections to the two-point correlation function (43) are given by Feynman diagrams shown in Fig. 4.

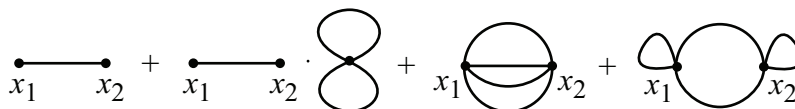


Fig. 4: Illustration of (43)

One should add a symmetry factor, which corresponds to possible permutations of equivalent lines. If one takes the Fourier transformation, one can formulate the rules in momentum space, which are usually used in practical computations. In momentum space the integral

$$\int \frac{dp}{(2\pi)^4}$$

corresponds to each loop and in each vertex the momentum conservation law takes place.

We will be more specific and precise later in the formulation of Feynman rules for the case of the SM.

But now we need to consider a few more properties of the generating functional. As we have seen already from the two-point correlator the perturbative expansion contains disconnected diagrams, which are not really needed in computations. The way out of this problem is to consider the logarithmic function of the generating functional:

$$iW[J] = \ln Z[J], \quad Z[J] = e^{iW[J]}.$$

Then, for the functional derivatives, one gets

$$\frac{\delta W}{\delta J} = \frac{1}{Z} \frac{\delta Z}{i\delta J}, \quad \frac{\delta^2 W}{\delta J_1 \delta J_2} = i \frac{1}{Z} \frac{\delta^2 Z}{i\delta J_1 i\delta J_2} - \frac{1}{Z^2} \frac{\delta Z}{i\delta J_1} \frac{\delta Z}{i\delta J_2} \dots,$$

where additional terms exactly cancel out disconnected pieces in the Green functions. The property that the logarithmic function leads to connected diagrams is a particular example of a more general theorem in graph theory.

The functional

$$W[J] = \frac{1}{i} \ln Z[J] \tag{43}$$

is called the generating functional for connected Green functions.

The next important observation is related to the functional Legendre transformation:

$$\Gamma[\varphi_{\text{cl}}] = W[J] - \int d^4x J(x)\varphi_{\text{cl}}(x), \quad (44)$$

where $\varphi_{\text{cl}} = \delta W/\delta J$ with a formal solution $J = J(\varphi_{\text{cl}})$. If we take functional derivatives of the functional $\Gamma[\varphi_{\text{cl}}]$, we get the so-called one-particle irreducible Green functions (Feynman diagrams corresponding to such functions cannot be split up into disconnected pieces by cutting only one internal line):

$$\frac{\delta \Gamma}{\delta \varphi_{\text{cl}}} = \frac{\delta W}{\delta J} \frac{\delta J}{\delta \varphi_{\text{cl}}} - J(x) - \varphi_{\text{cl}} \frac{\delta J}{\delta \varphi_{\text{cl}}}. \quad (45)$$

The functional Γ is called the effective action. Two terms in (45) are cancelled out and we get

$$\frac{\delta \Gamma}{\delta \varphi_{\text{cl}}(x)} = -J(x). \quad (46)$$

Now one can take functional derivatives from both sides of (46) and get the following relations:

$$\frac{\delta^2 \Gamma}{\delta \varphi_{\text{cl}}(x_1) \delta \varphi_{\text{cl}}(x_2)} = -\frac{\delta J(x_1)}{\delta \varphi_{\text{cl}}(x_2)} = -\left[\frac{\delta W}{\delta J(x_1) \delta J(x_2)} \right]^{-1}. \quad (47)$$

If we introduce notation for the connected Green function:

$$G_n(x_1, \dots, x_n) = -i \frac{\delta W[J]}{i \delta J(x_1) \dots i \delta J(x_n)} \Big|_{J=0}$$

and for the one-particle irreducible Green function:

$$\Gamma_n(x_1, \dots, x_n) = -i \frac{\delta^{(n)} \Gamma[\varphi_{\text{cl}}]}{\delta \varphi_{\text{cl}}(x_1) \dots \delta \varphi_{\text{cl}}(x_n)} \Big|_{\varphi_{\text{cl}}=0},$$

then the formula (47) can be written in the following form:

$$\Gamma_2 = G_2^{-1},$$

which means that the irreducible two-point Green function is nothing but the inverse propagator. If we take more derivatives on both sides of (47) we obtain relations between connected and one-particle irreducible Green functions. For the three-point Green function, it is presented schematically in Fig. 5. If we restore the Planck constant \hbar , the generating functional has the form

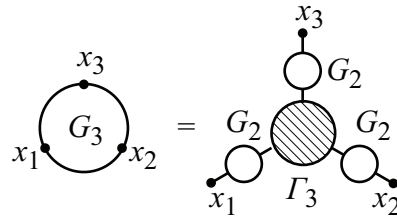


Fig. 5: Connected three-point Green function is equal to one-particle irreducible three-point function convoluted with three propagators.

$$Z[J] = \int D(\varphi) \exp \left(\frac{i}{\hbar} S[\varphi] + i \int dx J(x)\varphi(x) \right),$$

where $S[\varphi]$ is the action. In the quasiclassical limit ($\hbar \rightarrow 0$), the functional integral is dominated by the stationary trajectory:

$$\left. \frac{\delta S[\varphi]}{\delta \varphi(x)} \right|_{\varphi=\varphi_{\text{cl}}} + J = 0$$

Therefore

$$Z[J] \sim \exp \left(\frac{i}{\hbar} S[\varphi_{\text{cl}}] + i \int dx J(x) \varphi_{\text{cl}}(x) \right)$$

and one can see from (43) that

$$W[J] = S[\varphi_{\text{cl}}] + \int dx J(x) \varphi_{\text{cl}}$$

and, by comparing with (44), we obtain

$$\Gamma[\varphi_{\text{cl}}] = S[\varphi_{\text{cl}}].$$

So, we can conclude that the irreducible Green functions are $\Gamma_n(x_1, \dots, x_n)$, the effective vertices of the theory:

$$\Gamma_n(x_1, \dots, x_n) = (-i) \frac{\delta^{(n)} S[\varphi_{\text{cl}}]}{\delta \varphi_{\text{cl}}(x_1) \dots \delta \varphi_{\text{cl}}(x_n)}. \quad (48)$$

We obtain this formula for the case of scalar fields as an example, but it remains true for any theory with corresponding obvious changes. Note that the formula (48) is very useful to get Feynman rules for complicated vertices in the interaction Lagrangian. For example, three- and four-gluon vertices are obtained with all needed symmetry properties.

Up to now we have considered basic ingredients of the quantum field theory for the case of scalar fields. However, there are many other fields and corresponding particles which have different spin properties. In the SM there are leptons and quarks, being fermions with the spin 1/2, and the boson fields with the spin 1. We begin our brief consideration with spin-1/2 fermion fields.

8 Fermion fields

Spin-1/2 particles with mass m are described by the four-component field Ψ . The Lagrangian for the field has the well-known form

$$L = \bar{\Psi} i \partial_\mu \gamma^\mu \Psi - m \bar{\Psi} \Psi. \quad (49)$$

The least-action principle leads to the famous Dirac equation of motion:

$$(i \gamma^\mu \partial_\mu - m) \Psi = 0, \quad (50)$$

where γ^μ ($\gamma^0, \gamma^1, \gamma^2, \gamma^3$) are the Dirac (4×4) matrices. The matrices γ_μ obey the anticommutation relation

$$\{\gamma_\mu \gamma_\nu\} = 2\eta_{\mu\nu}. \quad (51)$$

There are several representations for γ -matrices. In the SM chiral or Weyl spinors are of particular importance. Therefore, we use the Weyl representation of γ -matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \quad (52)$$

where $\sigma^0 = I$, $\sigma^i = \tau^i$, $\bar{\sigma}^0 = I$, $\bar{\sigma}^i = -\tau^i$ and τ^i are the (2×2) Pauli matrices.

In this representation the γ^5 -matrix has the following form:

$$\gamma^5 = i \gamma^1 \gamma^2 \gamma^3 \gamma^4 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \quad (53)$$

Chiral spinors

$$\Psi_{L,R} = \frac{1 \mp \gamma_5}{2} \Psi \quad (54)$$

in this notation

$$\Psi_L = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_R = \begin{pmatrix} 0 \\ 0 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}$$

are in fact two-component objects.

In momentum space the function $\Psi(x)$ is decomposed into positive- and negative-energy parts

$$u_\lambda(p)e^{-ipx} \text{ and } v_\lambda(p)e^{ipx} \quad (55)$$

and the spinors $u(p)$ and $v(p)$ obey the Dirac equation in the following form:

$$\begin{aligned} (p_\mu \gamma^\mu - m) u_\lambda(p) &= 0, \\ (p_\mu \gamma^\mu + m) v_\lambda(p) &= 0. \end{aligned} \quad (56)$$

The concrete form of spinors is different in different parametrizations of γ -matrices, and in the Weyl representation the spinors are

$$u_\lambda = \begin{pmatrix} \sqrt{p^0 + \vec{p}\vec{\sigma}\xi_\lambda} \\ \sqrt{p^0 - \vec{p}\vec{\sigma}\xi_\lambda} \end{pmatrix}, \quad (57)$$

$$v_\lambda = \begin{pmatrix} \sqrt{p^0 + \vec{p}\vec{\sigma}\eta_\lambda} \\ -\sqrt{p^0 - \vec{p}\vec{\sigma}\eta_\lambda} \end{pmatrix}, \quad (58)$$

where ξ and η are two-component spinors determined by fixing some quantization axis. Left and right chiral spinors are then

$$\begin{aligned} u_{L,R} &= \frac{1 \mp \gamma_5}{2} u_\lambda, \\ v_{L,R} &= \frac{1 \mp \gamma_5}{2} v_\lambda. \end{aligned}$$

Normalization conditions and summation over indices are as follows:

$$\begin{aligned} \bar{u}_\lambda u_{\lambda'} &= 2m\delta_{\lambda\lambda'}, \quad \bar{v}_\lambda v_{\lambda'} = -2m\delta_{\lambda\lambda'}, \\ \sum_\lambda u_\lambda \bar{u}_\lambda &= p_\mu \gamma^\mu + m, \quad \sum_\lambda v_\lambda \bar{v}_\lambda = p^\mu \gamma_\mu - m. \end{aligned}$$

Quantization of the Dirac field is similar to the scalar case considered above with a very important difference. In order to have correct Fermi statistics and obey the Pauli principle, the commutation relations in the scalar case should be replaced by corresponding anticommutation relations:

$$\{\hat{\pi}_\alpha(t, \vec{x}), \Psi_\beta(t, \vec{x}')\} = -i\delta_{\alpha\beta}\delta(\vec{x} - \vec{x}'), \quad (59)$$

where $\alpha = 1, 2, 3, 4$ and the field momentum is

$$\pi_\alpha(t, \vec{x}) = \frac{\partial L}{\partial \dot{\Psi}_\alpha} = i\Psi_\alpha^\dagger.$$

The fermionic field operator may be constructed with the help of spinors obeying the Dirac equation

$$\Psi(x) = \int \frac{d\vec{p}}{(2\pi)^3 p^0} \sum_{\lambda=1,2} \left[\hat{b}_\lambda(p) u_\lambda(p) e^{-ipx} + \hat{d}_\lambda^\dagger(p) v_\lambda(p) e^{ipx} \right],$$

$$\bar{\Psi}(x) = \Psi^\dagger \gamma^0 = \int \frac{d\vec{p}}{(2\pi)^3 p^0} \sum_{\lambda=1,2} \left[\hat{b}_\lambda^\dagger(p) \bar{u}_\lambda(p) e^{ipx} + \hat{d}_\lambda(p) \bar{v}_\lambda(p) e^{-ipx} \right].$$

It is easy to check that from the anticommutators (59) the creation and annihilation operators satisfy the anticommutation relations in the following form:

$$\left\{ \hat{b}_\lambda(\vec{p}), \hat{b}_{\lambda'}(\vec{p}') \right\} = \left\{ \hat{d}_\lambda(\vec{p}), \hat{d}_{\lambda'}(\vec{p}') \right\} = (2\pi)^3 2p^0 \delta(\vec{p} - \vec{p}') \delta_{\lambda\lambda'}. \quad (60)$$

Then one-particle and one-antiparticle states are obtained from the vacuum state $|0\rangle$ by acting of the creation operators:

$$\hat{b}_\lambda^\dagger(\vec{p}) |0\rangle \text{ and } \hat{d}_\lambda^\dagger(\vec{p}) |0\rangle.$$

In the same way, by acting of creation operators for particle and antiparticle on the vacuum state, one gets two-, three-, ... particle states. Because of zero anticommutators, for any creation operator one gets nicely the Pauli principle:

$$\left\{ \hat{b}_\lambda^\dagger(\vec{p}), \hat{b}_\lambda^\dagger(\vec{p}') \right\} = 0 \Rightarrow \hat{b}_\lambda^\dagger(\vec{p}), \hat{b}_\lambda^\dagger(\vec{p}') |X\rangle \equiv 0 \quad (61)$$

for any state $|X\rangle$. From the field operators one can get the Feynman propagator (T-ordered correlator) in a similar way as was done for the scalar case:

$$\langle 0| T (\bar{\Psi}(x_1) \Psi(x_2)) |0\rangle = \frac{-1}{i} \int \frac{dp}{(2\pi)^4} \frac{p_\mu \gamma^\mu + m}{p^2 - m^2 + i0}. \quad (62)$$

Considering the path-integral method for the fermion field, one can construct the holomorphic representation similar to the scalar case. However, now we have to deal with anticommuting numbers called Grassmann numbers, which form the Grassmann algebra:

$$\begin{aligned} (a_\alpha)^* &= a_\alpha^*, & (a_\alpha^*)^* &= a_\alpha, \\ \{a_\alpha a_\beta\} &= \{a_\alpha^* a_\beta^*\} = \{a_\alpha a_\beta^*\} = 0, \\ ca_\alpha &= a_\alpha c; & ca_\alpha^* &= a_\alpha^* c, \end{aligned}$$

where $\alpha = 1, \dots, n$, c are the usual numbers.

A function of Grassmann variables has therefore the generic form

$$\begin{aligned} f(a, a^*) &= f_{00} + \sum_{\alpha_1} f_{\alpha_1|0} a_{\alpha_1} + \sum_{\alpha_1} f_{0|\alpha_1} a_{\alpha_1}^* \\ &+ \sum_{\alpha_1 \alpha_2} f_{\alpha_1|\alpha_2} a_{\alpha_1} a_{\alpha_2}^* + \dots + f_{1\dots n|n\dots 1} a_1 \dots a_n a_n^* \dots a_1^*. \end{aligned} \quad (63)$$

The expression (63) reminds us of the norm-ordering operator products we have used already. The operations of differentiation and integration are defined as follows:

$$\begin{aligned} \frac{\partial}{\partial a_\alpha} a_\beta &= \frac{\partial}{\partial a_\alpha^*} a_\beta^* = \delta_{\alpha\beta}, & \frac{\partial}{\partial a_\alpha} a_\beta^* &= \frac{\partial}{\partial a_\alpha^*} a_\beta = 0, \\ \frac{\partial}{\partial a_\alpha} f &= f_\alpha; & \frac{\partial}{\partial a_\alpha^*} f &= \bar{f}_\alpha, & \int da_\alpha f &= f_\alpha; & \int da_\alpha^* f &= \bar{f}_\alpha, \end{aligned}$$

where f_α does not depend on a_α and \bar{f}_α does not depend on a_α^* . One can check the anticommutation relation for differentials:

$$\{da_\alpha da_\beta\} = \{da_\alpha^* da_\beta^*\} = \{da_\alpha da_\beta^*\} = 0.$$

If we denote

$$da^* da = da_1^* \dots da_n^* da_n \dots da_1$$

and take into account that for the function defined in (63)

$$\int da^* da f(a, a^*) = f_{1\dots n|n\dots 1},$$

we can easily prove the following relation by expanding in series in a and a^* :

$$\int da^* da \exp \left(\sum a_\alpha^* A_{\alpha\beta} a_\beta + \sum \eta_\alpha^* a_\alpha + i \sum \eta_\alpha a_\alpha^* \right) = \det A \exp \left(\eta_\alpha^* A_{\alpha\beta}^{-1} \eta_\beta \right). \quad (64)$$

One can see from (64) that in contrast to the integration with the usual complex numbers (see (9)) the determinant appears in the numerator in the case of anticommuting Grassmann numbers (64).

Similar to the case of the scalar field, one can get a formula for the S-matrix normal symbol:

$$S_\eta(b^\dagger, d^\dagger, b, d) = \frac{1}{N} \exp \left(-i \int \bar{\eta} S_C \eta + i \int (\bar{\eta} \Psi_0 + \bar{\Psi}_0 \eta) \right),$$

where $\bar{\Psi}_0 = \Psi_0^\dagger \gamma^0$, Ψ_0 is a solution of the free Dirac equation

$$\Psi_0(x) = \int \frac{d\vec{k}}{(2\pi)^3 2k_0} \left[\sum_{\lambda=1,2} b_\lambda(\vec{k}) u_\lambda(\vec{k}) e^{-ikx} + \sum_{\lambda=1,2} d_\lambda^*(\vec{k}) v_\lambda(\vec{k}) e^{ikx} \right],$$

$\eta(x)$ is the fermion source and

$$S_C(x) = \frac{i}{(2\pi)^4} \int dk e^{-ikx} \frac{\hat{k} + m}{k^2 - m^2 + i0}$$

is the Feynman propagator for the fermion field.

Taking into account the following relation for the functional measure:

$$\prod_{t,k,\lambda} db_\lambda^*(t, \vec{k}) db_\lambda(t, \vec{k}) dd_\lambda^*(t, \vec{k}) dd_\lambda(t, \vec{k}) = \text{Const.} \prod_{x,\alpha} d\bar{\Psi}_\alpha(x) d\Psi_\alpha(x),$$

one gets the answer for the generating functional for the fermion Green functions in compact form similar to the scalar case:

$$Z(\bar{\eta}, \eta) = N^{-1} \int \exp \left(i \int d^4x (L(x) + \bar{\eta} \Psi + \bar{\Psi} \eta) \right) \prod_x d\bar{\Psi}(x) d\Psi(x).$$

9 Quantization of theories with the gauge fields

The gauge field was first introduced in QED when the Maxwell equations were rewritten in terms of the 4-vector potential $A_\mu(x)$.

The equation in terms of the field A_μ ,

$$\partial_\mu F^{\mu\nu} = 0,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is invariant under the local $U(1)$ transformation:

$$A'_\mu = A_\mu + \partial_\mu \alpha(x).$$

It is easy to check the $U(1)$ invariance of the QED Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} (i\hat{D} - m) \Psi, \quad \Psi \rightarrow e^{ie\alpha} \Psi, \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha, \quad (65)$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative.

In the SM we deal not only with the abelian $U(1)$ group but also with non-abelian $SU(N)$ groups, $SU(2)$ for the EW and $SU(3)$ for the strong forces.

We are not discussing here generic structure and properties of Lie groups but introduce briefly the $SU(N)$ group. $SU(N)$ is a group of unitary matrices U ($U^\dagger U = 1$) with determinant equal to 1 ($\det U = 1$). Elements of the group $U(x)$ may depend on the space-time point x^μ .

If we want the theory to be invariant under $SU(N)$ transformation the covariant derivative in the Lagrangian

$$L = \bar{\Psi}(i\hat{D} - m)\Psi \quad (66)$$

should transform as

$$\begin{aligned} D_\mu \Psi &\rightarrow (D_\mu \Psi)^U = U D_\mu \Psi, \\ (\partial_\mu - igA_\mu^U) U \Psi &= U (\partial_\mu - igA_\mu) \Psi. \end{aligned}$$

From this, one gets the following transformation form for the potential A :

$$A_\mu^U = U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1}. \quad (67)$$

The kinetic term for the non-abelian A_μ field is constructed as the gauge-invariant operator

$$L_A = -\frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}). \quad (68)$$

In the SM all gauge fields are taken in the adjoint representation:

$$A_\mu(x) = A_\mu^a(x) t^a,$$

where t^a ($a = 1, \dots, N-1$ for $SU(N)$) are so-called generators of the group. As do all generators for the Lie group, the generator t^a obeys the following commutation relation:

$$[t^a, t^a] = f^{abc} t^c, \quad \text{Tr}(t^a) = 0.$$

One may choose normalization conditions as

$$\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}. \quad (69)$$

Then the Lagrangian for the gauge field takes the form

$$L = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}, \quad (70)$$

where the field strength tensor is

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c. \quad (71)$$

One can express the unitary matrix $U(1)$ in the form $U(x) = e^{ig\alpha^a(x)t^a}$. Then the transformation for the gauge field A_μ^a takes the form

$$A_\mu^a \rightarrow (A^\alpha)_\mu^a = A_\mu^a + \partial_\mu \alpha^a + g f^{abc} A_\mu^b \alpha^c = A_\mu^a + D_\mu^{ac} \alpha^c, \quad (72)$$

where the covariant derivative in components is

$$D_\mu^{ac} = \partial_\mu \delta^{ac} + g f^{abc} A_\mu^b.$$

Part of the SM describing strong interactions is based on the $SU_C(3)$ group. It is called QCD with the Lagrangian

$$L_{\text{QCD}} = -\frac{1}{4}G_{\mu\nu}^a G^{\mu\nu a} + \bar{q}_i \left(i(D_\mu)_{ij} \gamma_j^\mu - m\delta_{ij} \right) q_j, \quad (73)$$

where $q = 1, 2, 3$, $(D_\mu)_{ij} = \partial_\mu \delta_{ij} - ig(t^a)_{ij} A_\mu^a$.

On the classical level abelian and non-abelian theories look very elegant. However, problems appear when the theories are quantized. The reason can be seen already in QED, where we know that in a theory described by the field A_μ we have four components but only two of them are physical degrees of freedom corresponding to two polarizations of the physics photon. This problem is manifested in the fact that the quadratic form of the Lagrangian

$$A^\mu D_{\mu\nu}^{-1} A^\nu = A^\mu (\square g_{\mu\nu} - \partial_\mu \partial_\nu) A^\nu$$

or, in momentum space,

$$(k^2 g_{\mu\nu} - k_\mu k_\nu)$$

does not have an inverse form. As we have seen in cases of scalar and fermion fields the inverse of the differential operator, the Green function, is the propagator. So, here one cannot get the propagator in such a way.

The way out of this problem is a correct quantization procedure called the quantization of constrained systems. The reason that the functional integral

$$\int \prod_{\mu,x} dA_\mu(x) \exp \left\{ i \int dx \left(-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \right) \right\} \quad (74)$$

does not give a reasonable result is that there are an infinite number of gauge configurations $(A^\alpha)_\mu^a$, which differ only by the gauge transformation, leading to identical physics results since the action is gauge invariant. So, one should perform the functional integration taking only one representative from such gauge configuration.

Without going into details, the final recipe is as follows:

$$\int DA \delta(F(A)) \det(\Delta_{gh}^F) e^{iS(A)}, \quad (75)$$

where $DA = \prod_{\mu,x} dA_\mu(x)$ and the so-called Faddeev–Popov determinant $\det(\Delta_{gh}^F)$ is introduced to ensure the gauge invariance of the functional measure. The δ -function fixes the gauge condition of gauge choice $F(A) = 0$.

We recall briefly the main ideas of the method proposed by Faddeev and Popov. Let us introduce a functional integral that is equal to unity:

$$1 = \int D\alpha \cdot \delta(F(A^\alpha)) \det\left(\frac{\delta F}{\delta\alpha}\right), \quad (76)$$

where $D\alpha = \prod_x d\alpha(x)$. Substituting (76) into the integral (16) and performing the gauge transformation $(A^\alpha)_\mu^a(x) \rightarrow A_\mu^a(x)$, one gets

$$\int D\alpha \int DA e^{iS[A]} \cdot \delta(F(A)) \cdot \det\left(\frac{\delta F}{\delta\alpha}\right). \quad (77)$$

The factor $\det\left(\frac{\delta F}{\delta\alpha}\right)$ does not depend on α and therefore the integration over the gauge group is factorized out. This infinite factor is included into the normalization factor of the functional measure and therefore could be dropped.

The main idea of quantization and the Faddeev–Popov method could be illustrated in a very simple example with the usual integrals. Let us consider an integral

$$I = \int \int_{-\infty}^{\infty} dx_1 dx_2 e^{-x_1^2 - x_2^2 + 2x_1 x_2} = \int dx_1 dx_2 e^{-x_i A_{ij} x_j}, \quad (78)$$

where

$$A_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } \det A = 0.$$

We can obviously see that by making a substitution of variables we get

$$I = c \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-x^2} \text{ when } x = x_1 - x_2, \quad y = x_1 + x_2.$$

The ‘action’ in the integral I is invariant under translation:

$$x_1 \rightarrow x_1 + a, \quad x_2 \rightarrow x_2 + a.$$

And, the integral $\int_{-\infty}^{\infty} dy$ gives simply the infinite volume of the algebra corresponding to the translation group.

Now let us substitute $\hat{1}$ into the integral I :

$$\int_{-\infty}^{\infty} d\omega \delta(F(x_i + \omega)) \det \left(\frac{\delta F}{\delta \omega} \right) = 1,$$

$$\int d\omega \int dx_1 dx_2 e^{-x_i A_{ij} x_j} \delta(F(x_i + \omega)) \det \left(\frac{\delta F}{\delta \omega} \right).$$

After the ‘gauge’ substitution $x_i \rightarrow x_i - \omega$, one can drop the infinite group integral $\int d\omega$ and get the final integral in a form symmetric with respect to integration variables:

$$I_G = \int dx_1 \int dx_2 e^{-x_i A_{ij} x_j} \delta(F(x_i)) \left| \frac{\partial F}{\partial \omega} \right|.$$

The determinant $\det \left(\frac{\delta F}{\delta \alpha} \right)$ is the Faddeev–Popov determinant and $\Delta_{\text{ch}} = \frac{\delta F}{\delta \alpha}$ is the so-called ghost operator.

Obviously, in the abelian case the Lorentz gauge condition, $F = \partial_\mu A_\mu$, leads to the ghost operator $\Delta_{\text{ch}} = \partial_\mu \cdot \partial^\mu$. Since the operator does not depend on the field, it is cancelled in the connected Green functions and does not give any contribution.

However, this is not so in the non-abelian case where Δ_{ch} is a non-trivial operator depending on the gauge field. Technically it is convenient to express $\det(\Delta_{\text{ch}})$ also as a functional integral. As we know, a determinant in the numerator appears when one integrates over anticommuting fields:

$$\det(\Delta_{\text{ch}}) = \int \prod d\bar{c} dc e^{\int \bar{c} \Delta_{\text{ch}} c}, \quad (79)$$

where the anticommuting fields c are called Faddeev–Popov ghosts.

As an example, let us consider the gauge condition in the covariant form:

$$F(A) = \partial_\mu A^\mu - a(x) \quad (80)$$

with an arbitrary function $a(x)$. The functional integral (77) in this case takes the form

$$e^{iW[J]} = \int \prod_{\mu, x} dA_\mu^a(x) d\bar{c}^a(x) dc^a(x) \cdot \delta(\partial_\mu A^{\mu a} - a^a(x)).$$

$$\exp \left(i \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{c}^a \left(\square c^a - f^{abc} \partial_\mu (A_\mu^c c^b) \right) + J_\mu^a A^{a\mu} \right] \right).$$

The Green function does not depend on $a(x)$, so one can integrate (9) with the exponent

$$\int da \exp \left(-\frac{a^2}{2\xi} \right) \delta(\partial_\mu A^\mu - a) = \exp \left(-\frac{(\partial_\mu A^\mu)^2}{2\xi} \right).$$

As the result, we get the following quadratic part of the action in (9):

$$S = \int dx \left[-\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - \frac{1}{2\xi} \partial_\mu A^{a\mu} \partial_\nu A^{a\nu} + \bar{c} \square c \right],$$

where the numerical parameter ξ is the gauge parameter. Now there is the inverse form, and we get the following propagators for non-abelian gauge and ghost fields, respectively:

$$D_{\mu\nu}^{ab}(k) = -i \frac{\delta^{ab}}{k^2 + i0} \left[g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right], \quad (81)$$

$$D_{\text{ch}}^{ab}(k) = i \frac{\delta^{ab}}{k^2 + i0}. \quad (82)$$

There are a few famous choices of the parameter ξ used in concrete computations:

$\xi = 1$ —the 't Hooft–Feynman gauge,

$\xi = 0$ —the Landau gauge,

$\xi = 3$ —the Frautschi–Yenni gauge.

Of course, any computed physics observable such as cross-section or distribution does not depend on ξ .

As we see,

$$k^\mu D_{\mu\nu}^{ab} = -\xi k_\nu i \frac{\delta^{ab}}{k^2 + i0} = -\xi k_\nu D_{\text{ch}}^{ab}(k).$$

So, ghosts are acting in a way to cancel a dependence on ξ in physics quantities.

Now one can use our formula (48) in momentum space:

$$\Gamma_{\mu_1 \dots \mu_n}^{a_1 \dots a_n}(p_1 \dots p_n) \cdot (2\pi)^4 \delta(p_1 + \dots + p_n) = i \frac{\delta^{(n)} S}{\delta A_{\mu_1}^{a_1}(p_1) \dots \delta A_{\mu_n}^{a_n}(p_n)} \quad (83)$$

in order to get Feynman rules in momentum space for all the vertices in abelian and non-abelian theories. Note that by taking functional derivatives in (83) one gets vertex functions with all needed symmetries. In the formula (83), μ and a indicate proper Lorentz and other group indices identifying the field.

As an example, let us consider the QED Lagrangian

$$L = \bar{\Psi} (iD_\mu \gamma^\mu - m) \Psi, \quad D_\mu = \partial_\mu + ieQA_\mu. \quad (84)$$

For the interaction vertex of the fermion field Ψ with the photon field A_μ , we obtain




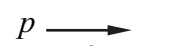



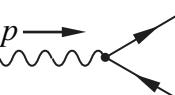
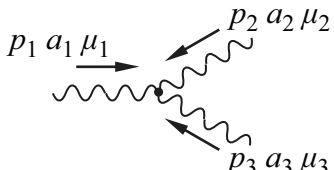
$$\begin{aligned} & \Gamma_\mu(p_1, p_2; p_3) (2\pi)^4 \delta(p_1 + p_2 + p_3) \\ &= i \frac{\delta^{(3)}}{\delta \bar{\Psi}(p_1) \delta \Psi(p_2) \delta A_\mu(p_3)} \cdot \int dx i \cdot (ieQ) \bar{\Psi}(x) \gamma^\mu A_\mu(x) \Psi(x) \\ &= -ieQ \frac{\delta^{(3)}}{\delta \bar{\Psi}(p_1) \delta \Psi(p_2) \delta A_\mu(p_3)} \cdot \\ & \cdot \int dx dq_1 dq_2 dq_3 \exp(-iq_1 x - iq_2 x - iq_3 x) \bar{\Psi}(q_1) \gamma^\mu \Psi(q_2) A_\mu(p_3) \end{aligned}$$

$$= -ieQ(2\pi)^4 \delta(p_1 + p_2 + p_3) \gamma_\mu$$

and so $\Gamma_\mu(p_1, p_2; p_3) = -ieQ\gamma_\mu$, where $Q = -1$ for the fermion field (say, an electron) and $+1$ for the antifermion field (say, a positron) etc.

In a similar way, one gets all other interaction vertices in the case of the SM Lagrangian, which will be considered later. Also, similar to the scalar field theory, the reduction formula allows us to compute S-matrix elements from the corresponding connected Green functions by cutting out propagators on all legs and multiplying by corresponding free-particle wave functions.

A well-known visual way for presenting and computing S-matrix elements is given by Feynman rules—lines for different types of propagators and external particles, and points for vertices:

| | |
|---|--|
|  | $\frac{-i}{k^2+i0} \delta^{ab} [g_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2}]$ for massless gauge field |
|  | $\frac{i}{k^2+i0} \delta^{ab}$ for the ghost field |
|  | $u(p)$ for an incoming fermion |
|  | $\bar{v}(p)$ for an incoming antifermion |
|  | $u(p)$ for an outgoing fermion |
|  | $\bar{v}(p)$ for an outgoing antifermion |
|  | $i \frac{\hat{p} + m}{p^2 - m^2 + i0}$ for a fermion propagator |
|  | $ig\gamma_\mu(t^a)$ for a fermion–gauge boson vertex |
|  | $gf^{a_1 a_2 a_3} [g_{\mu_1 \mu_2} (p_1 - p_2)_{\mu_3} + g_{\mu_2 \mu_3} (p_2 - p_3)_{\mu_1} + g_{\mu_3 \mu_1} (p_3 - p_1)_{\mu_2}]$ |

We do not derive here formulas for cross-sections and decay widths; they are given in many textbooks. We use notation of the Particle Data Group:

$$d\sigma_{ab} = \frac{|M|^2}{4\sqrt{(p_a p_b)^2 - m_a^2 m_b^2}} d\Phi_n,$$

where

$$d\Phi_n = (2\pi)^4 \delta(p_i - p_f) \cdot \frac{d^3 \vec{p}_1}{(2\pi)^3 2p_1^0} \cdots \frac{d^3 \vec{p}_n}{(2\pi)^3 2p_n^0},$$

$$d\Gamma = \frac{|M|^2}{2m_a} d\Phi_n.$$

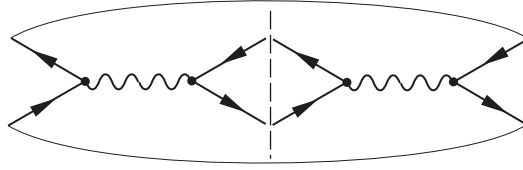


Fig. 6: $\mu^+\mu^-$ production in e^+e^- collisions

As we see, one needs to compute the matrix element squared in order to get a scattering cross-section and a decay width.

Also, one can formulate the Feynman rules for the matrix element squared directly.

In Fig. 6, the square diagram is shown for $\mu^+\mu^-$ -pair production in e^+e^- collisions in QED. The crossed lines correspond to external particles summed over polarizations. Concrete expressions for the sums over polarizations for particles and antiparticles with different spins are as follows:

1 for scalar particles

$$\begin{cases} \sum_{\lambda} u_{\lambda}(p) \times \bar{u}_{\lambda}(p) = p_{\mu}\gamma^{\mu} + m \\ \sum_{\lambda} v_{\lambda}(p) \times \bar{v}_{\lambda}(p) = -p_{\mu}\gamma^{\mu} + m \end{cases} \text{ for spin-1/2 Dirac particles}$$

$$\sum_{\lambda} e_{\lambda}^{\mu}(k) e_{\lambda}^{*\nu}(k) = g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2} \quad \text{for massless gauge fields in the Landau gauge}$$

$$\sum_{\lambda} e_{\lambda}^{\mu}(k) e_{\lambda}^{*\nu}(k) = g_{\mu\nu} \quad \text{in the 't Hooft–Feynman gauge}$$

$$\sum_{\lambda} e_{\lambda}^{\mu}(k) e_{\lambda}^{*\nu}(k) = g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{M^2} \quad \text{for vector fields in the unitary gauge}$$

The Feynman rules for propagators and vertices in the case of matrix elements squared are the same as for the case of amplitudes. Note that the sums over polarizations represent the spin-density matrix and coincide with numerators of the propagators of corresponding particles. Note also, in computations using the Feynman rules for matrix elements squared, that the ghosts should be added into initial and final lines together with corresponding gauge-boson lines. Each loop with crossed ghost lines should include extra factor (-1) with respect to the corresponding gauge-boson loop. This -1 sign reflects the anticommuting property of the ghost fields.

10 Electroweak interactions in the SM

As we know from school textbooks, the weak interactions are responsible for decay of elementary particles. As we shall see, there are also scattering processes due to weak interactions, as were predicted by the SM and discovered in experiments. Studies of weak interactions started from decays, and have a long history, which we do not describe here. From various experimental studies it was realized that (1) electron and muon neutrinos are not the same, and the electron neutrino and antineutrino are different. There are processes $\nu_{e n} \rightarrow e^{-} p$, $\bar{\nu}_{e p} \rightarrow e^{+} n$, $\nu_{\mu n} \rightarrow \mu^{-} p$, $\bar{\nu}_{\mu p} \rightarrow \mu^{+} n$, but there are no processes $\bar{\nu}_{e n} \not\rightarrow e^{-} p$, $\nu_{e p} \not\rightarrow e^{+} n$, $\bar{\nu}_{\mu n} \not\rightarrow \mu^{-} p$, $\nu_{\mu p} \not\rightarrow \mu^{+} n$;

(2) the decays $\mu \rightarrow eX$ have never been observed;

(3) only left-handed leptons and right-handed antileptons participate in the process with $|\Delta Q| = 1$ for leptons from the same generations;

(4) three generations have been observed.

$$\begin{array}{cc} \ell & \bar{\ell} \\ \longrightarrow p & \longrightarrow p \\ \longleftarrow \bar{J} & \longrightarrow \bar{J} \end{array}$$

The observations lead to the assumption that the lepton interactions with $|Q| = 1$ occur via charge current in so-called V–A form:

$$J_\ell \sim \bar{\ell} \gamma_\mu (1 - \gamma_5) \nu_\ell. \quad (85)$$

The corresponding four-fermion interaction Lagrangian for muon and electron currents is

$$L = \frac{G_F}{\sqrt{2}} \bar{\mu} \gamma_\sigma (1 - \gamma_5) \nu_\mu \bar{e} \gamma_\sigma (1 - \gamma_5) \nu_e + \text{h.c.}, \quad (86)$$

where the notation μ, e, ν stands for the corresponding fermion fields and G_F is the well-known Fermi constant with dimension $[m]^{-2}$. With the help of the Lagrangian (86), one can easily compute the decay width of $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$, the Feynman diagram for which is shown in Fig. 7.

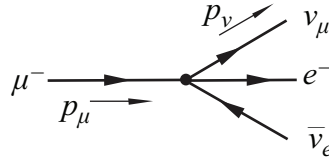


Fig. 7: Feynman diagram for μ^- decay due to four fermion interaction

From dimensional analysis, one can say without any computations that $\Gamma \sim G_F^2 \cdot m_\mu^5$.

The formula for the width is

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} \cdot f \left(\frac{m_e^2}{m_\mu^2} \right), \quad (87)$$

where $f = 1 + 0 \left(\frac{m_e^2}{m_\mu^2} \right)$. Today's G_F is measured from the muon decay very precisely:

$$G_F = (1.166371 \pm 6 \cdot 10^{-6}) \cdot 10^{-5} \text{ GeV}^{-2}.$$

As we know, any fermion field $\Psi(x)$ may be presented as

$$\Psi = \frac{1 - \gamma_5}{2} \Psi + \frac{1 + \gamma_5}{2} \Psi = \Psi_L + \Psi_R.$$

Therefore, the current (85) involves the left component of the fermion field only and has the form

$$J_\ell = \bar{\Psi}_L \gamma_\mu \Psi_L. \quad (88)$$

We want to construct a quantum field theory (the SM) which obeys a few requirements:

- correct electromagnetic neutral currents and V–A charge currents (Fermi);
- three generations without chiral anomalies;
- gauge-invariant dimension-four operators.

Because there are two leptons (charge lepton and corresponding neutrino) in each generation and the left components interact, a very natural assumption is to choose the gauge group for the EW part of the SM to be

$$SU_L(2) \otimes U_Y(1), \quad (89)$$

where $SU_L(2)$ is called the weak isospin group (the weak isospin is an analogue of the usual isospin introduced by Heisenberg to describe the proton and the neutron) and $U_Y(1)$ is the weak hypercharge group. The hypercharge group is needed because we need to have somehow the $U(1)$ group in order to describe the electromagnetic interactions, as we already know. But if one includes simply the electromagnetic group $U_{em}(1)$ instead of $U_Y(1)$ the construction would not give us interactions correctly for all the fermions. The EW part $SU_L(2) \otimes U_Y(1)$ and the $SU_C(3)$ group for strong interactions are combined into the gauge group of the SM.

The fermion fields are taken to be in three generations, and in each generation the left components are combined into $SU(2)$ doublets and the right components transform as $SU(2)$ singlets:

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \quad (90)$$

$e_R \quad \mu_R \quad \tau_R.$

Right-handed neutrinos are present in the original version of the SM.

For the quarks, a similar structure of representations is assumed but with singlet right-handed components for up and down types of quarks:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \quad \begin{pmatrix} c \\ s \end{pmatrix}_L \quad \begin{pmatrix} t \\ b \end{pmatrix}_L \quad (91)$$

$u_R, d_R \quad c_R, s_R \quad t_R, b_R.$

We begin with the construction of the gauge and fermion parts. Now the requirements of the gauge invariance and lowest possible dimension four of terms fix uniquely the form of the EW interaction Lagrangian. The strong interactions are described by the $SU_C(3)$ gauge group:

$$\begin{aligned} L = & -\frac{1}{4}W_{\mu\nu}^i(W^{\mu\nu})^i - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}G_{\mu\nu}^a(G^{\mu\nu})^a \\ & + \sum_{f=\ell,q} \bar{\Psi}_L^f(iD_\mu^L \gamma^\mu) \Psi_L^\dagger + \sum_{f=\ell,q} \bar{\Psi}_R^f(iD_\mu^R \gamma^\mu) \Psi_R^\dagger, \end{aligned} \quad (92)$$

where the field strength tensors and covariant derivatives have forms very familiar to us:

$$\begin{aligned} W_{\mu\nu}^i &= \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g_2 \varepsilon^{ijk} W_\mu^j W_\nu^k \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ G_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_S f^{abc} A_\mu^b A_\nu^c, \\ D_\mu^L &= \partial_\mu - ig_2 W_\mu^i \tau^i - ig_1 B_\mu \left(\frac{Y_L^f}{2} \right) - ig_S A_\mu^a t^a, \\ D_\mu^R &= \partial_\mu - ig_1 B_\mu \left(\frac{Y_R^f}{2} \right) - ig_S A_\mu^a t^a, \end{aligned} \quad (93)$$

$$(94)$$

where $i = 1, 2, 3$, $a = 1, \dots, 8$; W_μ^i are gauge fields for the weak isospin group, B_μ are gauge fields for the weak hypercharge group and A_μ are gluon gauge fields for the strong $SU_C(3)$ colour group. The gauge fields are taken in the adjoint representations and the lepton and the quark fields are in the fundamental representation of $SU_L(2)$ and $SU_C(3)$ groups. The strongly interacting part of the SM related to the $SU_C(3)$ colour group is called QCD, which is covered in a separate course of lectures.

Often when the SM is described the weak hypercharges Y_L^f and Y_R^f are chosen from the beginning such that the Gell-Mann–Nishijima formula is satisfied for each of the left and right chiral fermions:

$$\begin{aligned} Q_f &= (T_3^f)_L + \frac{Y_L^f}{2}, \\ Q_f &= (T_3^f)_R + \frac{Y_R^f}{2}, \end{aligned} \quad (95)$$

where $(T_3)_{\nu_\ell} = 1/2$ and $(T_3)_\ell = -1/2$ are projections of weak isospin operators, $+1/2$ for the up-type and $-1/2$ for the down-type fermions.

However, we do not know at this moment why the Gell-Mann–Nishijima formula should work in our case for the EW part. So, we do not assume from the beginning the Gell-Mann–Nishijima relations for weak hypercharges. Let us take weak hypercharges as free parameters for a moment, and try to fix them from two physics requirements:

1. correct electromagnetic interactions;
2. V–A weak charge currents.

Let us consider for simplicity only fermions from the first generation. We will consider the case of three generations later by introducing the quark mixing matrix.

From the covariant derivatives for the left and right chiral fields (94), one gets the following Lagrangian for leptons of the first generation:

$$L^\ell = -i^2 (\bar{\nu}_{e_L} \bar{e}_L) \gamma_\mu \begin{pmatrix} \frac{1}{2} g_2 W_\mu^3 + g_1 \frac{Y_L^\ell}{2} B_\mu & g_2 \frac{W_\mu^+}{\sqrt{2}} \\ g_2 \frac{W_\mu^-}{\sqrt{2}} & -\frac{1}{2} g_2 W_\mu^3 + g_1 \frac{Y_L^\ell}{2} B_\mu \end{pmatrix} \begin{pmatrix} \nu_{e_L} \\ e_L \end{pmatrix} \quad (96)$$

$$+ \bar{e}_R \gamma_\mu g_1 \frac{Y_R^\ell}{2} B_\mu e_R.$$

Here the relation following from the Pauli matrices is used:

$$\tau^i W^i = \frac{\sigma^i}{2} W^i = \frac{1}{2} \begin{pmatrix} W_\mu^3 & \sqrt{2} W_\mu^+ \\ \sqrt{2} W_\mu^- & -W_\mu^3 \end{pmatrix}, \quad (97)$$

where $W_\mu^\pm = (W_\mu^1 \mp iW_\mu^2) / \sqrt{2}$.

Products of non-diagonal elements give us the form of the charge current:

$$L_{CC}^\ell = \frac{g_2}{\sqrt{2}} \bar{\nu}_{e_L} \gamma_\mu W_\mu^+ e_L + \text{h.c.} = \frac{g_2}{2\sqrt{2}} \bar{\nu}_e \gamma_\mu (1 - \gamma_5) W_\mu^+ e + \text{h.c.} \quad (98)$$

The interaction Lagrangian (98) contains the lepton charge current with the needed V–A structure.

Products of diagonal elements in (96) lead to the neutral current interaction Lagrangian¹:

$$L_{NC}^\ell = \bar{\nu}_{e_L} \gamma_\mu \left(\frac{1}{2} g_2 W_\mu^3 + g_1 \frac{Y_L^\ell}{2} B_\mu \right) \nu_{e_L} \\ + \bar{e}_L \gamma_\mu \left(-\frac{1}{2} g_2 W_\mu^3 + g_1 \frac{Y_L^\ell}{2} B_\mu \right) e_L \quad (99) \\ + \bar{e}_R \gamma_\mu g_1 \frac{Y_R^\ell}{2} B_\mu e_R.$$

Generically, the neutral component of the W field and the B field can mix with some mixing angle θ_W :

$$W_\mu^3 = Z_\mu \cos \theta_W + A_\mu \sin \theta_W, \quad (100) \\ B_\mu = -Z_\mu \sin \theta_W + A_\mu \cos \theta_W.$$

The angle θ_W is called the Weinberg mixing angle.

¹Question for students: why can arbitrary hypercharge not exist in the case of non-abelian gauge symmetry?

One of these fields, say A , we try to identify with the photon—it should not interact with the neutrino and should have the well-known Dirac interaction with the electron field as needed in QED. These two physics requirements lead to three simple equations:

$$\frac{1}{2} \left(-\frac{g_2}{2} \sin \theta_W + \frac{g_1}{2} Y_L^\ell \cos \theta_W \right) + \frac{1}{2} \frac{g_1}{2} Y_R^\ell \cos \theta_W = Q_e e \quad (Q_e = -1), \quad (101)$$

$$\frac{1}{2} \left(\frac{g_2}{2} \sin \theta_W - \frac{g_1}{2} Y_L^\ell \cos \theta_W \right) + \frac{1}{2} \frac{g_1}{2} Y_R^\ell \cos \theta_W = 0, \quad (102)$$

$$\frac{g_2}{2} \sin \theta_W + \frac{g_1}{2} Y_L^\ell \cos \theta_W = 0. \quad (103)$$

The first equation (101) comes from the coefficient in front of the γ_μ structure in the interaction of the electron with the A_μ field, the second equation (102) follows from the coefficient in front of the $\gamma_\mu \gamma_5$ structure and the third one (103) comes from the absence of the interaction of the neutrino field with A_μ .

Therefore,

$$-\frac{g_2}{2} \sin \theta_W + \frac{g_1}{2} Y_L^\ell \cos \theta_W = \frac{g_1}{2} Y_R^\ell \cos \theta_W = Q_e e. \quad (104)$$

From the equations (104), we get

$$g_1 Y_L^\ell \cos \theta_W = -e,$$

$$g_2 \sin \theta_W = e$$

and

$$Y_R^\ell = 2Y_L^\ell. \quad (105)$$

As we can see, the hypercharges of the left and right chiral leptons are proportional but not fully fixed.

In the quark sector there are both left and right chiral components for up and down quarks:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \quad u_R, \quad d_R.$$

Then, from the Lagrangian (92), the interaction of the quarks with gauge fields is

$$\begin{aligned} (\bar{u}d)_L \gamma_\mu \left(\begin{array}{cc} \frac{1}{2} g_2 W_\mu^3 + g_1 \frac{Y_L^q}{2} B_\mu & g_2 \frac{W_\mu^+}{\sqrt{2}} \\ g_2 \frac{W_\mu^-}{\sqrt{2}} & -\frac{1}{2} g_2 W_\mu^3 + g_1 \frac{Y_L^q}{2} B_\mu \end{array} \right) \begin{pmatrix} u \\ d \end{pmatrix}_L \\ + \bar{u}_R \gamma_\mu g_1 \frac{Y_R^u}{2} B_\mu u_R + \bar{d}_R \gamma_\mu g_1 \frac{Y_R^d}{2} B_\mu d_R. \end{aligned} \quad (106)$$

The charge current has, as expected, the needed V–A form:

$$L_{CC}^q = \frac{g_2}{2\sqrt{2}} \bar{u} \gamma_\mu (1 - \gamma_5) W_\mu^+ d + \frac{g_2}{2\sqrt{2}} \bar{d} \gamma_\mu (1 - \gamma_5) W_\mu^- u. \quad (107)$$

In the same way as was done for the lepton case, one should require correct electromagnetic interactions for both u and d quarks. This means that one should have a QED electromagnetic Lagrangian with electric charges $\frac{2}{3}$ for up-quark and $-\frac{1}{3}$ for down-quark fields. Substituting W_μ^3 and B_μ in terms of A_μ and Z_μ fields (99) into (106), we get the following equalities:

$$\frac{1}{2} g_2 \sin \theta_W + \frac{1}{2} g_1 Y_L^q \cos \theta_W = \frac{1}{2} Y_R^u g_1 \cos \theta_W = \frac{2}{3} e, \quad (108)$$

$$-\frac{1}{2} g_2 \sin \theta_W + \frac{1}{2} g_1 Y_L^q \cos \theta_W = \frac{1}{2} Y_R^d g_1 \cos \theta_W = -\frac{1}{3} e.$$

From these four equations one gets the following four equalities:

$$\begin{cases} g_2 \sin \theta_W = e, \\ g_1 Y_L^q \cos \theta_W = \frac{1}{3}e, \\ Y_R^u = -2Y_R^d, \\ Y_R^u + Y_R^d = 2Y_L^q, \end{cases} \quad (109)$$

which are consistent with equalities we obtained from the lepton sector:

$$\begin{cases} g_2 \sin \theta_W = e, \\ g_1 Y_L^\ell \cos \theta_W = -e, \\ Y_R^\ell = 2Y_L^\ell. \end{cases} \quad (110)$$

Note that, as follows from (109) and (110),

$$Y_L^\ell = -3Y_L^q,$$

which means that there is only one independent hypercharge, say Y_L^ℓ , and all the others may be expressed in terms of it.

Let us recall that up to now we did not assume any additional relations such as ($Q = T_3 + Y/2$), which are usually assumed from the very beginning.

Now we can write the generic Lagrangian for neutral current interactions with introduced bosons A_μ and Z_μ in the following form:

$$L_{\text{NC}} = e \sum_f Q_f J_{f\mu}^{\text{em}} A^\mu + \frac{e}{4 \sin \theta_W \cos \theta_W} \cdot \sum_f J_{f\mu}^Z Z^\mu, \quad (111)$$

where $J_{f\mu}^{\text{em}} = \bar{f} \gamma_\mu f$, $Q_\nu = 0$, $Q_e = -1$, $Q_u = 2/3$, $Q_d = -1/3$,

$$J_{f\mu}^Z = \bar{f} \gamma_\mu [v_f - a_f \gamma_5] f,$$

$$v_\nu = 1, \quad a_\nu = 1, \quad v_e = -1 + 4 \sin^2 \theta_W, \quad a_e = -1;$$

$$v_u = 1 - \frac{1}{3} \left(4 + \frac{Y_R^u}{Y_L^q} \right) \sin^2 \theta_W, \quad a_u = 1 - \frac{1}{3} \left(4 - \frac{Y_R^u}{Y_L^q} \right) \sin^2 \theta_W,$$

$$v_d = -1 + \frac{1}{3} \left(2 - \frac{Y_R^d}{Y_L^q} \right) \sin^2 \theta_W, \quad a_d = -1 + \frac{1}{3} \left(2 + \frac{Y_R^d}{Y_L^q} \right) \sin^2 \theta_W.$$

Since the structure of all three generations is the same, the equality (111) is the same for all leptons and quarks. Vector and axial-vector couplings v_f and a_f may be expressed for all the fermions in a compact common form via the fermion charge Q_f and a component of the fermion weak isospin T_3^f :

$$v_f = 2T_3^f - 4Q_f \sin^2 \theta_W, \quad a_f = 2T_3^f.$$

Note that the hypercharge parameters Y_L^ℓ and Y_R^ℓ are not present in (111) while in the quark sector one free parameter, which as we saw may be expressed in terms of Y_L^ℓ , remains taking into account (109),

$$\frac{Y_R^u}{Y_L^q} + \frac{Y_R^d}{Y_L^q} = 2.$$

We have obtained the Lagrangian, the sum of (98), (107), (111) and the gauge boson kinetic terms from (92), which contains QED with massless fermions, an additional vector particle Z_μ interacting with new neutral currents, two charge massless vector particles W_μ^\pm interacting with V-A charge currents and self-interactions of the gauge particles.

Since our Lagrangian has a rather non-trivial chiral structure, an important question arises as to whether or not our construction is free of chiral anomalies, which is absolutely needed for a theory to be self-consistent.

11 Anomalies

Detailed discussion of anomalies is not a subject of our brief notes and can be found in a number of textbooks (see for example [6, 8]).

Generically, anomalies correspond to a situation in the field theory when some symmetry takes place at the level of a classical Lagrangian but it is violated at quantum loop level. For us an important anomaly is the chiral anomaly. In short, it means that, for example, the Lagrangian

$$L = \bar{\Psi}_L i\gamma^\mu (\partial_\mu - igA_\mu^a t^a) \Psi_L = \bar{\Psi} i\gamma^\mu (\partial_\mu - igA_\mu^a t^a) \frac{1 - \gamma_5}{2} \Psi \quad (112)$$

is invariant under the transformation

$$\Psi \rightarrow \exp\left(i\alpha^a t^a \frac{1 - \gamma_5}{2}\right) \Psi, \quad A_\mu \rightarrow A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a + f^{abc} A_\mu^b \alpha^c. \quad (113)$$

This invariance according to the Noether theorem leads to conserving of the current:

$$j_\mu^a = \bar{\Psi} \gamma_\mu \frac{1 - \gamma_5}{2} t^a \Psi. \quad (114)$$

However, after quantization one finds that the current (114) cannot be conserved due to the triangle loop contributions shown in Fig. 8.

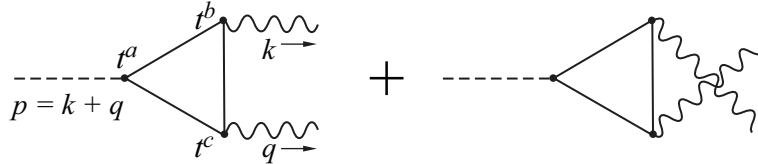


Fig. 8: Loop corrections

The diagrams in Fig. 8, after a convolution with the momentum p_μ instead of being equal to zero, are proportional to the factor

$$\frac{g^2}{8\pi^2} \varepsilon^{\mu\nu\alpha\beta} k^\alpha q^\beta \cdot \text{Tr} \left[t^a \{t^b t^c\} \right]. \quad (115)$$

If the anomaly is not vanishing the theory loses its gauge invariance and therefore cannot be acceptable. (However, in cases of some currents external with respect to the theory, which have nothing to do with symmetries of the theory and the Noether theorem, anomalies or such currents may take place. This does not lead to any problems. Moreover, such type of anomalies may have very important physics consequences, as in the case of π^0 decay to two photons.) In the SM there are simultaneously contributions from left and right chiral fermions which contribute to the anomaly with opposite signs. The anomaly is then proportional to the differences between traces of group generators coming from fermions with left and right chiralities:

$$\text{Anom} \sim \text{Tr} \left[t^a \{t^b t^c\} \right]_L - \text{Tr} \left[t^a \{t^b t^c\} \right]_R. \quad (116)$$

In the theories like QED or QCD there are no γ^5 matrices involved in the Lagrangians. Therefore, the left and right chiral contributions exactly compensate each other, the anomaly is equal to zero and the theories perfectly make sense.

In the EW part of the SM left and right states couple to $U_Y(1)$ gauge bosons with different hypercharges, and only the left components couple to the $SU_2(2)$ gauge bosons. So, it is not obvious a priori that chiral anomalies vanish. In fact, zero anomalies is a request to the SM to be a reasonable quantum theory.

Since the generators of the $SU_L(2)$ group are the matrices $t^i = \sigma^i/2$, then any combination of three $SU_C(2)$ generators gives zero traces in (116) and therefore zero anomalies. The only potentially dangerous ones are

$$(SU_L(2))^2 \cdot U_Y(1) \text{ and } U_Y(1)^3$$

anomalies. For the first type we have to take into account $\{t^i, t^j\} = \frac{1}{2}\delta^{ij}$ and then the only non-zero contribution is

$$\text{Anom} \sim \text{Tr} [Y\{t^i, t^j\}]_L = \frac{1}{2}\delta^{ij}\text{Tr}Y_L = \frac{1}{2}\delta^{ij} [N_C \cdot 2Y_L^q + 2Y_L^\ell]. \quad (117)$$

From the relations (108) and (109), as already was mentioned, we have the following relation between hypercharges:

$$Y_L^\ell = -3Y_L^q. \quad (118)$$

After substitution of (118) into (117), we get

$$\text{Anom} \sim \frac{1}{2}\delta^{ij}2Y_L^q(N_C - 3). \quad (119)$$

It is very interesting that the anomaly vanishes only for the number of colours equal to three, as it does in QCD. However, the value of the hypercharge Y_L^q is not fixed.

The second type $((U_Y(2))^3)$ of anomaly for the fermions for each generation is proportional to

$$\begin{aligned} \text{Anom} &\sim \text{Tr}(Y_L^3) - \text{Tr}(Y_R^3) \\ &= N_C(Y_L^q)^3 \cdot 2 + (Y_L^\ell)^3 \cdot 2 - N_C [(Y_R^u)^3 + (Y_R^d)^3] - (Y_R^\ell)^3, \end{aligned} \quad (120)$$

where the factor (2) in the left-hand contribution comes from two (u and d) quarks and two (e and ν_e) leptons. Taking into account from (118) and (108) that $Y_R^u + Y_R^d = 2Y_L^q$, one gets from (120) the following:

$$\begin{aligned} \text{Anom} &\sim Y_L^\ell [2N_C(\frac{1}{3}Y_L^\ell + Y_R^u)^2 - 6(Y_L^\ell)^2] \\ &= Y_L^\ell \cdot 6(\frac{1}{3}Y_L^\ell + Y_R^u - Y_L^\ell)(\frac{1}{3}Y_L^\ell + Y_R^u + Y_L^\ell). \end{aligned} \quad (121)$$

In order to get zero for the anomaly,

$$Y_R^u = \frac{2}{3}Y_L^\ell \text{ or } Y_R^u = -\frac{4}{3}Y_L^\ell. \quad (122)$$

At this point one cannot prefer one of the relations in (122). This value will be finally fixed only after $SU_L(2) \times U_Y(1)$ symmetry breaking.

So, we note once more that we have constructed a theory with the Lagrangian for massless fermions and gauge bosons, which gives us:

1. correct V-A charge currents;
2. correct electromagnetic interactions;
3. no chiral anomalies;

4. predictions of additional neutral currents observed experimentally.

However, obviously such a theory cannot describe nature correctly. We do not observe massless EW vector particles, except for the photon, and we do not observe massless fermions, except for, maybe, the neutrinos (or one of the neutrinos). Massive W and Z bosons, massive leptons and quarks are observed experimentally.

So, we have to introduce masses in the theory. But we cannot do it directly without violation of the basic principle of gauge invariance. Indeed, the mass term for the vector field $m_V^2 V^\mu V_\mu$ is not invariant under the gauge transformation $V_\mu \rightarrow V_\mu + \partial_\mu \alpha$, and the mass term for fermions $m_\Psi \bar{\Psi} \Psi$, the Dirac mass, is equal to $m (\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L)$, and it is also not gauge invariant. Indeed, the left field Ψ_L is the doublet and the right field Ψ_R is the singlet with respect to the group $SU_L(2)$. How to make massive particles without violation of the basic principle of gauge invariance? There is a way to resolve this problem, which is related to spontaneous symmetry breaking, the Nambu–Goldstone theorem and the Brout–Englert–Higgs–Hagen–Guralnik–Kibble mechanism.

12 Spontaneous symmetry breaking and the Brout–Englert–Higgs–Hagen–Guralnik–Kibble mechanism

The situation when the Lagrangian is invariant under some symmetry while the spectrum of the system is not invariant is very common for spontaneous symmetry breaking (for example, Ginzburg–Landau theory). But a naive realization of ideas of spontaneous symmetry breaking leads to a problem manifested in the appearance of so-called Nambu–Goldstone bosons with zero masses.

To illustrate this, let us consider a very simple scalar model with the Lagrangian

$$L = \partial_\mu \varphi^\dagger \partial^\mu \varphi - \mu^2 \varphi^\dagger \varphi - \lambda (\varphi^\dagger \varphi)^2. \quad (123)$$

The Lagrangian (123) is invariant under the phase shift $\varphi \rightarrow \varphi e^{i\omega}$ with $\omega = \text{Const}$. The case with $\mu^2 > 0$ is trivial and not interesting for us. In the case $\mu^2 = -|\mu^2| < 0$, the potential shown in Fig.9

$$V(\varphi) = \mu^2 \varphi^\dagger \varphi + \lambda (\varphi^\dagger \varphi)^2 \quad (124)$$

has a non-trivial minimum:

$$\left. \frac{dV}{d\varphi^\dagger} \right|_{\varphi_0} = -|\mu^2| \varphi_0 + 2\lambda (\varphi_0^\dagger \varphi_0) \varphi_0 = 0 \Rightarrow |\varphi_0| = \sqrt{\frac{|\mu^2|}{2\lambda}} = \frac{v}{\sqrt{2}} > 0.$$

The system takes some concrete value for the vacuum solution, say $\varphi_0 = +v/\sqrt{2}$, which violates the phase-shift symmetry.

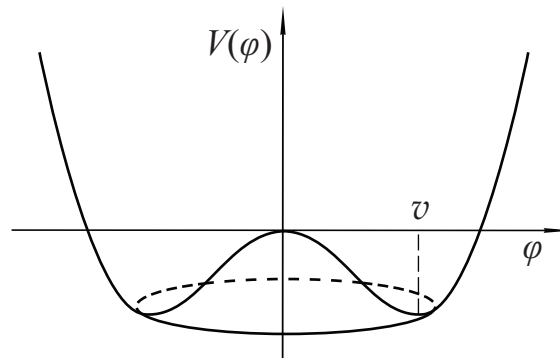


Fig. 9: Higgs potential

A complex scalar field can be parametrized by two real fields

$$\varphi = \frac{1}{\sqrt{2}}(v + h(x))e^{-i\xi(x)/v}. \quad (125)$$

In terms of the new fields $h(x)$ and $\xi(x)$, the Lagrangian has the following form:

$$\begin{aligned} L = & \frac{1}{2}\partial_\mu h \partial^\mu h - \lambda v^2 h^2 - \lambda v h^3 - \lambda h^4/4 \\ & + \frac{1}{2}\partial_\mu \xi \partial^\mu \xi + \frac{2}{v}\partial_\mu \xi \partial^\mu \xi h + \frac{1}{v^2}\partial_\mu \xi \partial^\mu \xi h^2 + \lambda v^4/4. \end{aligned} \quad (126)$$

The Lagrangian (126) describes the system of a massive scalar field h with mass $m_h^2 = 2\lambda v^2$ interacting with a massless scalar field $\xi(x)$. The field $\xi(x)$ is called the Nambu–Goldstone boson field.

This is a particular case of the generic Goldstone theorem. If the theory is invariant under a global group with m generators but the vacuum is invariant under transformations generated only by ℓ -generators, then in theory there exist $m - \ell$ massless Nambu–Goldstone bosons.

Consider the system described by the Lagrangian

$$L = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - V(\varphi).$$

Let the Lagrangian be invariant under $i = 1, \dots, m$ transformations:

$$\varphi \rightarrow \varphi' = \varphi + \delta\varphi, \quad \delta\varphi_i = i\delta\Theta_A t_{ij}^A \varphi_j.$$

The invariance of the potential means that

$$\delta V = (\partial V / \partial \varphi_i) \delta\varphi_i = i\delta\Theta^A (\partial V / \partial \varphi_i) t_{ij}^A \varphi_j = 0. \quad (127)$$

Let the potential have a minimum (vacuum) at some field value $\varphi_i = \varphi_i^0$:

$$(\partial V / \partial \varphi_i)(\varphi_i = \varphi_i^0) = 0. \quad (128)$$

We consider the case that the vacuum is invariant under transformations generated only by ℓ -generators from all m generators corresponding to the symmetry, which means that

$$t_{ij}^A \varphi_j^0 = 0 \quad (129)$$

only for $i = 1, \dots, \ell$ ($\ell < m$).

The second derivative from the invariance condition (127) at the minimum leads to

$$\frac{\partial^2 V}{\partial \varphi_k \partial \varphi_i}(\varphi_i = \varphi_i^0) t_{ij}^A \varphi_j^0 + \frac{\partial V}{\partial \varphi_i}(\varphi_i = \varphi_i^0) t_{ij}^A = 0.$$

The second term here is equal to zero due to (128) and therefore

$$\frac{\partial^2 V}{\partial \varphi_k \partial \varphi_i}(\varphi_i = \varphi_i^0) t_{ij}^A \varphi_j^0 = 0.$$

For the first ℓ , this equality takes place because of (129). However, for other fields with $i = \ell + 1, \dots, m$, the following equality has to be valid:

$$\frac{\partial^2 V}{\partial \varphi_k \partial \varphi_i}(\varphi_i = \varphi_i^0) = 0. \quad (130)$$

But the second derivative of the potential in (130) is nothing but the mass term for these $i = \ell + 1, \dots, m$ fields. And, the equation (130) tells us that the masses of these fields are equal to zero. So, in such a situation when the vacuum of a system is not invariant under all the symmetry transformations of the Lagrangian there are $m - \ell$ massless fields (Nambu–Goldstone bosons) corresponding to the number of generators violating the symmetry.

Let us return to the SM gauge group $SU_L(2) \times U_Y(1)$ and add to the system one additional complex scalar field $\Phi(x)$, being an $SU_L(2)$ doublet and a $U_Y(1)$ singlet:

$$L_\Phi = D_\mu \Phi^\dagger D^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^4. \quad (131)$$

The Lagrangian L_Φ is gauge invariant; the covariant derivative has the form

$$D_\mu = \partial_\mu - ig_2 W_\mu^i \tau^i - ig_1 \frac{Y_H}{2} B_\mu. \quad (132)$$

As in our previous example, let the mass parameter squared be negative, $\mu^2 = -|\mu^2|$, and therefore the field potential has a non-trivial minimum at $\Phi = v/\sqrt{2}$.

One can parametrize the complex field doublet $\Phi(x)$ by four real fields in the following generic way:

$$\Phi(x) = \exp\left(-i \frac{\xi^i(x) t^i}{v}\right) \begin{pmatrix} 0 \\ (v+h)/\sqrt{2} \end{pmatrix}, \quad (133)$$

where four scalar fields ξ_1, ξ_2, ξ_3 and h are introduced.

The Lagrangian (131) is invariant under an $SU_L(2)$ transformation:

$$\Phi(x) \rightarrow \Phi'(x) = \exp(ig_2 \alpha^i t^i) \Phi(x), \quad (134)$$

where $t^i = \sigma^i/2$ are the generators of the $SU_L(2)$ gauge group. If we compare (133) and (134), we can choose a special gauge $g_2 \alpha^i(x) = \xi^i(x)/v$ such that the unitary factor $\exp(-i\xi^i(x)t^i/v)$ disappears from all the formulas. This gauge is called the unitary gauge. The Higgs field $\Phi(x)$ in this gauge takes therefore the following form:

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h(x) \end{pmatrix}. \quad (135)$$

The field Φ has a non-zero vacuum expectation value and as we know it leads to a violation of the symmetry of the system.

After such spontaneous symmetry breaking, the substitution of (135) into the Lagrangian (131) with the covariant derivative (132) yields the following Lagrangian in terms of fields W_μ^\pm, A_μ and Z_μ introduced before:

$$L = \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}(2\lambda v^2)h^2 - \lambda v h^3 - \frac{\lambda}{4}h^4 \quad (136)$$

$$+ M_W^2 W_\mu^+ W^{\mu-} (1+h/v)^2 + \frac{1}{2} M_Z^2 Z_\mu Z^\mu (1+h/v)^2,$$

where

$$M_h^2 = 2\lambda v^2 \quad (137)$$

is the mass of the scalar field h called the Higgs boson and

$$M_W = \frac{1}{2} g_2 v \quad \text{and} \quad M_Z = \frac{1}{2} (g_2 \cos \theta_W + g_1 Y_H \sin \theta_W) v \quad (138)$$

are masses of the vector fields W_μ^\pm and Z_μ . The field A_μ is not present in the Lagrangian (136) and therefore remains massless only in the case where the corresponding coefficient in front of it is equal to zero:

$$-\frac{1}{2} g_2 \sin \theta_W + g_1 \frac{Y_H}{2} \cos \theta_W = 0.$$

The condition from the lepton sector (103)

$$g_2 \sin \theta_W = -g_1 Y_L^\ell \cos \theta_W$$

tells us that the field A has the correct electromagnetic interactions and has zero mass simultaneously if the charged lepton and Higgs field hypercharges have equal moduli and opposite signs:

$$Y_H = -Y_L^\ell.$$

One should note that if the vacuum is invariant under some group transformation, then the generator of the group gives zero acting on the vacuum. Indeed, the invariance of the vacuum means that

$$e^{iT_i \Theta_i} \Phi_{\text{vac}} = \Phi_{\text{vac}};$$

therefore, the generator T_i is given by

$$T_i \Phi_{\text{vac}} = 0.$$

In our case

$$\Phi_{\text{vac}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

and the generator of unbroken symmetry should have a generic form:

$$T \Phi_{\text{vac}} = \frac{1}{\sqrt{2}} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0, \quad \Rightarrow \quad T = \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

taking into account the fact that the generator should be Hermitian. For the group $SU_L(2) \times U_Y(1)$, such a generator is

$$T_3 + \frac{1}{2} Y_H = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2} Y_H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{only if } Y_H = +1.$$

This reflects the fact that the vacuum should be neutral, and the remaining group is naturally the unbroken electromagnetic group $U_{\text{em}}(1)$:

$$\begin{aligned} SU_L(2) \times U_Y(1) &\rightarrow U_{\text{em}}(1), \\ T_3 + \frac{1}{2} Y_H &= Q_H = 0, \quad Y_H = +1. \end{aligned} \quad (139)$$

Because $Y_H = 1$, one gets the following relation:

$$g_2 \sin \theta_W = g_1 \cos \theta_W. \quad (140)$$

If one substitutes (140) into the equality for M_Z (138), one gets the well-known relation between masses of W and Z bosons:

$$M_W = M_Z \cos \theta_W. \quad (141)$$

The value of the Higgs hypercharge $Y_H = 1$ fixes the lepton hypercharge $Y_L^\ell = -1$. Now, from the connection between hypercharges (109, 110) and following from them

$$Y_L^\ell = -3Y_L^q,$$

all the values for hypercharges of leptons and quarks with left and right chiralities are fixed:

$$Y_R^\ell = -2, Y_L^q = Y_L^u = Y_L^d = 1/3, Y_R^u = 4/3, Y_R^d = -2/3.$$

This confirms the Gell-Mann–Nishijima relation

$$I_3 + \frac{Y}{2} = Q$$

for all three leptons and for all quarks with both chiralities. It should be so from the relation between $SU_L(2)$ and $U_Y(1)$ generators leading to an unbroken $U_{\text{em}}(1)$ generator.

Coming back to the Lagrangian (136) and adding to it the kinetic terms and self-interactions of the gauge fields W_μ^\pm , A_μ and Z_μ , which come from the terms of the SM Lagrangian

$$-\frac{1}{4}W_{\mu\nu}^i W^{i\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu},$$

we get the Lagrangian describing the massive Higgs boson h , massive vector fields W_μ^i and Z_μ and massless field A_μ . From the Goldstone theorem we expect $4 - 1 = 3$ massless Nambu–Goldstone bosons. But they are not present in the Lagrangian. Three would-be Nambu–Goldstone bosons ξ_1 , ξ_2 and ξ_3 are ‘eaten’ by three longitudinal components of the fields W_μ^- , W_μ^+ and Z_μ . One should stress that while the symmetry is spontaneously broken, the gauge symmetry of the Lagrangian itself remains unbroken.

This is the famous Brout–Englert–Higgs mechanism of spontaneous symmetry breaking (Nobel prize for 2012) confirmed by the discovery of the Higgs-like boson in ATLAS and CMS experiments at the LHC.

Now we consider the fermions, leptons and quarks, of the SM, and show how the mechanism of spontaneous symmetry breaking allows us to get massive fermions without violation of the gauge invariance.

As we know, in the SM the left fermions are the $SU(2)$ doublets and the right fermions are the singlets. There are only two gauge-invariant dimension-four operators of the Yukawa-type preserving the SM gauge invariance:

$$\bar{Q}_L \Phi d_R \quad \text{and} \quad \bar{Q}_L \Phi^C u_R, \quad (142)$$

where

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

is the doublet of left fermions and

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix} \quad \text{and} \quad \Phi^C = i\sigma^2 \Phi^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} v + h \\ 0 \end{pmatrix}$$

are the Higgs and conjugated Higgs $SU_L(2)$ doublet fields in the unitary gauge. Corresponding to (142), charge conjugated operators have the form

$$(\bar{Q}_L \Phi d_R)^\dagger = d_R^\dagger \Phi^\dagger (\bar{Q}_L)^\dagger = d_R^\dagger \gamma^0 \gamma^0 \Phi^\dagger \gamma^0 Q_L = \bar{d}_R \Phi^\dagger Q_L$$

and

$$(\bar{Q}_L \Phi^C u_R)^\dagger = \bar{u}_R (\Phi^C)^\dagger Q_L.$$

As one can easily see, the operators of (142) type lead after spontaneous symmetry breaking to the needed terms for the fermion masses. Indeed,

$$(\bar{u}_L \bar{d}_L) \begin{pmatrix} 0 \\ v \end{pmatrix} d_R + \bar{d}_R (0 \ v) \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \bar{d}_L d_R + v \bar{d}_R d_L = v (\bar{d}_L d_R + \bar{d}_R d_L) = v \bar{d}d,$$

which is the Dirac mass term for the fermion. Similarly, the operator with the Φ^C field leads to the correct Dirac mass term for the up-type fermions:

$$v \bar{u}u.$$

However, most general operators preserving the SM gauge invariance may include mixing of the fermion fields from various generations. The most general interaction Lagrangian including operators of (142) type has the following form:

$$L_{\text{Yukawa}} = -\Gamma_d^{ij} \bar{Q}_L^i \Phi d_R^j + \text{h.c.} - \Gamma_u^{ij} \bar{Q}_L^i \Phi^C u_R^j + \text{h.c.} - \Gamma_e^{ij} \bar{L}_L^i \Phi e_R^j + \text{h.c.}, \quad (143)$$

where there are no terms with a right-handed neutrino, and $\Gamma_{u,d,e}$ are generically possible mixing coefficients. After spontaneous symmetry breaking, one can rewrite the Lagrangian (143) in the unitary gauge as follows:

$$L_{\text{Yukawa}} = - \left[M_d^{ij} \bar{d}_L^i d_R^j + M_u^{ij} \bar{u}_L^i u_R^j + M_e^{ij} \bar{e}_L^i e_R^j + \text{h.c.} \right] \cdot \left(1 + \frac{h}{v} \right), \quad (144)$$

where $M^{ij} = \Gamma^{ij} v / \sqrt{2}$.

The physics states are the states with definite mass. So, one should diagonalize the matrices in order to get the physics states for quarks and leptons. This can be done by unitary transformations for all left- and right-handed fermions:

$$\begin{aligned} d'_{Li} &= (U_L^d)_{ij} d_{Lj}; & d'_{Ri} &= (U_R^d)_{ij} d_{Rj}; & u'_{Li} &= (U_L^u)_{ij} u_{Lj}; & u'_{Ri} &= (U_R^u)_{ij} u_{Rj} \\ \ell'_L &= (U_L^\ell) \ell_L; & \ell'_R &= (U_R^\ell) \ell_R \\ U_L U_L^\dagger &= 1, & U_R U_R^\dagger &= 1, & U_L^\dagger U_L &= 1. \end{aligned}$$

The matrices U are chosen such that

$$\begin{aligned} (U_L^u)^\dagger M_u U_R^u &= \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}; & (U_L^d)^\dagger M_d U_R^d &= \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \\ (U_L^\ell)^\dagger M_\ell U_R^\ell &= \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \end{aligned}$$

Therefore, the Yukawa-type Lagrangian (144) is

$$\begin{aligned} L_{\text{Yukawa}} &= - \left[m_d^i \bar{d}_L^i d_R^i + m_d^{*i} \bar{d}_R^i d_L^i + \right. \\ &\quad \left. + m_u^i \bar{u}_L^i u_R^i + m_u^{*i} \bar{u}_R^i u_L^i + m_\ell^i \bar{\ell}_L^i \ell_R^i + m_\ell^{*i} \bar{\ell}_R^i \ell_L^i \right] \cdot \left(1 + \frac{h}{v} \right). \end{aligned}$$

We consider only real mass parameters $m \equiv m^*$. So, the Yukawa Lagrangian after diagonalization of the mass matrices contains masses of fermions and their interactions with the Higgs boson:

$$\implies L_{\text{Yukawa}} = - \left[m_d^i \bar{d}^i d^i + m_u^i \bar{u}^i u^i + m_\ell^i \bar{\ell}^i \ell^i \right] \cdot \left(1 + \frac{h}{v} \right). \quad (145)$$

Now one can easily see what the fermion interactions with the gauge bosons look like in the basis of the fermion physics state with definite masses.

Neutral currents have the same structure (110) with respect to flavours as the mass terms. And they, after the unitary rotation $\Psi' \rightarrow U\Psi$, become diagonal simultaneously with the mass terms:

$$\bar{\Psi}' \hat{O}_N \Psi' \rightarrow \bar{\Psi} \hat{O} \Psi.$$

However, charge currents

$$J_C \sim \bar{u}_L \hat{O}_{\text{ch}} d_L + \text{h.c.}$$

contain fermions rotated by different unitary matrices for the up- and down-type fermions:

$$u' \rightarrow (U_L^u)u, \quad d' \rightarrow (U_L^d)d.$$

Therefore, after the rotation one gets for the charge current

$$J_C \sim (U_L^u)^\dagger U_L^d \bar{u}_L \hat{Q} d_L.$$

The unitary matrix

$$V_{\text{CKM}} = (U_L^u)^\dagger U_L^d$$

is called the Cabibbo–Kobayashi–Maskawa (CKM) mixing matrix:

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \quad (146)$$

Concrete values for the elements of the CKM matrix are not predicted in the SM. One can show that an arbitrary unitary matrix with $N \times N$ complex elements may be parametrized by $N(N-1)/2$ real angles and $(N-1)(N-2)/2$ complex phases. So, the CKM 3×3 matrix contains three real parameters and one complex phase. The presence of this phase leads to parity and charge (CP) violation, which in this sense is a prediction of the SM with three generations. In this lecture we do not discuss physics of the CKM matrix. The flavour physics is covered in a special lecture course at the School.

Now we have all parts of the EW part of the SM Lagrangian expressed in terms of physics fields in unitary gauge:

$$L_{\text{SM}} = L_{\text{Gauge}} + L_{\text{FG}} + L_{\text{H}}. \quad (147)$$

Here, as was mentioned, the self-interactions of the gauge fields W_μ^\pm , A_μ and Z_μ come from the terms of the SM Lagrangian $-\frac{1}{4}W_{\mu\nu}^i W^{i\mu\nu} - \frac{1}{4}B_{\mu\nu} B^{\mu\nu}$:

$$\begin{aligned} L_{\text{Gauge}} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2}W_{\mu\nu}^+ W^{-\mu\nu} \\ & + e [W_{\mu\nu}^+ W^{-\mu} A^\nu + \text{h.c.} + W_\mu^+ W_\nu^- F^{\mu\nu}] \\ & + e \frac{c_W}{s_W} [W_{\mu\nu}^+ W^{-\mu} Z^\nu + \text{h.c.} + W_\mu^+ W_\nu^- Z^{\mu\nu}] \\ & - e^2 \frac{1}{4s_W^2} [(W_\mu^- W_\nu^+ - W_\nu^- W_\mu^+) W^{-\mu} W^{+\nu} + \text{h.c.}] \\ & - \frac{e^2}{4} (W_\mu^+ A_\nu - W_\nu^+ A_\mu)(W^{-\mu} A^\nu - W^{-\nu} A^\mu) \\ & - \frac{e^2}{4} \frac{c_W^2}{s_W^2} (W_\mu^+ Z_\nu - W_\nu^+ Z_\mu)(W^{-\mu} Z^\nu - W^{-\nu} Z^\mu) \\ & - \frac{e^2}{2} \frac{c_W}{s_W} (W_\mu^+ A_\nu - W_\nu^- A_\mu)(W^{+\mu} Z^\nu - W^{-\nu} Z^\mu) + \text{h.c.}, \end{aligned} \quad (148)$$

where $c_W = \cos \theta_W$, $s_W = \sin \theta_W$; the gauge for a photon field may be taken differently, for example $(\partial_\mu A^\mu) = 0$. The Lagrangian for the interactions of fermions with the gauge bosons is

$$L_{\text{FG}} = \sum_f \bar{f}(i\hat{D})f + L_{\text{NC}} + L_{\text{CC}}, \quad (149)$$

where L_{NC} and L_{CC} are given by (111), (99) and (107). The Lagrangian for the Higgs boson and its interactions with the gauge and fermion fields is

$$\begin{aligned} L_{\text{H}} = & \frac{1}{2}(\partial^\mu h)(\partial_\mu h) + \frac{M_h^2}{2}h^2 - \frac{M_h^2}{2v}h^3 - \frac{M_h^2}{8v^2}h^4 \\ & + (M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu) \left(1 + \frac{h}{v}\right)^2 - \sum_f m_f \bar{f}f \left(1 + \frac{h}{v}\right). \end{aligned} \quad (150)$$

All Feynman rules following from the Lagrangian (147) can be obtained with the help of the formula (83). The kinetic parts of the Lagrangians L_{Gauge} (148) and L_{FG} (148) being taken together with the gauge and fermion mass terms from L_{H} (150) give the propagators for fermions, massless photons, massive W^\pm and Z bosons, and for the Higgs boson by inverting the corresponding quadratic forms, as we know already².

The propagator for the massive gauge bosons requires special care. The propagators for massive vector fields follow directly from the Lagrangian by inverting the quadratic form:

$$V^\mu \left(\square g_{\mu\nu} - \partial_\mu \partial^\nu + g_{\mu\nu} M_V^2 \right) V^\nu. \quad (151)$$

Then the propagator for massive vector fields ($V = W, Z$) has the following structure in the unitary gauge:

$$D_{\mu\nu}(p) = \frac{-i}{p^2 - M_V^2} \left[g_{\mu\nu} - \frac{p_\mu p_\nu}{M_V^2} \right]. \quad (152)$$

However, the term $p_\mu p_\nu / M_V^2$ has a bad ultraviolet behaviour. This leads to the problem of proving renormalizability of the SM. To resolve the problem, one can use another gauge in which the bad ultraviolet behaviour is absent. It is convenient to express the Higgs field as follows:

$$\Phi(x) = \begin{pmatrix} -i w_g^+ \\ (v + h + i z_g) / \sqrt{2} \end{pmatrix} \quad (153)$$

and Φ^\dagger contains w_g^- , where the notation w_g^\pm and z_g for the Goldstone bosons is introduced. The covariant derivative, being expressed in terms of the fields W_μ^\pm , Z_μ and A_μ , and constants e and $\sin \theta_W = s_W$, takes the form

$$D_\mu \Phi = \begin{pmatrix} \partial_\mu - i \frac{e(1 - 2s_W^2)}{2s_W c_W} Z_\mu - i e A_\mu & -i \frac{e}{\sqrt{2} s_W} W_\mu^+ \\ -i \frac{e}{\sqrt{2} s_W} W_\mu^- & \partial_\mu + i \frac{e}{2s_W c_W} Z_\mu \end{pmatrix} \Phi \quad (154)$$

Simple algebraic manipulation leads to the following Lagrangian for the Higgs-gauge part of the SM:

$$\begin{aligned} L &= (D_\mu \Phi)^\dagger (D^\mu \Phi) - \lambda (\Phi \Phi^\dagger - v^2/2)^2 \\ &= \frac{1}{2} (\partial^\mu h) (\partial_\mu h) + M_W^2 W_\mu^+ W^{\mu-} (1 + h/v)^2 + \frac{1}{2} M_Z^2 Z_\mu Z^\mu (1 + h/v)^2 \\ &\quad - M_h^2 h^2 - \lambda v h^3 - \frac{\lambda}{4} h^4 \\ &\quad - M_W \partial_\mu w_g^+ W^{\mu-} - M_W \partial_\mu w_g^- W^{\mu+} - M_Z \partial_\mu z_g Z^\mu \\ &\quad + \partial_\mu w_g^+ \partial^\mu w_g^- + \frac{1}{2} \partial_\mu z_g \partial^\mu z_g \\ &\quad - \lambda h (h + 2v) (w_g^- w_g^+ + z_g/2) - \lambda (w_g^- w_g^+ + z_g/2)^2 \\ &\quad + \text{more cubic and quadratic terms involving } w_g^\pm \text{ and } z_g \text{ fields.} \end{aligned} \quad (155)$$

The first two lines in (155) are exactly the same as in the SM in the unitary gauge. The fourth line involves massless scalar fields, the Goldstone bosons w_g^\pm and z_g . There are many terms describing interactions of the Goldstone fields. But we would like to draw attention to the third line in (155) describing the kinetic mixing of the w_g^\pm and z_g fields with W^\pm and Z fields, respectively. Such a mixing should be removed from the Lagrangian. This can be achieved by choosing proper gauge conditions.

²The complete list of the Feynman rules for interaction vertices can be found in many books and listed explicitly, for example, in the model files for the SM used in computer codes like CompHEP, Grace, CalcHEP, MadGraph, Wizard, Sherpa etc.

Indeed, if we add to (155) the following gauge fixing terms:

$$L_{\text{GF}} = -\frac{1}{\xi} (\partial_\mu W_\mu^+ - \xi M_W w_g^+) (\partial_\nu W^{\mu-} - \xi M_W w_g^-) - \frac{1}{2\xi} (\partial_\mu Z^\mu - \xi M_Z z_g)^2, \quad (156)$$

the mixing terms are cancelled out. Then the quadratic part of the SM Lagrangian for W_μ^\pm and Z_μ fields which gives their propagators is

$$\begin{aligned} & -\frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + \frac{1}{2} M_Z^2 Z_\nu Z^\nu - \frac{1}{2\xi} (\partial_\mu Z^\mu)^2 \\ & -\frac{1}{2} W_{\mu\nu}^+ W^{-\mu\nu} + M_W^2 W_\nu^+ W^{-\nu} - \frac{1}{\xi} (\partial_\mu W^{+\mu}) (\partial_\nu W^{-\nu}), \end{aligned} \quad (157)$$

where terms with

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu, \quad W_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm$$

come from the kinetic part of the SM Lagrangian, as in the unitary case. Inverting the quadratic form (157), one gets the propagator of the massive gauge field in the so-called R_ξ gauge:

$$D_{\mu\nu}^\xi = \frac{-i}{k^2 - M_V^2} \left[g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 - \xi M_V^2} \right], \quad (158)$$

where M_V is M_W or M_Z and ξ is the gauge parameter.

The unitary gauge is restored by the formal limit $\xi \rightarrow \infty$. In the Landau gauge $\xi = 0$, we get the transverse structure $(g_{\mu\nu} - k_\mu k_\nu / k^2)$, while in the 't Hooft–Feynman gauge $\xi = 1$ the propagator contains only the part with the $g_{\mu\nu}$ tensor. However, one should stress that in both these gauges as well as in the generic R_ξ gauge one should take into account the appearance of Faddeev–Popov ghost fields. This is done with the help of the Faddeev–Popov method in the functional integral, which we have described already. Without going into details, the following ghost fields appear: c_W^\pm , c_Z and c_A , corresponding to the gauge fixing terms (156) and the gauge fixing $-(\partial_\mu A^\mu)^2 / 2\xi$ term for the photon field. In contrast to pure electrodynamics where the photon ghost fields do not interact and can be omitted, in the SM the photon ghost field c_A has non-trivial interactions with the ghost c_W^\pm and Goldstone w_g^\pm fields.

Propagators of all the ghost fields have the following form:

$$D^c = \frac{i}{p^2 - \xi M_V^2},$$

where M_V^2 is equal to M_Z^2 for c_Z , M_W^2 for c_W and 0 for c_A ghost fields. (The complete set of all Feynman rules for interaction vertices of Goldstone and ghost fields in the SM in the R_ξ gauge is rather long and can be found in the mentioned computer codes such as CompHEP.)

So, all the propagators for massive gauge, Goldstone and ghost fields have good ultraviolet behaviour, and therefore the SM is a renormalizable quantum field theory.

13 Phenomenology of the SM in lowest order

The Fermi constant G_F is measured with high precision from the muon lifetime:

$$G_F = 1.166\,378\,7(6) \times 10^{-5} \text{ GeV}^{-2}. \quad (159)$$

The decay is described in the SM by the Feynman diagram shown in Fig. 10.

Since the muon mass $m_\mu \ll M_W$, one can neglect the W-boson momentum in the propagator, and one immediately gets the following relation:

$$\frac{g_2^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}. \quad (160)$$

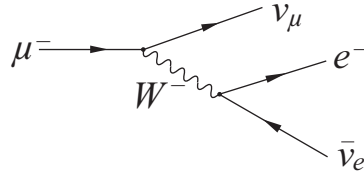


Fig. 10: $\mu \rightarrow e^- \bar{\nu}_e \nu_\mu$ decay diagram

As we have seen, the W-boson mass is obtained in the SM due to the Higgs mechanism and is proportional to the Higgs field vacuum expectation value v :

$$M_W^2 = \frac{1}{4} g_2^2 v^2. \quad (161)$$

From these two relations, we obtain

$$v = \frac{1}{\sqrt{\sqrt{2}G_F}} = 246.22 \text{ GeV}. \quad (162)$$

The Higgs field expectation value v is determined by the Fermi constant G_F introduced long before the Higgs mechanism appeared. At this point one can see the power of the gauge invariance principle; g_2 is the same gauge coupling in the relations (160) and (161).

Now from (160) using the relation (140), $g_1 c_W = g_2 s_W = e$ and keeping in mind $M_W = M_Z c_W$, one gets

$$M_W^2 \left(1 - \frac{M_W^2}{M_Z^2}\right) = \frac{\pi \alpha_{\text{em}}}{\sqrt{2}G_F} \equiv A_0^2, \quad (163)$$

where $\alpha_{\text{em}} = e^2/4\pi$ is the usual electromagnetic fine structure constant. The low-energy

$$\alpha_{\text{em}} = (137.035\,999\,074(44))^{-1}$$

follows mainly from the electron anomalous magnetic measurements. One gets A_0 very precisely from low-energy experimental results:

$$A_0 = 37.2804 \text{ GeV}. \quad (164)$$

On the other hand, one gets A_0 from measured values of the masses of W and Z bosons:

$$\begin{aligned} M_W &= 80.385 \pm 0.015 \text{ GeV}, \\ M_Z &= 91.1876 \pm 0.0021 \text{ GeV} \end{aligned} \quad (165)$$

by substituting (165) into the left-hand side of (163):

$$A_0 = 37.95 \text{ GeV}. \quad (166)$$

The values (163) and (166) are rather close. The difference is about 1.5%. If one takes into account properly the higher-order corrections to the relation (163), the agreement between the two numbers will be improved further.

CC and NC interactions of the SM fermions, as has been shown in the previous section, have the following structure (see (111)):

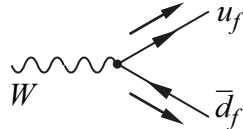
$$\begin{aligned} L_{\text{CC}} &= \frac{g_2}{2\sqrt{2}} \sum_{ij} V_{ij} \bar{u}_i \gamma_\mu (1 - \gamma_5) d_j = \frac{e}{2\sqrt{2}s_W} \sum_{ij} V_{ij} \bar{u}_i \gamma_\mu (1 - \gamma_5) d_j, \\ L_{\text{NC}} &= e \sum_f Q_f \bar{f} \gamma_\mu f A^\mu + \frac{e}{4s_W c_W} \sum_f \bar{f} \gamma_\mu (v_f - a_f \gamma_5) f Z^\mu, \end{aligned} \quad (167)$$

where V_{ij} are the CKM matrix elements, $i, j = 1, 2, 3$ the number of the SM fermion generations and

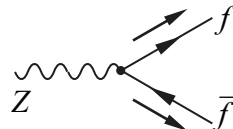
$$\begin{aligned} v_{u_i} &= 1 - \frac{8}{3}s_W^2, & a_{u_i} &= 1; & v_{d_i} &= -1 + \frac{4}{3}s_W^2, & a_{d_i} &= -1; \\ v_\ell &= -1 + 4s_W^2, & a_\ell &= -1; & v_\nu &= 1, & a_\nu &= 1 \end{aligned}$$

are the vector and axial-vector coupling constants.

The Feynman rules following from (166) allow us to get tree level formulas for the W- and Z-boson partial decay widths, as shown below at tree level:



$$\Gamma(W \rightarrow u_f \bar{d}_f) = |V_{ij}|^2 N_c \frac{\alpha}{12s_W^2} M_W, \quad (168)$$



$$\Gamma(Z \rightarrow f \bar{f}) = N_c \frac{\alpha M_Z}{12 \sin^2(2\theta_W)} [v_f^2 + a_f^2], \quad (169)$$

where the number of colours $N_c = 3$ for quarks and $N_c = 1$ for leptons.

The total W- and Z-boson widths are obtained by summing up all the partial widths (168) and (169). Since CCs for all SM fermions have the same V-A structure, one can very easily obtain branching fractions for W-decay modes:

$$\begin{aligned} \sum_q \text{Br}(W \rightarrow q\bar{q}) &= 2N_c \cdot \frac{1}{9} = \frac{2}{3}, \\ \sum_\ell \text{Br}(W \rightarrow \ell\nu) &= 3 \cdot \frac{1}{9} = \frac{1}{3}. \end{aligned} \quad (170)$$

The measured $\text{Br}(W \rightarrow \ell\nu) = (10.80 \pm 0.09)\%$ is in reasonable agreement with the simple tree level result

$$\text{Br}(W \rightarrow \ell\nu) = \frac{1}{9} = (11.11)\%. \quad (171)$$

QCD corrections to the branching ratio $\text{Br}(W \rightarrow q\bar{q})$ improve the agreement.

The decay width of the Z boson to neutrinos, the invisible decay mode, allows us to measure the number of light ($m_\nu < M_Z/2$) neutrinos by comparing

$$\Gamma_{\text{inv}}^Z = \Gamma_{\text{tot}}^Z - \Gamma_{\text{had}}^Z - \Gamma_{\ell^+\ell^-}^Z \quad (172)$$

with the tree level formula obtained from (169):

$$\Gamma_{\text{inv}}^Z = \Gamma_{\nu\bar{\nu}}^Z = N_\nu \cdot \frac{\alpha M_Z}{12 \sin^2(2\theta_W)} (1 + 1). \quad (173)$$

Experimentally, Γ_{tot} is measured from the shape of the Z-boson resonance according to the well-known Breit-Wigner formula

$$\Gamma_{\text{tot}}^Z = 2.4952 \pm 0.0023 \text{ GeV}.$$

Decay widths to hadrons and charged leptons are measured directly in e^+e^- collisions (LEP1) to be

$$\Gamma_{\text{had}}^Z = 1744.4 \pm 2.0 \text{ MeV},$$

$$\Gamma_{\ell^+\ell^-}^Z = 83.984 \pm 0.086 \text{ MeV}.$$

As a result, Γ_{inv}^Z obtained from (172) is

$$\Gamma_{\text{inv}}^Z = 0.4990 \pm 0.0015 \text{ GeV}.$$

This gives for N_ν

$$N_\nu = 2.984 \pm 0.008,$$

which is close to the number of known neutrinos. The test is an important confirmation of three generations of fermions assumed in the SM and observed in experiments as shown in Fig. 11.

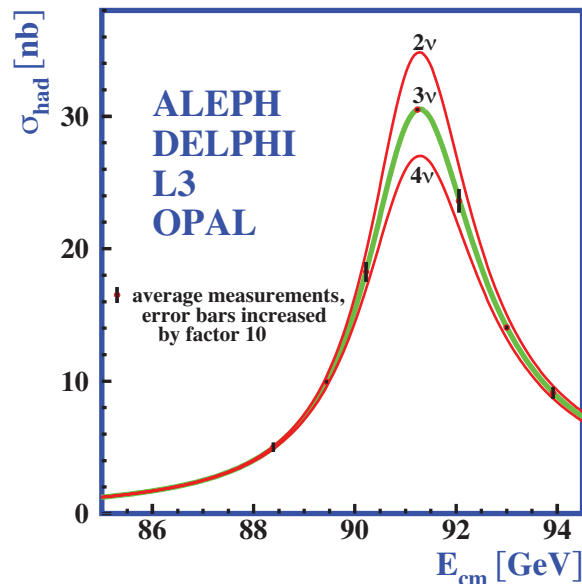


Fig. 11: N_ν from LEP measurements

One can make this test by looking at the ratio $\Gamma_{\text{inv}}^Z/\Gamma_{e^+e^-}^Z$. In the SM, the ratio follows from (169) and (173):

$$\frac{\Gamma_{\text{inv}}^Z}{\Gamma_{e^+e^-}^Z} = \frac{2N_\nu}{1 + (1 - 4s_W^2)^2}. \quad (174)$$

The measured value (5.942 ± 0.016) is in agreement with 5.970 coming from the formula (174) for $N_\nu = 3$ and $s_W^2 = 0.2324$.

An important part of information about the EW interactions and couplings of the SM fermions comes from e^+e^- annihilation to fermion–antifermion pairs. The differential cross-section computed from the diagrams shown in Fig. 12 has a well-known form neglecting fermion masses compared to the

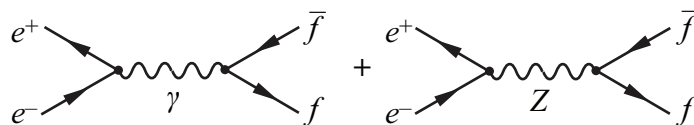


Fig. 12: e^+e^- diagram

centre of mass energy \sqrt{s} :

$$\begin{aligned} \frac{d\sigma}{d\cos\theta} &= \frac{2\pi\alpha^2}{4s} N_C \{ (1 + \cos^2\theta) \cdot \\ &\cdot [Q_f^2 - 2\chi_1 v_e v_f Q_f + \chi_2 (a_e^2 + v_e^2)(a_f^2 + v_f^2)] \\ &+ 2\cos\theta [-2\chi_1 a_e a_f Q_f + 4\chi_2 a_e a_f v_e v_f] \}, \end{aligned} \quad (175)$$

where

$$\chi_1 = \frac{1}{16s_W^2 c_W^2} \frac{s(s - M_Z^2)}{(s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2},$$

$$\chi_2 = \frac{1}{256s_W^2 c_W^2} \frac{s^2}{(s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2}.$$

The cross-section obtained from the differential form (175) is in good agreement with the experimental data, as shown in Fig. 13. In the region far below the Z-boson pole, one can neglect the Z-boson exchange

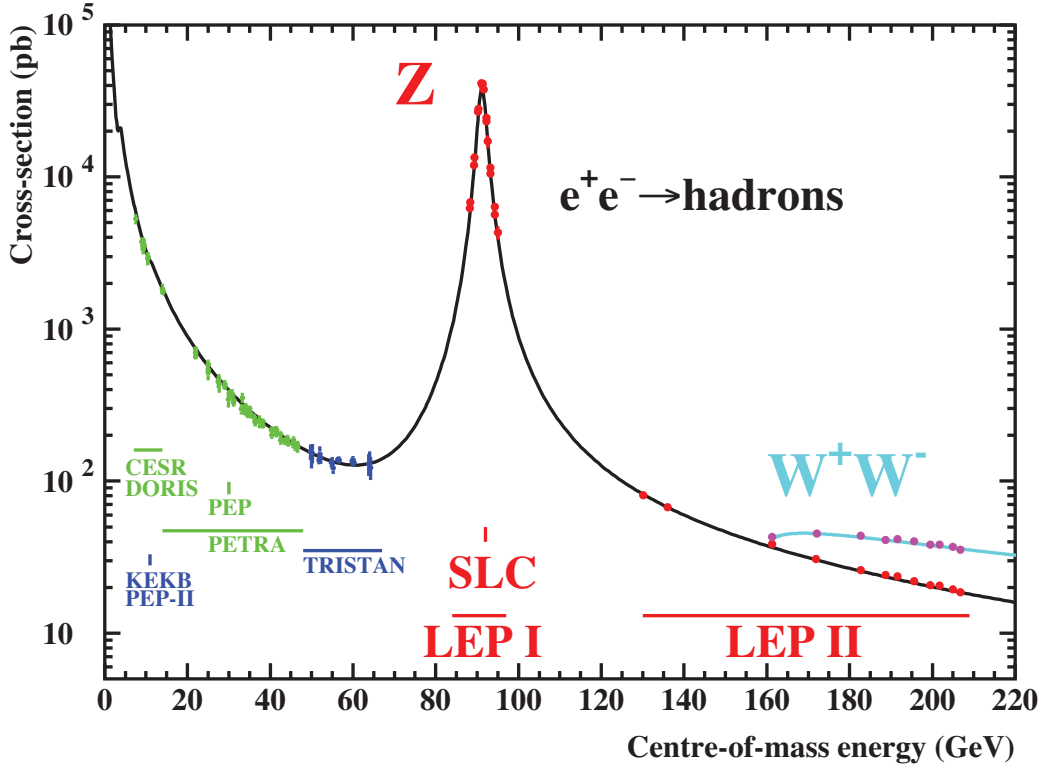


Fig. 13: The cross section of hadron production in e^+e^- collisions showing a good agreement between the SM computation and various experiments in the energy range up to 220 GeV including the contribution of the Z-boson resonance.

diagram and restore the well-known QED formula

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s} Q_f^2 N_C (1 + \cos^2\theta), \quad \sigma = \frac{4\pi\alpha^2}{3} Q_f^2 N_C. \quad (176)$$

From the formula (175), one can get a number of asymmetries, which have been measured, in particular, at LEP1 and SLC. In the region close to the Z-boson pole the photon exchange part is small and can be neglected. Then the forward-backward asymmetry is

$$A_{FB} \equiv \frac{N_F - N_B}{N_F + N_B},$$

where

$$N_F = \int_0^1 d(\cos\theta) \frac{d\sigma}{d\cos\theta}, \quad N_B = \int_{-1}^0 d(\cos\theta) \frac{d\sigma}{d\cos\theta}.$$

Simple integration of (175) gives the following result:

$$A_{FB} = \frac{3}{2} A_e \cdot A_f, \quad A_{e,f} = \frac{2a_{e,f} v_{e,f}}{a_{e,f}^2 + v_{e,f}^2}.$$

Measurements of decay widths being proportional to $(a_f^2 + v_f^2)$ and asymmetries for different fermions f allow us to extract the coefficients a_f and v_f . Then one can get a precise value for the Weinberg mixing angle from the relation involving the lepton couplings:

$$\sin^2 \theta_{\text{eff}}^{\text{lept}} = \frac{1}{4} \left(1 - \frac{v_l}{a_l} \right).$$

Results of the measurements are shown in Fig.14 as obtained by the EW working group [16].

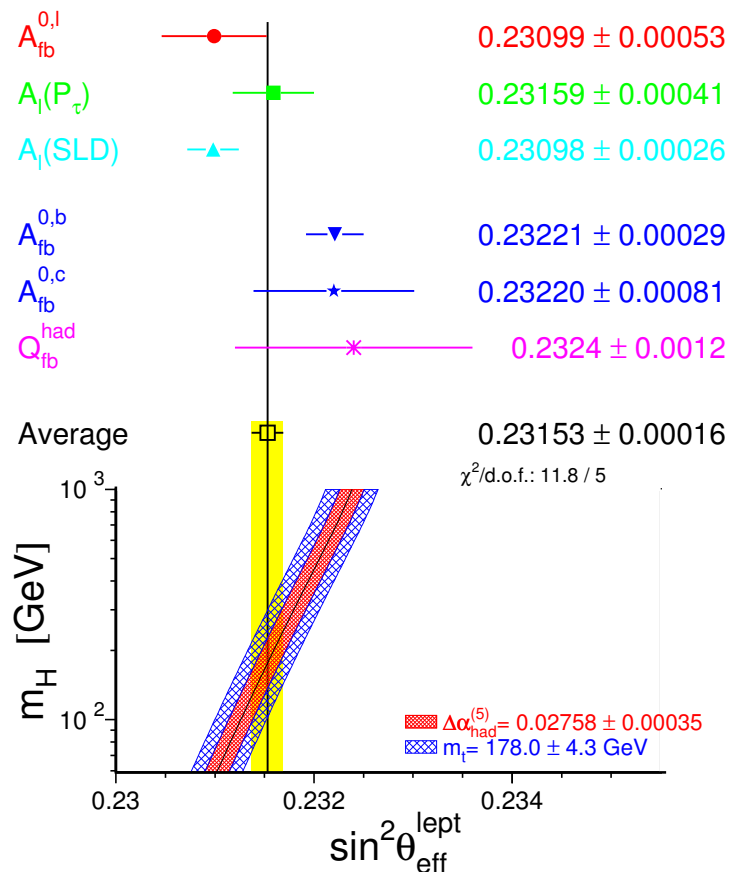


Fig. 14: Effective electroweak mixing angle from various observables [16]

In the SM there are no $1 \rightarrow 2$ decays of fermions to the real Z boson due to absence of FCNCs (flavour-changing neutral currents). The top quark is heavy enough to decay to a W boson, as shown in Fig. 15. The decay mode to the b-quark is dominant in the top decays due to the CKM mixing matrix

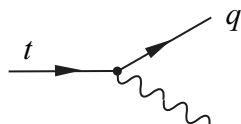


Fig. 15: Top-quark decay

structure, where

$$V_{tb} \sim 1 \gg V_{ts}, V_{td}.$$

A direct tree level computation leads to

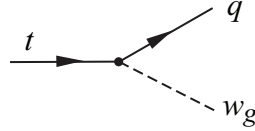
$$\Gamma_{\text{top}} = \frac{G_F M_t^3}{8\pi\sqrt{2}} \left(1 - \frac{M_W^2}{M_t^2}\right)^2 \left(1 + 2\frac{M_W^2}{M_t^2}\right), \quad (177)$$

where one neglects the b-quark mass:

$$\Gamma(t \rightarrow bW)_{\text{LO}} \simeq 1.53 \text{ GeV}, \quad \Gamma(t \rightarrow bW)_{\text{corr}} = 1.42 \text{ GeV}.$$

The top-quark lifetime $\tau_{\text{top}} = 1/\Gamma_{\text{top}}$ is about $5 \cdot 10^{-25}$ s, which is much smaller than the typical time of strong bound state formation $\tau_{\text{QCD}} \sim 1/\Lambda_{\text{QCD}} \sim 3 \cdot 10^{-24}$ s. The top quark decays before hadronization. Therefore, there are no hadrons containing the top quark.

Since the top-quark mass is larger than the W-boson and b-quark masses, one can use the EW equivalence theorem to get the leading top width up to the term m_W^2/m_t^2 . According to the EW equivalence theorem, amplitudes with external W and Z bosons are dominated by the longitudinal polarization of the bosons ($e_L^{W,Z} \sim p^0/M_{W,Z}$). But the longitudinal component in the SM appears by ‘eating’ the Goldstone bosons w_g, z_g . So, one can compute simply the diagram

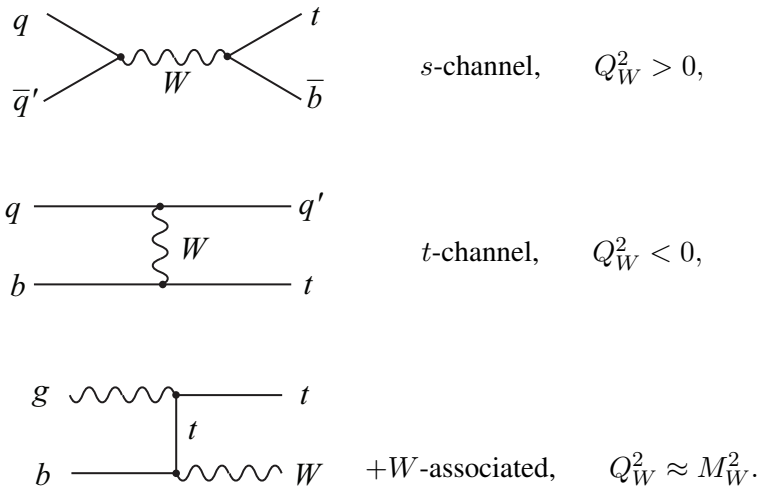


with the Yukawa vertex $M_t/(v\sqrt{2})$. Then one immediately obtains for the top width the following formula:

$$\Gamma = \frac{2}{32\pi} \left(\frac{M_t}{v}\right)^2 \cdot M_t = \frac{G_F M_t^3}{8\pi\sqrt{2}},$$

which is exactly equal to the first term in (177), as expected.

The EW single top quark production is another confirmation of the EW fermion structure of the SM. There are three mechanisms of single top production at hadron colliders differing by the typical virtuality (Q_W^2) of the W boson involved:



quark t -channel and s -channel production mechanisms have been observed at the Tevatron, while t -channel and W -associated production was observed at the LHC. There are a number of important QCD

next-to-leading-order (NLO) and next-to-next-to-leading-order (NNLO) corrections which are needed to be taken into account in order to get SM predictions with needed accuracy to be compared to experimental results. Up to now a good agreement with SM computations was observed.

A well-known example demonstrating correctness of the Yang–Mills interactions of the gauge bosons is the gauge boson pair production. Triple gauge boson vertices $WW\gamma$ and WWZ have been tested at LEP2 ($e^+e^- \rightarrow W^+W^-$) and at the Tevatron ($q\bar{q} \rightarrow W^+W^-$, $q\bar{q}' \rightarrow W\gamma$, $q\bar{q}' \rightarrow WZ$). The diagrams for the process $e^+e^- \rightarrow W^+W^-$ form the so-called CC3 set of diagrams, as shown in Fig. 16.



Fig. 16: $e^+e^- \rightarrow W^+W^-$

The triple vertex of the Yang–Mills interaction:

$$\Gamma_{m_1 m_2 m_3}^{WW\gamma/Z}(p_1 p_2 p_3) = g_{\gamma,Z} [(p_1 - p_2)_{m_3} g_{m_1 m_2} + (p_3 - p_1)_{m_2} g_{m_1 m_3} + (p_2 - p_3)_{m_1} g_{m_2 m_3}],$$

where $g_\gamma = e$, $g_Z = g_2 c_W = e \frac{c_W}{s_W}$, is confirmed perfectly experimentally, as shown in Fig. 17.

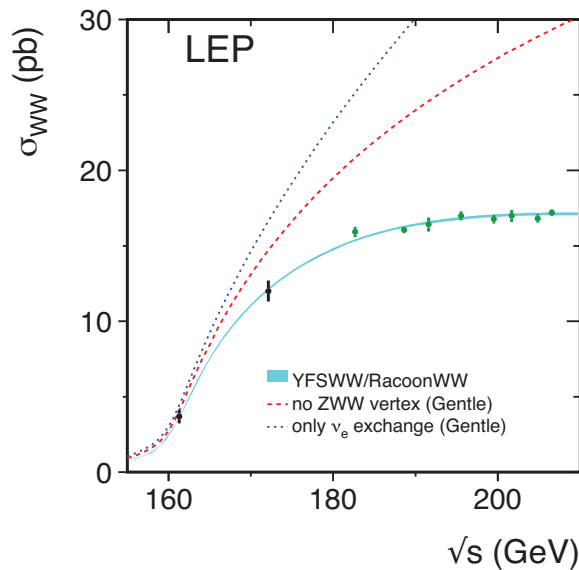


Fig. 17: Measurements of the W -pair production cross-section, compared to the theoretical predictions. For explanations, see [17].

The quartic gauge coupling $WW\gamma\gamma$ has been tested recently at the Tevatron and the LHC in W -pair production in association with two protons or a proton and an antiproton. The quartic couplings $WW\gamma Z$, $WWZZ$ have not been tested yet. This is a challenging task for the LHC and will require a high-luminosity regime at a linear collider.

14 The electroweak SM beyond the leading order

All the above examples confirming the structure of the SM interactions are leading order processes. However, in many cases a high accuracy of experimental measurements requires the SM computations

beyond the leading order. In the SM, being a quantum field theory, computations of higher-order corrections face divergences of ultraviolet and infrared/collinear nature. In general, the ultraviolet (hard) divergences are treated with the help of the renormalization procedure while the infrared/collinear (soft) divergences are cancelled out due to the Kinoshita–Lee–Nauenberg theorem in summing virtual and real contributions to squared matrix elements.




The renormalization procedure is the usual way to deal with the ultraviolet divergences. We describe briefly only the main ideas of the procedure. In the SM the dimensions of all the coupling constants are equal to zero. This property has important consequences making the theory renormalizable. In simple words, the renormalizability means that all the UV divergences may be incorporated into a redefinition of a few constants such as coupling constants, masses and field normalization constants. In renormalizable theories only a few diagrams are UV divergent.

As an example, let us consider QED.

The divergency index of a diagram depends only on the number of external legs, and for QED can be expressed in a well-known form

$$w = 4 - L_\gamma - 3/2L_e,$$

where L_γ is the number of external photon lines and L_e is the number of external electron lines.

So, there are only three types of divergent diagrams with two external photon lines, the photon self-energy , with two external electron lines, the electron self-energy  and, with one photon and two electron external lines, the electron–photon vertex .

It is convenient to consider the renormalization procedure in the functional integral approach already familiar to us. The generating functional integral in QED in the covariant gauge is given by

$$\begin{aligned} Z[J, \eta, \bar{\eta}] &= \int D(\bar{\Psi}\Psi A) \exp \left(i \int d^4x \bar{\Psi}(i\rlap{\not{D}} - m)\Psi + ieA + J_\mu A^\mu \right. \\ &\quad \left. + \bar{\eta}\Psi + \bar{\Psi}\eta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\xi} \int d^4x (\partial_\mu A^\mu)^2 \right). \end{aligned} \quad (178)$$

The photon propagator is obtained from (178) by taking two functional derivatives on J and setting the sources J , η and $\bar{\eta}$ to zero:

$$\begin{aligned} iD_{\alpha\beta}(x_1, x_2) &= \int D(\bar{\Psi}\Psi A) A_\mu(x_1) A_\mu(x_2) \\ &\quad \cdot \exp \left(i \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\rlap{\not{D}} - m)\Psi + e\bar{\Psi}A\Psi \right] \right). \end{aligned} \quad (179)$$



The Dyson–Schwinger equation for the photon propagator is obtained as a consequence of the invariance of the measure of the functional integral with respect to the shift $A_\mu(x) \rightarrow A_\mu(x) + \varepsilon_\mu(x)$.

The equation for the inverse propagator takes, after Fourier transformation, the form

$$D_{\alpha\beta}^{-1}(k) = (D_0)_{\alpha\beta}^{-1} + \Pi_{\alpha\beta} \quad (180)$$

or, graphically,

$$\left(\text{wavy line with loop} \right)^{-1} = \left(\text{wavy line} \right)^{-1} + \left(\text{circle with loop} \right),$$

where  denotes the dressed fermion propagator and  is the truncated one-particle irreducible vertex function $\Gamma_\mu(p_1, p_2, k)$.

At one-loop level the function $\Pi_{\alpha\beta}(k)$ is given by the following Feynman integral:

$$(-ie)^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[\frac{\not{p} + m}{p^2 - m^2 + i0} \gamma_\alpha \frac{(\not{p} - \not{k}) + m}{(p - k)^2 - m^2 + i0} \gamma_\beta \right]. \quad (181)$$

The integral (181) is quadratically divergent from formal power counting. In order to deal with divergent integrals, we need to introduce some regularization. We use the dimensional regularization

$$d^4p \rightarrow d^D p (\mu^2)^{2-D/2}.$$

One can see that (181) gives zero, being convoluted with the external moments k_α or k_β . Indeed,

$$k_\alpha \gamma^\alpha = (\not{p}) - (\not{p} - \not{k}) = (\not{p} - m) - [(\not{p} - \not{k}) - m]. \quad (182)$$

If we substitute (182) into (181), we get

$$(\mu^{2-D/2})(-ie)^2 \int \frac{d^D p}{(2\pi)^D} \text{Tr} \left[\left(\frac{\not{p} + m}{p^2 - m^2 + i0} - \frac{(\not{p} - \not{k}) + m}{(p - k)^2 - m^2 + i0} \right) \gamma_\beta \right] = 0. \quad (183)$$

In fact, this result is valid to all perturbation orders due to the Ward identity

$$k^\mu \Gamma_\mu(p_1, p_2, k) = S^{-1}(p_1) - S^{-1}(p_2). \quad (184)$$

The identity (183) can be easily derived from $U(1)$ gauge invariance of (178).

The property means that $\Pi_{\alpha\beta}$ has the following form:

$$\Pi_{\alpha\beta}(k) = (g_{\alpha\beta} k^2 - k_\alpha k_\beta) \Pi(k^2). \quad (185)$$

Therefore, the dressed photon propagator can be written as

$$D_{\alpha\beta}(k) = -\frac{i}{k^2} \left[\frac{1}{1 + \Pi_\gamma(k^2, \varepsilon, \mu^2)} \left(g_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) + \xi \frac{k_\alpha k_\beta}{k^2} \right]. \quad (186)$$

Of course, in the case where $\Pi_\gamma = 0$, we obtain the free photon propagator.

The factor $(1 + \Pi_\gamma(k^2, \varepsilon, \mu^2))^{-1}$, being taken at zero momentum, should be removed for the correct normalization of the kinetic term. It can be done by rescaling the field $A_\mu(x)$ in the following way:

$$A_\mu(x) \rightarrow \frac{1}{\sqrt{Z_3}} A_\mu, \text{ where } Z_3^{(a)} = (1 + \Pi_\gamma(0, \varepsilon))^{-1}. \quad (187)$$

Direct computation of (181) with well-known Feynman techniques gives

$$\begin{aligned} \Pi_\gamma(k^2, \varepsilon, \mu^2) &= \frac{\alpha}{3\pi\varepsilon} + \Pi_{\text{finite}} \\ \text{and therefore } Z_3^{-1} &= 1 + \frac{\alpha}{3\pi\varepsilon}, \end{aligned} \quad (188)$$

where $\varepsilon = (4 - D)/2$.

Now let us consider the fermion propagator taking functional derivatives of (178) on the fermion sources $\bar{\eta}$ and η .

Now the expression for the dressed inverse fermion propagator is

$$S^{-1}(p) = S_0^{-1}(p) - \Sigma(p). \quad (189)$$

The formula (188) is also the Dyson–Schwinger equation for the dressed fermion propagator, graphically presented as

$$\left(\text{---} \bigcirc \text{---} \right)^{-1} = \left(\text{---} \bullet \text{---} \right)^{-1} - \frac{k_\mu \text{---} \bigcirc \text{---} \Gamma}{p_1 \text{---} p}$$

The equation involves the same truncated vertex function $\Gamma_\mu(p_1 p_2; k)$. In the second order of perturbation theory, $\Sigma^{(2)}(p)$ is as follows:

$$-i\Sigma^{(2)}(p) = (-ie)^2 \int \frac{d^D k}{(2\pi)^D} (\mu^2)^{2-D/2} \cdot \gamma^\mu D_{\mu\nu}(k) \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2} \gamma^\nu. \quad (190)$$

Direct computation gives the following answer:

$$\Sigma_2 = \frac{\alpha}{8\pi} (4m - \not{p}) \frac{2}{\varepsilon} + \Sigma_{\text{finite}}. \quad (191)$$

The generic structure of $\Sigma(p)$ is as follows:

$$\Sigma(p) = \not{p} f_1(p^2) - m f_2(p^2).$$

Due to (189), this means that the fermion propagator has the form

$$S(p) = \frac{1}{\not{p}(1 - f_1(p^2)) - m(1 - f_2(p^2))} = -\frac{1}{1 - f_1(p^2)} \frac{1}{\not{p} - m \frac{1 - f_2(p^2)}{1 - f_1(p^2)}}. \quad (192)$$

Close to physics mass, one should have

$$m_{\text{phys}} = m \frac{1 - f_2(m_{\text{phys}}^2)}{1 - f_1(m_{\text{phys}}^2)}.$$

So, the fermion propagator has the following form close to physics mass:

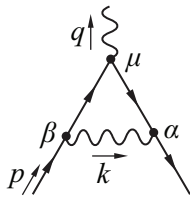
$$S(p) = \frac{Z_2(\varepsilon, \mu)}{\hat{p} - m_{\text{phys}}(\varepsilon, \mu)}, \quad (193)$$

where $Z_2 = (1 - f_1)^{-1}$, $m_{\text{phys}} = m \frac{Z_2}{Z_m}$ and $Z_m = (1 - f_2)^{-1}$. From the one-loop result (191), we obtain

$$Z_2^{-1} = 1 + \frac{\alpha}{4\pi\varepsilon} + O(\alpha), \quad (194)$$

$$Z_m^{-1} = 1 + \frac{\alpha}{\pi\varepsilon} + O(\alpha). \quad (195)$$

The remaining divergent QED diagram is the vertex function correction given by the integral



$$\begin{aligned} ie\Gamma_\mu^{(2)}(p, q) &= (-ie)^3 (\mu^2)^{2-D/2} \int \frac{d^D k}{(2\pi)^D} \\ &\cdot \gamma^\alpha(i) \frac{\not{p} - \not{q} - \not{k} + m}{(p-q-k)^2 - m^2 + i0} \\ &\cdot \gamma_\mu(i) \frac{\not{p} - \not{q} + m}{(p-q)^2 - m^2 + i0} \cdot \gamma^\beta \frac{-i}{k^2 + i0} g^{\alpha\beta} \end{aligned} \quad (196)$$

In order to compute the divergent part, one can simplify the problem and compute (196) in the limit $q \rightarrow 0$. The answer is

$$\Gamma_\mu^{(2)}(p, 0) = \gamma_\mu \left[\frac{\alpha}{4\pi\varepsilon} + O(\alpha) \right]. \quad (197)$$

Therefore, the vertex function including one-loop correction may be written in the form

$$-ie\Gamma_\mu = -ieZ_1\gamma_\mu, \quad (198)$$

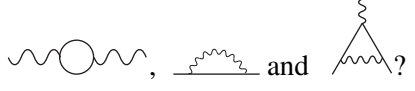


where

$$Z_1 = 1 - \frac{\alpha}{4\pi} \frac{1}{\varepsilon} + O(\alpha).$$

We can see from (194) and (198) that $Z_1 = Z_2$ including the one-loop part. The equality $Z_1 = Z_2$ takes place to all orders of perturbation theory due to the Ward identity

$$\Gamma_\mu(p, 0) = \partial_\mu S^{-1}(p), \quad (199)$$

as follows from (183) in the limit of the photon momentum $k \rightarrow 0$.

Why did we do all the above computations of divergent graphs ,  and ? Let us rewrite our initial (before renormalization) QED Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}^0 F^{0\mu\nu} + \bar{\Psi}_0(i\not{D}^0 - m_0)\Psi_0, \quad (200)$$

where $F_{\mu\nu}^0 = \partial_\mu A_\nu^0 - \partial_\nu A_\mu^0$, $D_\mu^0 = \partial_\mu - ie_0 A_\mu^0$ (we use the symbol (0) to stress that all the objects are not renormalized, or bare, as one usually says) in terms of physical fields and parameters labelled by the symbol ph and additional terms ΔL :

$$L = -\frac{1}{4}F_{\mu\nu}^{\text{ph}} F^{\text{ph}\mu\nu} + \bar{\Psi}_{\text{ph}}(i\not{D}_{\text{ph}} - m_{\text{ph}})\Psi_{\text{ph}} + \Delta L, \quad (201)$$

where $A_\mu^{\text{ph}} = Z_3^{-1/2} A_\mu^0$, $\Psi_{\text{ph}} = Z_2^{-1/2} \Psi$, $m_{\text{ph}} = (Z_2/Z_m)m_0$, $e_0 = Z_1 Z_2^{-1} Z_3^{-1/2} (\mu)^{D/2-2} e_{\text{ph}}$, $D_\mu^{\text{ph}} = \partial_\mu - ie_{\text{ph}} A_\mu^{\text{ph}}$ and

$$\begin{aligned} \Delta L = & -(Z_3 - 1) \frac{1}{4} F_{\mu\nu}^{\text{ph}} F^{\text{ph}\mu\nu} + (Z_2 - 1) \bar{\Psi}_{\text{ph}}(i\not{\partial})\Psi_{\text{ph}} \\ & + (Z_m - 1) m_{\text{ph}} \bar{\Psi}_{\text{ph}} \Psi_{\text{ph}} + (Z_1 - 1) e_{\text{ph}} \bar{\Psi}_{\text{ph}} (\not{A}_{\text{ph}}) \Psi_{\text{ph}}. \end{aligned} \quad (202)$$

The Lagrangian ΔL contains so-called counter-terms. In the leading order we computed all the coefficients in front of the counter-terms. Now when one computes some effect using the Lagrangian (201) all UV divergences are cancelled out order by order in perturbation theory by contributions of the counter-terms.

Let us look in more detail at the relation for the coupling constant

$$e_0 = Z_1 Z_2^{-1} Z_3^{-1/2} (\mu)^{D/2-2} e_{\text{ph}}(\mu),$$

where $(\mu)^{D/2-2}$ is the dimension of the charge. As we discussed above, $Z_1 = Z_2$ due to the Ward identity, and

$$e_0 = Z_3^{-1/2} (\mu)^{D/2-2} e_{\text{ph}}(\mu). \quad (203)$$

Note that $e_{\text{ph}}(\mu)$ is a function of the dimension regularization parameter μ , while e_0 does not depend on μ . From (203), one gets the following equality for $\alpha = \frac{e^2}{4\pi}$ ($D = 4 - 2\varepsilon$):

$$\alpha_0 = Z_3^{-1} (\mu^2)^{-\varepsilon} \alpha_{\text{ph}}(\mu). \quad (204)$$

Taking the derivative $\mu \frac{\partial}{\partial \mu}$ on both sides of (204), we get the following equation:

$$\mu \frac{\partial \alpha}{\partial \mu} = \frac{2\alpha^2}{3\pi} \equiv \beta(\alpha). \quad (205)$$

The equation (205) is a particular example of the renormalization group equation, which we do not discuss in this brief lecture course. The function on the right-hand side of (205) is called the β -function. So, at the one-loop level the β -function in QED is given by the following formula:

$$\beta(\alpha) = \frac{b_0}{\pi} \alpha^2, \quad b_0 = \frac{2}{3}. \quad (206)$$

One should stress that the coefficient $b_0 = \frac{2}{3}$ in QED is positive. The equation (205) can be easily solved:

$$\alpha(\mu) = \frac{\alpha(\mu_1)}{1 - \frac{\alpha(\mu_1)}{3\pi} \ln(\mu/\mu_1)^2}. \quad (207)$$

This is the running coupling constant. If one measures the constant α at some scale μ_1 , one gets values for the constant at other scales. The coupling constant $\alpha = 1/137$ being measured at very small scale (small momentum transfer or large distance) in Thompson scattering increases with the scale growing and becomes $\alpha(M_Z^2) \approx \frac{1}{129}$ at the Z mass. This fact was confirmed nicely in LEP experiments. Note that, in order to get $1/129$, one should take into account the contribution of all SM charged particles to the photon vacuum polarization function Π_γ . This means that the charged particle–antiparticle pairs screen the bare charge at small μ^2 or at large distances. The QED running coupling constant is illustrated in Fig.18 [18].

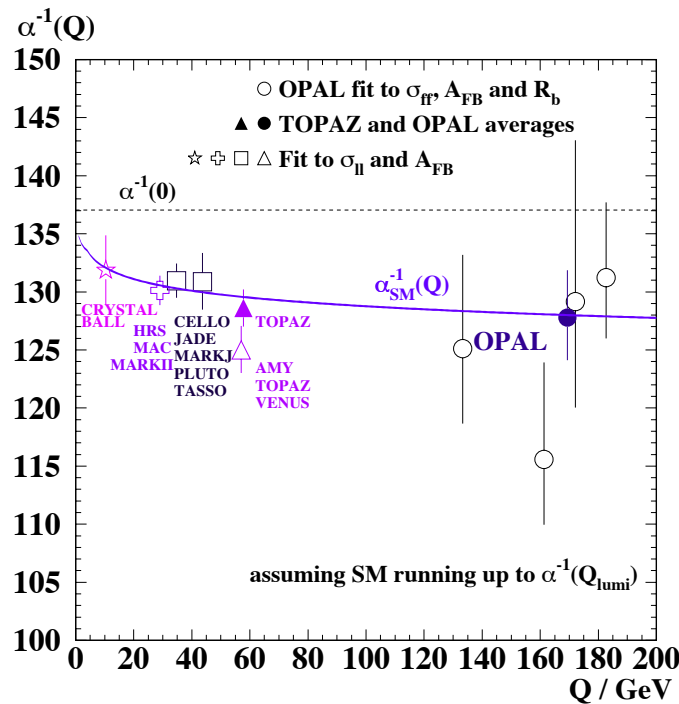


Fig. 18: Running electromagnetic coupling as a function of collision energy measured at various energies, in particular, at LEP by the OPAL collaboration [18].

Note that in QCD the β -function is negative, leading to an antiscreening effect; α_S becomes smaller with increasing of the momentum scale (momentum transfer) or decreasing distances. This is a famous asymptotic freedom property in QCD. In QED the situation is the opposite. If the scale μ increases to very large values the well-known Landau pole

$$\frac{b_0}{\pi} \ln(\mu/\mu_1)^2 = 1$$

is approached where the perturbation picture in QED breaks down.

One should note that in QED all terms in the four-dimensional Lagrangian (gauge-invariant operators) have dimension four. As a result, the coupling constant in QED is dimensionless. This is crucial to have renormalizable theory, where all UV divergences are cancelled to all orders with the help of a finite number of counter-terms. This is also the case in the SM. All terms of the SM Lagrangian have dimension four and all the coupling constants are dimensionless. So, the SM is a renormalizable theory in the same manner as QED.

Naively, one may think that the EW higher-order corrections are not that important. The perturbation theory expansion parameters α/π with $\alpha_{\text{em}} \sim 1/129$ and $\alpha_{\text{weak}} \sim 1/30$ are very small. However, the experimental accuracies are so high in various cases that even one-loop EW corrections might not be sufficient. Indeed, selected lists of measured parameters by LEP1, LEP2, SLD and Tevatron are given below:

| | | | | | | |
|------------------|---|---------|---|--------|-----|--------|
| M_Z | = | 91.1875 | ± | 0.0021 | GeV | 0.002% |
| Γ_Z | = | 2.4952 | ± | 0.0023 | GeV | 0.09% |
| M_W | = | 80.385 | ± | 0.015 | GeV | 0.02% |
| M_{top} | = | 173.2 | ± | 0.9 | GeV | 0.52% |

The most important higher-order corrections come from resummation of the large logarithms, $\log \frac{M_t^2}{m_e^2} \approx 24.2$, as we have seen with running α ($1/137 \rightarrow 1/129$). The second class of large corrections comes from contributions of the order of M_{top}^2/M_W^2 , which originate from EW Goldstone boson (or longitudinal W/Z polarization state) couplings to the quarks of the third generation. The later corrections lead to the shifts in W- and Z-boson masses coming from the diagram in Fig. 19.

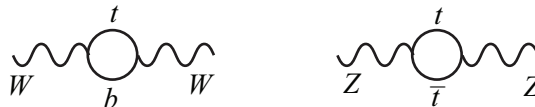


Fig. 19: Loop corrections

Loop corrections lead to the fact that SM parameters, coupling constants and masses, are running parameters, and they are non-trivial functions of each other. A famous example is given in Fig.20 [17] showing the dependence of the W-boson mass as a function of the top-quark mass at different values of the Higgs boson mass.

One should recall that the top-quark mass has been determined indirectly from the analysis of loop corrections, it being

$$m_t = 178 \pm 8 \begin{matrix} +17 \\ -20 \end{matrix} \text{ GeV},$$

which is remarkably close to today's precise measured value 173.2 ± 0.9 GeV.

The low Higgs mass range was preferred by a similar analysis of the Tevatron and LEP data, as one can clearly see in Fig.20.

A summary of comparisons of the EW precision measurements at LEP1, LEP2, SLD and the Tevatron and a global parameter fit is given in the well-known plot shown in Fig.21 [16].

The only one discrepancy on the level of 3σ is observed for $b\bar{b}$ -pair forward-backward asymmetry.

15 Concluding remarks

1. The SM is a renormalizable anomaly-free gauge quantum field theory with spontaneously broken EW symmetry. Remarkable agreement with many experimental measurements is observed.

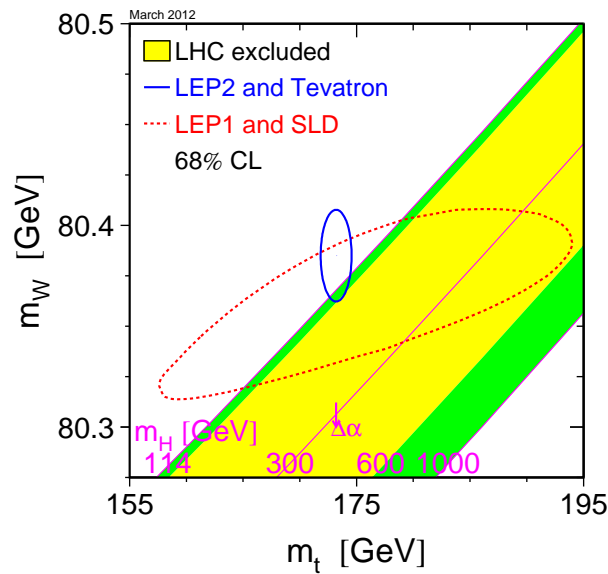


Fig. 20: W boson mass as a function of the top quark mass at various masses of the Higgs boson (see details in [17]).

2. All SM leptons, quarks, gauge bosons and, very probably, the Higgs boson have been discovered.
3. The SM predicts the structure of all interactions: fermion-gauge, gauge self-interactions, Higgs-gauge, Higgs-fermion and Higgs self-interactions. As a result, the SM allows us to compute various cross-sections, distributions and decay rates taking into account higher-order corrections. However, not yet all of the interactions were tested experimentally. Parameters of the theory are not predicted by the theory itself but extracted from the measurements.
4. The EW SM has 17 parameters and QCD has one more parameter fixed from experiments:
 - the gauge-Higgs sector contains four parameters g_1, g_2, μ^2, λ or in terms of best measured $\alpha_{\text{em}}, G_F, M_Z, M_h$;
 - six quark masses, three lepton masses;
 - three mixing angles and one phase of the CKM matrix (more parameters come from the neutrino mixing matrix, which we do not consider here);
 - the QCD coupling constant α_s .
5. There are facts which cannot be explained in the SM:
 - fermions have very much different masses ($M_{\text{top}} = 173 \text{ GeV}, M_e = 0.5 \text{ MeV}$) coming from the same mechanism;
 - dark matter exists in the Universe, and there are no dark-matter candidates in the SM;
 - the CKM phase as a source of CP violation in the SM is too small to explain particle-antiparticle asymmetry in the Universe;
 - neutrino masses, mixing and oscillations cannot be understood in the framework of the SM EW symmetry breaking mechanism;
 - there is some tension in explaining the muon anomalous magnetic moment.
6. As is well known, the simplest Higgs mechanism in the SM is not stable with respect to quantum corrections (naturalness problem). In the SM there is no symmetry which protects a strong

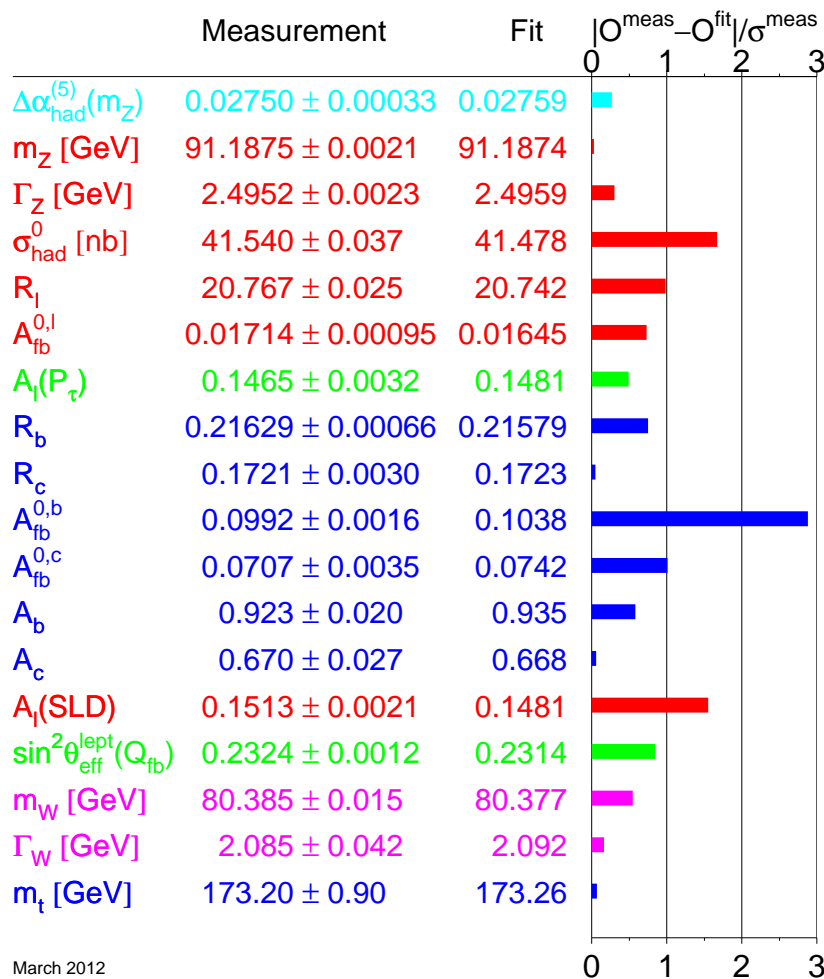


Fig. 21: Global fit of the EW precision measurements at LEP1, LEP2, SLD and the Tevatron by the SM computations including loop corrections [16] (the latest update version of the plot <http://lepewwg.web.cern.ch/LEPEWWG/plots/winter2012/>)

(quadratic) dependence of the Higgs mass on a possible new scale. Something is needed in addition to the SM to stabilize the mass parameter.

7. In addition, the SM does not give answers to many questions, such as:
- What is a generation? Why are there only three generations?
 - How are quarks and leptons related to each other?; what is the nature of the quark–lepton analogy?
 - What is responsible for gauge symmetries, why are charges quantized? Are there additional gauge symmetries?

- What is responsible for the formation of the Higgs potential?
- To which accuracy is the CPT (charge, parity, and time) symmetry exact?
- Why is gravity so weak compared to other interactions?

In our lecture we focused mainly on the EW part of the SM and aspects of the field theory needed clarifying, and we did not discuss QCD physics, Higgs boson physics, neutrino physics, flavour physics, problems of the SM models leading to BSM (beyond the Standard Model) scenarios and sequences for cosmology. These are the subjects of the lectures by Z. Trócsányi, J. Ellis, B. Gavelo, Z. Ligetti, C. Csaki and D. Gorbunov at the 2013 European School.

Acknowledgements

I would like to thank the organizers and the participants of the School for creating a very warm and productive atmosphere. Many thanks to the organizers for their help and support. The work was partially supported by RFBR grant 12-02-93108 and NSh grant 3042.2014.2.

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