# LANDAU DAMPING IN THE TRANSVERSE PLANE

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*Abstract*

In these proceedings we sketch a general theory for Landau damping in the transverse plane, in the particular case of beam-coupling impedance with linear (octupolar) detuning as the source of tune spread. Using a Hamiltonian formalism and perturbation theory, we will go beyond the traditional stability diagram approach to obtain the general non-linear determinant equation that leads to modes determination. Limiting cases are studied, and preliminary results of the full formalism are obtained in an attempt to generalize the concept of stability diagram.

## INTRODUCTION

In particle accelerators and storage rings, the beam can in principle be affected by various mechanisms that can lead to collective instabilities. Nevertheless, it typically remains stable thanks to a natural stabilization mechanism originating from the spread of the particles tune [1]. This phenomenon, first observed in plasmas, is called Landau damping [2].

When a beam is under the effect of a coherent, possibly unstable, mode, Landau damping can be understood intuitively as the de-synchronization of the beam individual particles from the collective motion, due to the fact that their oscillation frequency (or tune) gets modified, via the spread, as the amplitude of the unstable mode grows. There are simple ways to mathematically theorize Landau damping (see e.g. in Ref. [1]). For instance, in the transverse plane and with a dipolar beam-coupling impedance as the source of the coherent instability, one can simply use Hill equation, with a collective force depending on the beam average position on the right hand side of the equation, and integrate the solution over a continuous distribution of betatron frequencies. Such a formalism can be very handy to understand quickly the physics of the phenomenon, but lacks generality, as it is in particular not able to handle the case when the spread in frequency is in the same plane as the collective excitation, unless rather complicated developments are made on top of the theory [3].

Here we will use phase space distribution functions, Vlasov equation [4] and linear perturbation theory, to compute coherent modes originating from a beam-coupling dipolar impedance, similarly to what is done in Chao's book [1], but adding as additional ingredient a tune spread in the form of a linear, octupolar detuning. We will derive the complete formalism and get an extension of Sacherer equation, obtained first by Chin in 1985 [5]. The equation will be then transformed into a determinant equation, that can be reduced to the usual stability diagram theory as a limiting case. Finally, an indirect analysis of the general equation is performed as an application of the formalism, showing that the stability diagram theory can be potentially recovered also in the general case.

The conventions and notations are identical to those in Ref. [6], which were inspired from Chao's book [1].

## SACHERER EQUATION WITH LINEAR AMPLITUDE DETUNING

The system of beam particles is governed by an Hamiltonian $H$ split in two parts: the unperturbed Hamiltonian $H_0$ and a first order perturbation $\Delta H$

$$H = H_0 + \Delta H. \tag{1}$$

The phase space density is represented by a distribution function $\psi$, also separated into an stationary part $\psi_0$ (governed by $H_0$) and a perturbation $\Delta\psi$:

$$\psi = \psi_0 + \Delta\psi. \tag{2}$$

For a beam of particles, using the coordinates $(x, x' = \frac{dx}{ds}, y, y' = \frac{dy}{ds}, z, \delta)$ – with $s$ the longitudinal coordinate along the orbit and $\delta$ the relative deviation of the longitudinal momentum from that of the synchronous particle, we consider a lattice without coupling and use the smooth approximation, including the effect of chromaticities $(Q'_x, Q'_y)$ and of linear (octupolar) amplitude detuning:

$$\begin{aligned} H_0 = {}& \omega_0 \left( Q_{x0} + Q'_x\delta + \frac{a_{xx}}{2} J_x + \frac{a_{xy}}{2} J_y \right) J_x \\ &+ \omega_0 \left( Q_{y0} + Q'_y\delta + \frac{a_{xy}}{2} J_x + \frac{a_{yy}}{2} J_y \right) J_y - \omega_s J_z, \end{aligned} \tag{3}$$

with $Q_{x0}$ and $Q_{y0}$ the unperturbed transverse tunes, $\omega_0$ the angular revolution frequency, $\omega_s = Q_s\omega_0$ the angular synchrotron frequency, and the actions $(J_x, J_y, J_z)$ defined by

$$J_x = \frac{1}{2}\left( \frac{Q_{x0}}{R} x^2 + \frac{R}{Q_{x0}} x'^2 \right), \tag{4}$$

$$J_y = \frac{1}{2}\left( \frac{Q_{y0}}{R} y^2 + \frac{R}{Q_{y0}} y'^2 \right), \tag{5}$$

$$J_z = \frac{1}{2}\left( \frac{\omega_s}{v\eta} z^2 + \frac{v\eta}{\omega_s} \delta^2 \right), \tag{6}$$

with $R$ the machine physical radius, $v$ the beam velocity and $\eta = \alpha_p - \frac{1}{\gamma^2}$ the slippage factor. The corresponding angle variables are given by

$$\theta_x = \operatorname{atan}\left( \frac{Rx'}{Q_{x0}x} \right), \tag{7}$$

$$\theta_y = \operatorname{atan}\left( \frac{Ry'}{Q_{y0}y} \right), \tag{8}$$

$$\phi = \operatorname{atan}\left( \frac{v\eta\delta}{\omega_s z} \right). \tag{9}$$

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In the case of octupoles distributed all around the ring, the amplitude detuning coefficients are given by [7]

$$
\begin{aligned}
a_{xx} &= \frac{3}{8\pi}\int_0^{2\pi R} \mathrm{d}s\beta_x^2(s)\frac{O_3(s)}{\frac{p_0}{e}},\\
a_{yy} &= \frac{3}{8\pi}\int_0^{2\pi R} \mathrm{d}s\beta_y^2(s)\frac{O_3(s)}{\frac{p_0}{e}},\\
a_{xy} &= -\frac{3}{4\pi}\int_0^{2\pi R} \mathrm{d}s\beta_x(s)\beta_y(s)\frac{O_3(s)}{\frac{p_0}{e}},
\end{aligned}
\tag{10}
$$

with $(\beta_x, \beta_y)$ the beta functions, $e$ the elementary charge, $O_3 \equiv \frac{1}{6}\frac{\partial^3 B_y}{\partial x^3}$ the octupolar strength in T.m$^{-3}$ and $p_0$ the longitudinal momentum of the synchronous particle.

The stationary distribution, solution of Vlasov equation [4] for the unperturbed Hamiltonian, is in general a function of the invariants of motion. Since the Hamiltonian does not depend on the angles $\theta_x$ and $\theta_y$, $J_x$ and $J_y$ are invariants of motion. On the other hand, $H_0$ depends on $\phi$ through the chromatic term, but this dependency being much weaker than the main longitudinal motion (given by the term $-\omega_s J_z$), the standard approximation is to neglect it [1, chap. 6], which is equivalent to the assumption

$$
\frac{\partial H_0}{\partial \phi} \approx 0,
\tag{11}
$$

(see Ref. [6] for more details). Moreover, one can neglect the weak coupling induced by both the indirect amplitude detuning and the chromaticities, such that it is reasonable to assume that the stationary distribution can be written by separating all three degrees of freedom:

$$
\psi_0(x, x', y, y', z, \delta) = N f_{x0}(J_x) f_{y0}(J_y) g_0(J_z),
\tag{12}
$$

with $N$ the total number of particles in the full phase space. The normalization conditions are chosen as:

$$
\begin{aligned}
\int_0^{+\infty} \mathrm{d}J_x\, f_{x0}\left(J_x\right) &= \frac{1}{2\pi},\\
\int_0^{+\infty} \mathrm{d}J_y\, f_{y0}\left(J_y\right) &= \frac{1}{2\pi},\\
\int_0^{+\infty} \mathrm{d}J_z\, g_0\left(J_z\right) &= \frac{1}{2\pi}.
\end{aligned}
\tag{13}
$$

The perturbative part of the Hamiltonian $\Delta H$ is assumed to be responsible for a collective, $z$-dependent, vertical dipolar force $F_y^{coh}$, e.g. to be of the form [6]

$$
\Delta H = -\frac{y F_y^{coh}(z;t)}{p_0} = -\sqrt{\frac{2J_y R}{Q_{y0}}}\cos\theta_y \frac{F_y^{coh}(z;t)}{p_0},
\tag{14}
$$

using

$$
y = \sqrt{\frac{2J_y R}{Q_{y0}}}\cos\theta_y.
\tag{15}
$$

Having defined $H_0$, $\Delta H$ and $\psi_0$, we want to obtain now at first order the perturbation of the distribution $\Delta\psi$. Our starting point will be the linearized Vlasov equation expressed with Poisson brackets [8]:

$$
\frac{\partial \Delta\psi}{\partial t} + [\Delta\psi, H_0] + [\psi_0, \Delta H] = 0,
\tag{16}
$$

with the definition (using action-angle variables as expressed above):

$$
\begin{aligned}
[\mathcal{F}, \mathcal{G}] = &\frac{\partial \mathcal{F}}{\partial J_x}\frac{\partial \mathcal{G}}{\partial \theta_x} - \frac{\partial \mathcal{F}}{\partial \theta_x}\frac{\partial \mathcal{G}}{\partial J_x}\\
&+ \frac{\partial \mathcal{F}}{\partial J_y}\frac{\partial \mathcal{G}}{\partial \theta_y} - \frac{\partial \mathcal{F}}{\partial \theta_y}\frac{\partial \mathcal{G}}{\partial J_y} + \frac{\partial \mathcal{F}}{\partial J_z}\frac{\partial \mathcal{G}}{\partial \phi} - \frac{\partial \mathcal{F}}{\partial \phi}\frac{\partial \mathcal{G}}{\partial J_z},
\end{aligned}
\tag{17}
$$

for two differentiable functions $\mathcal{F}$ and $\mathcal{G}$. We refer the reader to Ref. [9] for more details on classical Hamiltonian mechanics, and to Refs. [10–12] for a detailed description of Hamiltonian dynamics in the case of single particle beam physics. Vlasov equation applied to a distribution of beam particles, in the context of linear perturbation theory, is thoroughly explained in Ref. [1, chap. 6]. A short primer on both Hamiltonian and Vlasov aspects can be found in Ref. [6].

Neglecting as above $\frac{\partial H_0}{\partial \phi}$, we can write

$$
\begin{aligned}
[\Delta\psi, H_0] &\approx -\frac{\partial \Delta\psi}{\partial \theta_x}\frac{\partial H_0}{\partial J_x} - \frac{\partial \Delta\psi}{\partial \theta_y}\frac{\partial H_0}{\partial J_y} - \frac{\partial \Delta\psi}{\partial \phi}\frac{\partial H_0}{\partial J_z}\\
&\approx -\omega_0 Q_x \frac{\partial \Delta\psi}{\partial \theta_x} - \omega_0 Q_y \frac{\partial \Delta\psi}{\partial \theta_y} + \omega_s \frac{\partial \Delta\psi}{\partial \phi},
\end{aligned}
\tag{18}
$$

with

$$
Q_x = Q_{x0} + Q_x'\delta + a_{xx}J_x + a_{xy}J_y,
\tag{19}
$$

$$
Q_y = Q_{y0} + Q_y'\delta + a_{xy}J_x + a_{yy}J_y.
\tag{20}
$$

The other Poisson brackets is given by (using the fact that $\Delta H$ does not depend on $\theta_x$ and that $\psi_0$ does not depend on any of the angles)

$$
\begin{aligned}
[\psi_0, \Delta H] &= N f_{x0}(J_x)\left(\frac{df_{y0}}{dJ_y} g_0(J_z)\frac{\partial \Delta H}{\partial \theta_y} + f_{y0}(J_y)\frac{dg_0}{dJ_z}\frac{\partial \Delta H}{\partial \phi}\right)\\
&= N f_{x0}(J_x)\frac{df_{y0}}{dJ_y} g_0(J_z)\sqrt{\frac{2J_y R}{Q_{y0}}}\sin\theta_y \frac{F_y^{coh}(z;t)}{p_0}\\
&\quad + N f_{x0}(J_x) f_{y0}(J_y)\frac{dg_0}{dJ_z}\frac{\partial \Delta H}{\partial z}\frac{\partial z}{\partial \phi}\\
&\approx N f_{x0}(J_x)\frac{df_{y0}}{dJ_y} g_0(J_z)\sqrt{\frac{2J_y R}{Q_{y0}}}\sin\theta_y \frac{F_y^{coh}(z;t)}{p_0}.
\end{aligned}
\tag{21}
$$

In the above we have neglected $\frac{\partial \Delta H}{\partial z}$. This is a standard approximation, made in Ref. [1, chap. 6], which has its grounds in the general idea that we neglect any effect of the transverse coherent force on the longitudinal motion. This should be valid as long as one remains far from low-order synchro-betatron resonances $Q_{y0} + lQ_s = n$ (and provided the transverse beam size is small enough).

The linearized Vlasov equation (16) then takes the form

$$\frac{\partial\Delta\psi}{\partial t} - \frac{\partial\Delta\psi}{\partial\theta_x}\omega_0 Q_x - \frac{\partial\Delta\psi}{\partial\theta_y}\omega_0 Q_y + \frac{\partial\Delta\psi}{\partial\phi}\omega_s + N f_{x0}(J_x)\frac{df_{y0}}{dJ_y}g_0(J_z)\sqrt{\frac{2J_yR}{Q_{y0}}}\sin\theta_y\frac{F_y^{coh}(z;t)}{p_0} = 0. \tag{22}$$

For convenience we switch to the $r$ coordinate instead of $J_z$, as in Ref. [1, chap. 6]:

$$r = \sqrt{\frac{2J_z v\eta}{\omega_s}}, \quad z = r\cos\phi, \quad \delta = \frac{\omega_s}{v\eta}r\sin\phi. \tag{23}$$

Then we express the perturbation assuming a single coherent mode of complex angular frequency $\Omega = Q_c\omega_0$, and decompose it using Fourier series for all three angles, in a completely general way:

$$\Delta\psi\left(J_x,\theta_x,J_y,\theta_y,r,\phi;t\right) = e^{j\Omega t}\sum_{m=-\infty}^{+\infty}e^{jm\theta_x}\sum_{p=-\infty}^{+\infty}e^{jp\theta_y} \times e^{-\frac{j\left(mQ'_x+pQ'_y\right)r\cos\phi}{\eta R}}\sum_{l=-\infty}^{+\infty}e^{-jl\phi}h^{m,p,l}\left(J_x,J_y,r\right), \tag{24}$$

where we have introduced, without loss of generality, the head-tail phase factor

$$e^{-\frac{j\left(mQ'_x+pQ'_y\right)r\cos\phi}{\eta R}},$$

similarly to what is done in Ref [1, chap. 6]. This phase factor is a convenience introduced to simplify the equation, as we will see below. To proceed further, we again assume that the dependency between the longitudinal and transverse actions is separable, in other words that

$$h^{m,p,l}\left(J_x,J_y,r\right) = f^{m,p,l}\left(J_x,J_y\right)R_l(r). \tag{25}$$

Now we expand the linearized Vlasov equation from Eq. (22) as

$$e^{j\Omega t}\sum_{m=-\infty}^{+\infty}\sum_{p=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}e^{jm\theta_x}e^{jp\theta_y}e^{-jl\phi}e^{-\frac{j\left(mQ'_x+pQ'_y\right)r\cos\phi}{\eta R}}\times \left[jQ_c - jmQ_x - jpQ_y - jlQ_s + \frac{j\omega_s\left(mQ'_x+pQ'_y\right)r\sin\phi}{\eta v}\right] \times\omega_0 f^{m,p,l}\left(J_x,J_y\right)R_l(r) = -Nf_{x0}(J_x)\frac{df_{y0}}{dJ_y}g_0(J_z) \times\sqrt{\frac{2J_yR}{Q_{y0}}}\frac{e^{j\theta_y}-e^{-j\theta_y}}{2j}\frac{F_y^{coh}(z;t)}{p_0}, \tag{26}$$

where we have used the identity $\omega_0 = \frac{v}{R}$. In the above, the expression within square brackets can be simplified further since the chromatic terms in Eqs. (19) and (20) cancel out with the term

$$\frac{j\omega_s\left(mQ'_x+pQ'_y\right)r\sin\phi}{\eta v} = j\left(mQ'_x+pQ'_y\right)\delta,$$

using Eq. (23) – this is the reason why we introduced the head-tail phase factor in the first place. Then, comparing the expressions in $\theta_x$ and $\theta_y$ on both sides of the equation, term by term identification in the Fourier series gives:

$$f^{m,p,l}\left(J_x,J_y\right) = 0 \quad \text{for any } m\neq 0 \text{ and any } p\neq\pm 1. \tag{27}$$

Equation (26) thus becomes

$$e^{j\Omega t}\sum_{p=\pm1}\sum_{l=-\infty}^{+\infty}e^{jp\theta_y}e^{-jl\phi}e^{-\frac{jpQ'_yr\cos\phi}{\eta R}}\times j\left[Q_c - p\left(Q_{y0}+a_{xy}J_x+a_{yy}J_y\right) - lQ_s\right] \times\omega_0 f^{0,p,l}\left(J_x,J_y\right)R_l(r) = -Nf_{x0}(J_x)\frac{df_{y0}}{dJ_y}g_0(J_z) \times\sqrt{\frac{2J_yR}{Q_{y0}}}\frac{e^{j\theta_y}-e^{-j\theta_y}}{2j}\frac{F_y^{coh}(z;t)}{p_0}. \tag{28}$$

Now we make the standard assumption that $Q_c \approx Q_{y0}$ such that

$$\left|Q_c + \left(Q_{y0}+a_{xy}J_x+a_{yy}J_y\right) - lQ_s\right| >> \left|Q_c - \left(Q_{y0}+a_{xy}J_x+a_{yy}J_y\right) - lQ_s\right|.$$

This means that the term for $p = -1$ can be neglected as well as $e^{-j\theta_y}$ on the right-hand side [1, chap. 6], and we end up with (after simplification by $e^{j\theta_y}$)

$$e^{j\Omega t}e^{-\frac{jQ'_yr\cos\phi}{\eta R}}\sum_{l=-\infty}^{+\infty}e^{-jl\phi}f^{0,1,l}\left(J_x,J_y\right)R_l(r) \times\omega_0\left[Q_c - \left(Q_{y0}+a_{xy}J_x+a_{yy}J_y\right) - lQ_s\right] = Nf_{x0}(J_x)\frac{df_{y0}}{dJ_y}g_0(J_z)\sqrt{\frac{2J_yR}{Q_{y0}}}\frac{F_y^{coh}(z;t)}{2p_0}. \tag{29}$$

Renaming $f^{0,1,l} \equiv f^l$, and re-arranging to put all terms in $J_x$ and $J_y$ on the left-hand side, we get

$$\frac{\omega_0}{N}\sum_{l=-\infty}^{+\infty}e^{-jl\phi}R_l(r) \times\left\{\frac{f^l\left(J_x,J_y\right)\left[Q_c - \left(Q_{y0}+a_{xy}J_x+a_{yy}J_y\right) - lQ_s\right]}{f_{x0}(J_x)\frac{df_{y0}}{dJ_y}\sqrt{\frac{2J_yR}{Q_{y0}}}}\right\} = e^{-j\Omega t}e^{\frac{jQ'_yr\cos\phi}{\eta R}}g_0(J_z)\frac{F_y^{coh}(z;t)}{2p_0}. \tag{30}$$

If we take the derivative with respect to $J_x$, the right-hand side goes to zero, which means that for any $l$ the term between curly brackets in the left-hand side must be a constant with respect to $J_x$. The same goes if we take instead the derivative with respect to $J_y$. Hence

$$f^l\left(J_x, J_y\right) \propto \frac{f_{x0}(J_x)\frac{df_{y0}}{dJ_y}\sqrt{\frac{2J_yR}{Q_{y0}}}}{Q_c - \left(Q_{y0} + a_{xy}J_x + a_{yy}J_y\right) - lQ_s}. \quad (31)$$

The proportionality constant may be included in $R_l(r)$ – thus we can write the above as an equality rather than a proportionality relation. We can finally write the full perturbation to the distribution as

$$\Delta\psi\left(J_x, \theta_x, J_y, \theta_y, r, \phi; t\right) = e^{j\Omega t}e^{j\theta_y}e^{-\frac{jQ'_y r\cos\phi}{\eta R}} \times \sum_{l=-\infty}^{+\infty} R_l(r)e^{-jl\phi}\frac{f_{x0}(J_x)\frac{df_{y0}}{dJ_y}\sqrt{\frac{2J_yR}{Q_{y0}}}}{Q_c - \left(Q_{y0} + a_{xy}J_x + a_{yy}J_y\right) - lQ_s}. \quad (32)$$

Up to now, the only assumptions on the coherent force considered were that it is vertical, dipolar and $z$-dependent. To proceed further, we will take the specific case of an impedance distributed along the ring, i.e. given by [6]

$$F_y^{coh}(z;t) = \frac{q^2}{2\pi R}\sum_{k=-\infty}^{+\infty}\iint \mathrm{d}\tilde{z}\mathrm{d}\tilde{\delta}\, W_y(\tilde{z} + 2\pi kR - z) \cdot \iiiint \mathrm{d}\tilde{J}_x\mathrm{d}\tilde{\theta}_x\mathrm{d}\tilde{J}_y\mathrm{d}\tilde{\theta}_y\Delta\psi\left(\tilde{J}_x, \tilde{\theta}_x, \tilde{J}_y, \tilde{\theta}_y, \tilde{r}, \tilde{\phi}; t - k\frac{2\pi R}{v}\right)\tilde{y},$$

with $q = Ze$ the charge of each particle, $W_y(z)$ the wake function, and where the infinite sum stands for the multiturn effect (i.e. the fact that the wake does not decay completely after one or several turns). Only the perturbed distribution $\Delta\psi$ enters the expression above as the stationary distribution is assumed to be perfectly centred and hence not giving rise to any dipolar force. Plugging Eqs. (15) and (32) into the above we get

$$F_y^{coh}(z;t) = \frac{q^2}{\pi Q_{y0}}\sum_{k=-\infty}^{+\infty} e^{j\Omega\left(t-k\frac{2\pi R}{v}\right)}\iint \mathrm{d}\tilde{z}\mathrm{d}\tilde{\delta} \cdot W_y(\tilde{z} + 2\pi kR - z)\iint \mathrm{d}\tilde{\theta}_x\mathrm{d}\tilde{\theta}_y e^{j\tilde{\theta}_y}\cos\tilde{\theta}_y e^{-\frac{jQ'_y\tilde{r}\cos\tilde{\phi}}{\eta R}} \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r})e^{-jl'\tilde{\phi}}\iint \frac{\mathrm{d}\tilde{J}_x\mathrm{d}\tilde{J}_y \cdot \tilde{J}_y \cdot f_{x0}\left(\tilde{J}_x\right)\cdot\frac{df_{y0}}{dJ_y}\left(\tilde{J}_y\right)}{Q_c - \left(Q_{y0} + a_{xy}\tilde{J}_x + a_{yy}\tilde{J}_y\right) - l'Q_s}.$$

This can be further simplified thanks to

$$\iint \mathrm{d}\tilde{z}\mathrm{d}\tilde{\delta} = \frac{\omega_s}{v\eta}\iint \tilde{r}\mathrm{d}\tilde{r}\mathrm{d}\tilde{\phi} = \frac{Q_s}{\eta R}\iint \tilde{r}\mathrm{d}\tilde{r}\mathrm{d}\tilde{\phi},$$
$$\int_0^{2\pi}\mathrm{d}\tilde{\theta}_x = 2\pi, \qquad \int_0^{2\pi}\mathrm{d}\tilde{\theta}_y e^{j\tilde{\theta}_y}\cos\tilde{\theta}_y = \pi,$$

and defining the dispersion integral $\mathcal{I}(Q_c - lQ_s)$ using[1]

$$\mathcal{I}(Q) = \iint \frac{\mathrm{d}J_x\mathrm{d}J_y \cdot J_y \cdot f_{x0}\left(J_x\right)\cdot\frac{df_{y0}}{dJ_y}}{Q - \left(Q_{y0} + a_{xy}J_x + a_{yy}J_y\right)}, \quad (33)$$

such that we get

$$F_y^{coh}(z;t) = \frac{2\pi q^2Q_s}{\eta RQ_{y0}}\sum_{k=-\infty}^{+\infty} e^{j\Omega\left(t-k\frac{2\pi R}{v}\right)} \cdot \iint \tilde{r}\,\mathrm{d}\tilde{r}\,\mathrm{d}\tilde{\phi}W_y(\tilde{r}\cos\tilde{\phi} + 2\pi kR - r\cos\phi)\cdot e^{-\frac{jQ'_y\tilde{r}\cos\tilde{\phi}}{\eta R}} \cdot \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r})e^{-jl'\tilde{\phi}}\mathcal{I}(Q_c - l'Q_s). \quad (34)$$

Note that analytical expressions exist for the dispersion integral expressed in Eq. (33), for various kinds of unperturbed distributions, e.g. Gaussian or parabolic [14, 15].

We can now plug Eq. (34) into Eq. (30), using also Eq. (31) (taken as an equality) and recalling that $\omega_0 = \frac{v}{R}$, to get

$$\sum_{l'=-\infty}^{+\infty} e^{-jl'\phi}R_{l'}(r) = \frac{N\pi q^2Q_s}{\eta vQ_{y0}p_0}e^{\frac{jQ'_y r\cos\phi}{\eta R}}g_0\left(J_z\right) \times \sum_{k=-\infty}^{+\infty} e^{-j2\pi kQ_c}\iint \tilde{r}\,\mathrm{d}\tilde{r}\,\mathrm{d}\tilde{\phi}W_y(\tilde{r}\cos\tilde{\phi} + 2\pi kR - r\cos\phi) \times e^{-\frac{jQ'_y\tilde{r}\cos\tilde{\phi}}{\eta R}}\sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r})e^{-jl'\tilde{\phi}}\mathcal{I}(Q_c - l'Q_s). \quad (35)$$

Multiplying each side of the equation by $\frac{e^{jl\phi}}{2\pi}$, integrating over $\phi$, and considering $g_0$ as a function of $r$, we can write:

$$R_l(r) = \frac{Nq^2Q_sg_0(r)}{2\eta vQ_{y0}p_0}\int_0^{2\pi}\mathrm{d}\phi e^{jl\phi}e^{\frac{jQ'_y r\cos\phi}{\eta R}}\sum_{k=-\infty}^{+\infty} e^{-j2\pi kQ_c} \times \iint \tilde{r}\,\mathrm{d}\tilde{r}\,\mathrm{d}\tilde{\phi}W_y(\tilde{r}\cos\tilde{\phi} + 2\pi kR - r\cos\phi)\cdot e^{-\frac{jQ'_y\tilde{r}\cos\tilde{\phi}}{\eta R}} \times \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r})e^{-jl'\tilde{\phi}}\mathcal{I}(Q_c - l'Q_s). \quad (36)$$

This is an extension of Sacherer integral equation, expressed with the wake function and including tune spread. Formally, this is an equation in $Q_c$ – the coherent complex tune shift looked for – and in the unknown functions $R_l(r)$.

The equation can also be expressed with the impedance $Z_y(\omega)$ instead of the wake function [1, chap. 6], using the

[1] Compared to the definition used in the code DELPHI [13], there is an additional factor $\frac{1}{4\pi^2}$ due to the normalization chosen in Eqs. (13), i.e.

$$\mathcal{I}^{\text{DELPHI}}(Q) = 4\pi^2\mathcal{I}^{\text{here}}(Q).$$

relation (see Ref [6] for a mathematical derivation):

$$\sum_{k=-\infty}^{+\infty} e^{-j2\pi k Q_c} W_y(\tilde{r}\cos\tilde{\phi} + 2\pi k R - r\cos\phi) \approx$$
$$\frac{-j\omega_0}{2\pi} \sum_{k=-\infty}^{+\infty} Z_y\left[\left(Q_{y0}+k\right)\omega_0\right] e^{j\left(Q_{y0}+k\right)\frac{\tilde{r}\cos\tilde{\phi}-r\cos\phi}{R}}, \quad (37)$$

where we have used again $Q_c \approx Q_{y0}$. This gives

$$R_l(r) = \frac{-jNq^2 Q_s g_0(r)}{4\pi\eta R Q_{y0} p_0} \int_0^{2\pi} \mathrm{d}\phi e^{jl\phi} e^{\frac{jQ'_y r\cos\phi}{\eta R}}$$
$$\times \sum_{k=-\infty}^{+\infty} e^{\frac{-j\left(Q_{y0}+k\right)r\cos\phi}{R}} Z_y\left[\left(Q_{y0}+k\right)\omega_0\right]$$
$$\times \iint \tilde{r}\,\mathrm{d}\tilde{r}\,\mathrm{d}\tilde{\phi}\, e^{\frac{j\left(Q_{y0}+k\right)\tilde{r}\cos\tilde{\phi}}{R}} \cdot e^{-\frac{jQ'_y\tilde{r}\cos\tilde{\phi}}{\eta R}}$$
$$\times \sum_{l'=-\infty}^{+\infty} R_{l'}(\tilde{r}) e^{-jl'\tilde{\phi}} \mathcal{I}(Q_c - l'Q_s), \quad (38)$$

which we can further simplify using

$$\int_0^{2\pi} \mathrm{d}\tilde{\phi} e^{-jl\tilde{\phi}} e^{j\chi\cos\tilde{\phi}} = 2\pi j^l J_l(\chi) \qquad \text{for any } \chi, \quad (39)$$

from Eq. (9.1.21) of Ref. [16], $J_l$ being the Bessel function of order $l$. This gives another version of Sacherer equation, expressed now with the impedance:

$$R_l(r) = \frac{-jN\pi q^2 Q_s g_0(r)}{\eta R Q_{y0} p_0} \sum_{k=-\infty}^{+\infty} J_l\left[\left(Q_{y0}+k-\frac{Q'_y}{\eta}\right)\frac{r}{R}\right]$$
$$\times Z_y\left[\left(Q_{y0}+k\right)\omega_0\right] \sum_{l'=-\infty}^{+\infty} j^{l'-l}\mathcal{I}(Q_c - l'Q_s)$$
$$\times \int_0^{+\infty} \tilde{r}\mathrm{d}\tilde{r} R_{l'}(\tilde{r}) J_{l'}\left[\left(Q_{y0}+k-\frac{Q'_y}{\eta}\right)\frac{\tilde{r}}{R}\right]. \quad (40)$$

A similar equation was obtained by Chin [5], with one-dimensional transverse tune spread. In the absence of tunespread, this reduces to the standard Sacherer equation [1, 6, 17, 18][2], if we put the coefficients $\mathcal{I}(Q_c - lQ_s)$ into the unknown functions $R_l(r)$ and notice that $\mathcal{I}(Q) = -1/\left(4\pi^2(Q - Q_{y0})\right)$ (see below).

## SOLVING THE EQUATION

One strategy to solve the equation is first to replace the unknown functions $R_l(r)$ thanks to

$$\rho_l(r) \equiv \mathcal{I}(Q_c - lQ_s)R_l(r).$$

This gives

$$\frac{\rho_l(r)}{\mathcal{I}\left(Q_c - lQ_s\right)} = \frac{-jN\pi q^2 Q_s g_0(r)}{\eta R Q_{y0} p_0}$$
$$\times \sum_{k=-\infty}^{+\infty} J_l\left[\left(Q_{y0}+k-\frac{Q'_y}{\eta}\right)\frac{r}{R}\right] Z_y\left[\left(Q_{y0}+k\right)\omega_0\right]$$
$$\times \sum_{l'=-\infty}^{+\infty} j^{l'-l} \int_0^{+\infty} \tilde{r}\mathrm{d}\tilde{r}\rho_{l'}(\tilde{r}) J_{l'}\left[\left(Q_{y0}+k-\frac{Q'_y}{\eta}\right)\frac{\tilde{r}}{R}\right]. \quad (41)$$

Then, one can expand $\rho_l(r)$ and $g_0(r)$ over generalized Laguerre polynomials $L_n^\alpha$, as in Refs. [5, 18], i.e.

$$\rho_l(r) = \left(\frac{r}{A}\right)^{|l|} e^{-br^2} \sum_{n=0}^{+\infty} c_{ln} L_n^{|l|}(ar^2), \quad (42)$$

$$g_0(r) = \frac{\eta R}{Q_s} e^{-br^2} \sum_{m=0}^{+\infty} g^m L_m^0(ar^2), \quad (43)$$

where $A$ , $a$ and $b$ are arbitrary constants that will be specified later. Note that the change of variable $J_z \to r$ introduces an additional factor $\frac{\eta R}{Q_s}$ in the normalization condition of $g_0$ expressed as an integral over $r$ (see Eqs. (13) and (23)), hence the proportionality constant in front of the expression for $g_0(r)$. Using the orthogonality relations of the generalized Laguerre polynomials [16, chap. 22], the $c_{ln}$ and $g^m$ coefficients can be expressed as

$$c_{ln} = \frac{2a^{1+|l|}A^{2|l|}n!}{(n+|l|)!} \int_0^{+\infty} r\mathrm{d}r \left(\frac{r}{A}\right)^{|l|} e^{(b-a)r^2} L_n^{|l|}(ar^2)\rho_l(r), \quad (44)$$

$$g^m = 2a \int_0^{+\infty} r\mathrm{d}r e^{(b-a)r^2} L_m^0(ar^2) g_0(r). \quad (45)$$

Note that any longitudinal distribution $g_0(r)$ can be dealt with in this way, generalizing the approach of Chin [5,19,20]. Multiplying both sides of Eq. (41) by

$$2a^{1+|l|}A^{2|l|}\frac{n!}{(n+|l|)!} r\left(\frac{r}{A}\right)^{|l|} e^{(b-a)r^2} L_n^{|l|}(ar^2),$$

and integrating from $r = 0$ to $r = +\infty$ we get

$$\frac{c_{ln}}{\mathcal{I}\left(Q_c - lQ_s\right)} = \frac{-j2\pi Nq^2 Q_s}{\eta R Q_{y0} p_0} \frac{a^{1+|l|}A^{2|l|}n!}{(n+|l|)!}$$
$$\times \sum_{k=-\infty}^{+\infty} Z_y\left[\left(Q_{y0}+k\right)\omega_0\right] \int_0^{+\infty} r\mathrm{d}r g_0(r)$$
$$\times \left(\frac{r}{A}\right)^{|l|} e^{(b-a)r^2} L_n^{|l|}(ar^2) J_l\left[\left(Q_{y0}+k-\frac{Q'_y}{\eta}\right)\frac{r}{R}\right]$$
$$\times \sum_{l'=-\infty}^{+\infty} j^{l'-l} \int_0^{+\infty} r\mathrm{d}r\rho_{l'}(r) J_{l'}\left[\left(Q_{y0}+k-\frac{Q'_y}{\eta}\right)\frac{r}{R}\right]. \quad (46)$$

[2] Note that in the impedance term of Eq. (11) in Ref. [18], the signs of the expressions between brackets in the Bessel functions have been incorrectly inverted: one should read $(\omega_p - \omega_\xi)\tau$ instead of $(\omega_\xi - \omega_p)\tau$. Moreover, the constant factor on the right-hand side of the equation depends on the normalization conditions chosen for each of the unperturbed functions $f_{x0}, f_{y0}$ and $g_0$, which is arbitrary (only the normalization of $\psi_0$ is fixed).

Now we expand $g_0(r)$ and $\rho_{l'}(r)$ using Eqs. (42) and (43), obtaining

$$\frac{c_{ln}}{\mathcal{I}\left(Q_c - lQ_s\right)} = \frac{-j2\pi N q^2}{Q_{y0}p_0}\frac{a^{1+|l|}A^{|l|}n!}{(n+|l|)!} \times \sum_{k=-\infty}^{+\infty} Z_y\left[\left(Q_{y0}+k\right)\omega_0\right]\sum_{m=0}^{+\infty} g^m$$
$$\times \int_0^{+\infty} r^{1+|l|}e^{-ar^2}L_m^0(ar^2)L_n^{|l|}(ar^2)J_l\left[\left(Q_{y0}+k-\frac{Q_y'}{\eta}\right)\frac{r}{R}\right]\mathrm{d}r$$
$$\times \sum_{l'=-\infty}^{+\infty} j^{l'-l}\sum_{n'=0}^{+\infty} c_{l'n'}$$
$$\times \int_0^{+\infty} r\left(\frac{r}{A}\right)^{|l'|}e^{-br^2}L_{n'}^{|l'|}(ar^2)J_{l'}\left[\left(Q_{y0}+k-\frac{Q_y'}{\eta}\right)\frac{r}{R}\right]\mathrm{d}r. \tag{47}$$

The two integrals above can be computed analytically using two formulas obtained from Hankel transforms [21, pp. 42-43][3](using also Eq. (9.1.5) from Ref. [16, p. 358]):

$$\int_0^{+\infty} r^{1+|l|}e^{-ar^2}L_m^0(ar^2)L_n^{|l|}(ar^2)J_l(\lambda r)\mathrm{d}r = \frac{(-1)^{m+n}}{2a}\left(\frac{\mathrm{sign}(l)\lambda}{2a}\right)^{|l|}e^{\frac{-\lambda^2}{4a}}L_n^{m-n}\left(\frac{\lambda^2}{4a}\right)L_m^{|l|+n-m}\left(\frac{\lambda^2}{4a}\right), \tag{48}$$

$$\int_0^{+\infty} r^{1+|l|}e^{-br^2}L_n^{|l|}(ar^2)J_l(\lambda r)\mathrm{d}r = \frac{\mathrm{sign}(l)^{|l|}}{2b}\left(\frac{\lambda}{2b}\right)^{|l|}\left(\frac{b-a}{b}\right)^n e^{\frac{-\lambda^2}{4b}}L_n^{|l|}\left(\frac{a\lambda^2}{4b(a-b)}\right), \tag{49}$$

valid for any $a > 0$, $b > 0$ and $\lambda$. Defining then

$$G_{ln}(\lambda) \equiv (2a)^{1+|l|}A^{|l|}\sum_{m=0}^{+\infty} g^m \int_0^{+\infty} r^{1+|l|}e^{-ar^2} \times L_m^0(ar^2)L_n^{|l|}(ar^2)J_l(\lambda r)\mathrm{d}r$$
$$= (\mathrm{sign}(l)\lambda A)^{|l|}e^{\frac{-\lambda^2}{4a}}\sum_{m=0}^{+\infty}(-1)^{m+n}g^m \times L_n^{m-n}\left(\frac{\lambda^2}{4a}\right)L_m^{|l|+n-m}\left(\frac{\lambda^2}{4a}\right), \tag{50}$$

and

$$I_{ln}(\lambda) \equiv A^{-|l|}\int_0^{+\infty} e^{-br^2}r^{1+|l|}L_n^{|l|}(ar^2)J_l(\lambda r)\mathrm{d}r$$
$$= \begin{cases} \frac{1}{2b}\left(\frac{\mathrm{sign}(l)\lambda}{2bA}\right)^{|l|}\left(1-\frac{a}{b}\right)^n e^{\frac{-\lambda^2}{4b}}L_n^{|l|}\left(\frac{a\lambda^2}{4b(a-b)}\right) & \text{if } a \neq b, \\ \left(\frac{\lambda}{2}\right)^{2n+|l|}\frac{\mathrm{sign}(l)^{|l|}}{2n!a^{n+|l|+1}A^{|l|}}e^{-\frac{\lambda^2}{4a}} & \text{if } a = b, \end{cases}$$

(taking the expression to its limit when $a = b$), Sacherer equation finally becomes

$$0 = \sum_{l'=-\infty}^{+\infty}\sum_{n'=0}^{+\infty}\left(\frac{\delta_{ll'}\delta_{nn'}}{\mathcal{I}\left(Q_c - l'Q_s\right)} + \mathcal{M}_{ln,l'n'}\right)c_{l'n'}, \tag{51}$$

with the matrix $\mathcal{M}$ defined by

$$\mathcal{M}_{ln,l'n'} = \frac{j^{1+l'-l}\pi N q^2\, n!}{Q_{y0}p_0\, 2^{|l|}\,(n+|l|)!}\sum_{k=-\infty}^{+\infty}\left\{ Z_y\left[\left(Q_{y0}+k\right)\omega_0\right] \times G_{ln}\left(\frac{Q_{y0}+k}{R}-\frac{Q_y'}{\eta R}\right)I_{l'n'}\left(\frac{Q_{y0}+k}{R}-\frac{Q_y'}{\eta R}\right)\right\}. \tag{52}$$

In the DELPHI code [13, 18], for an initially Gaussian longitudinal distribution, $b = \frac{1}{2\sigma_z^2}$ (with $\sigma_z$ the RMS bunch length) and all $g^m$ are zero except $g^0 = \frac{b}{\pi}$ to respect the normalization condition in Eqs. (13). Then the choices $a = b$ and $A = 4\sigma_z$ were found to optimize convergence with respect to the matrix size in the case without tunespread (see next section)[4].

Non-trivial solutions of Eq. (51) are found if and only if

$$\det\left(\left[\frac{\delta_{ll'}\delta_{nn'}}{\mathcal{I}\left(Q_c - l'Q_s\right)} + \mathcal{M}_{ln,l'n'}\right]\right) = 0. \tag{53}$$

This equation in $Q_c$ generalizes the similar determinant equation obtained in Ref. [5] to the case of a two-dimensional tune spread and any longitudinal distribution. Mathematically, it is a transcendental equation, because the dispersion integral is non-polynomial in general, hence there is a priori no general strategy to find all possible roots.

As a final remark, one can easily add two extensions to this formalism, both implemented in the DELPHI code:

- if the beam is made of $M > 1$ equidistant bunches, and assuming the intrabunch motion in all the bunches is identical (only the phase of the oscillation may differ from one bunch to another), the equation obtained is

[3] To obtain Eq. (48), we have corrected a typo in Ref [21, p. 43], Eq. (8): the two lower indices of the Laguerre polynomials on the right-hand side were inverted.

[4] In Ref. [18], the functions $G_{ln}$ and $I_{ln}$ were chosen slightly differently: $G_{ln}^{\text{Ref. [18]}}(\omega) = \nu^2\tau_b^{-|l|}G_{ln}^{\text{here}}\left(\frac{\omega}{\nu}\right)$ and $I_{ln}^{\text{Ref. [18]}}(\omega) = \nu^{-2}\tau_b^{|l|}I_{ln}^{\text{here}}\left(\frac{\omega}{\nu}\right)$, with $\tau_b = \frac{4\sigma_z}{\nu}$. The matrix $\mathcal{M}$ of Ref. [18] is also multiplied by $\frac{\omega_0}{4\pi^2}$ with respect to the one written in Eq. (52): the $\omega_0$ factor is because in Ref. [18] the problem is expressed in terms of angular frequency shifts rather than tune shifts, while the $\frac{1}{4\pi^2}$ factor gets compensated by the same factor in $\mathcal{I}(Q)$ here.

still of the form (51) but with a slightly modified matrix $\mathcal{M}$: one has to multiply the matrix given in Eq. (52) by $M$ and replace [22]

$$Q_{y0} + k,$$

by

$$[Q_{y0}] + kM + p,$$

in all the terms of the infinite sum over $k$, where $[\cdot]$ indicates the fractional part and $0 \leq p \leq M - 1$ is the coupled-bunch mode considered (one has to solve the problem for each coupled-bunch mode, in principle),

- an ideal, bunch-by-bunch transverse damper can be added by considering it as an impedance proportional to a delta function and replacing the infinite sum over $k$ in Eq. (52) by an integral. This gives a matrix $\mathcal{D}$ that can be added to $\mathcal{M}$ above, of the form

$$\mathcal{D}_{ln,l'n'} = \frac{j^{1+l'-l} N \pi q^2 \, n!}{Q_{y0} p_0 \, 2^{|l|} \, (n + |l|)!} \times \mu G_{ln} \left( -\frac{Q'_y}{\eta R} \right) I_{l'n'} \left( -\frac{Q'_y}{\eta R} \right), \quad (54)$$

with $\mu$ a constant adjusted, in the case without tune-spread (see next section), in such a way as to damp the rigid-bunch mode in $n_d$ turns, with a damping phase $\varphi$ (the origin $\varphi = 0$ being chosen as the perfectly resistive damper). This means that one chooses $\mu$ such that (see also the first limiting case below) [18]

$$\frac{1}{4\pi^2} \mathcal{D}_{00,00} = \frac{j e^{j\varphi}}{2\pi n_d}.$$

## LIMITING CASES

One can find two well-known limiting cases to the determinant equation (53). First, in the absence of tune spread, $a_{xy} = a_{yy} = 0$ and

$$\mathcal{I}(Q) = -\frac{1}{4\pi^2 (Q - Q_{y0})}, \quad (55)$$

from the normalization conditions in Eqs. (13). This means that Eq. (51) becomes

$$\left( Q_c - Q_{y0} \right) c_l^n = \sum_{l'=-\infty}^{+\infty} \sum_{n'=0}^{+\infty} \left( l' Q_s \delta_{ll'} \delta_{nn'} + \frac{1}{4\pi^2} \mathcal{M}_{ln,l'n'} \right) c_{l'}^{n'}, \quad (56)$$

which is the usual eigenvalue problem in the absence of tune spread [1, 18]. The coherent tune shifts looked for can then be obtained through a diagonalization.

Another limiting case appears in the absence of coupling between different modes, i.e. when all non-diagonal terms are zero in the matrix $\mathcal{M}$. Then the determinant equation becomes

$$\det \left( \left[ \delta_{ll'} \delta_{nn'} \left( \frac{1}{\mathcal{I}(Q_c - l' Q_s)} + \mathcal{M}_{ln,l'n'} \right) \right] \right) = \prod_{l=-\infty}^{+\infty} \prod_{n=0}^{+\infty} \left( \frac{1}{\mathcal{I}(Q_c - l Q_s)} + \mathcal{M}_{ln,ln} \right) = 0, \quad (57)$$

(as the matrix is now diagonal), which means that we get a set of equations of the form (for each $l$ and $n$):

$$-1 = \mathcal{M}_{ln,ln} \times \mathcal{I}(Q_c - l Q_s), \quad (58)$$

which gives one possible coherent tune $Q_c$ for each $(l, n)$. We hence can consider separately the coherent tune shift and the dispersion integral, in other words we recover the stability diagram theory [23, 24].

## RESULTS

Notwithstanding the difficulty to find all the roots of the general determinant equation (53), we can try to get a generalization of the concept of stability diagram. To do so, we simply compute the determinant along lines of constant real tune shift $\Re(Q_c) - Q_{y0}$ in the absence of imaginary tune shift ($\Im(Q_c) = 0$), and get the one-dimensional minimum along such curves. At the exact location of the stability diagram (if it exists), this minimum should get to zero. This is illustrated in Fig. 1, in a case with only a transverse damper that is set in antidamper mode [25] i.e. with a phase above $\pi/2$ in order to create instabilities.

Doing this exercise for a set of configurations that span a large portion of the complex plane of possible unperturbed coherent tune shifts (i.e. tune shifts in the absence of tune spread) thus allows us to find a (potential) stability diagram by plotting as a color the minimum of such 1D curves. The way to find a set of configurations that span a large part of the complex plane is to use an ideal (bunch-by-bunch) damper with arbitrary phase and gain [25]. The unperturbed tune shifts that serve as abscissa and ordinate of the plot, are obtained by diagonalization of the eigenvalue problem in Eq. (56), using routines from the DELPHI [18] code with LHC-like parameters, in the absence of impedance (see Table 1 for details).

In Fig. 2 we show the result of this exercise, in a case where the chromaticity is zero. Looking at the brightest region (which represents the closest to zero minima of the aforementioned 1D curves), we clearly recover the standard stability diagram as obtained in e.g. [14]. On the other hand, for $Q' = 5$ we see in Fig. 3 that one deviates from the usual stability diagram for an unperturbed tune shift close to $Q_s$. Note, still, that one cannot really tell at this stage if the brightest region really represents a stability diagram, i.e. that it delimits the boundary between the unstable and the stable region of unperturbed tune shifts – this is under investigation.

## CONCLUSION

We have derived an extension of Sacherer integral equation in the case of two-dimensional transverse tune spread, generalizing several approaches in the literature. This allowed us to obtain both the usual eigenvalue problem and the stability diagram theory as limiting cases. The final determinant equation turns out to be very challenging to solve in general; still, preliminary results were obtained, in an attempt to generalize the concept of stability diagram.

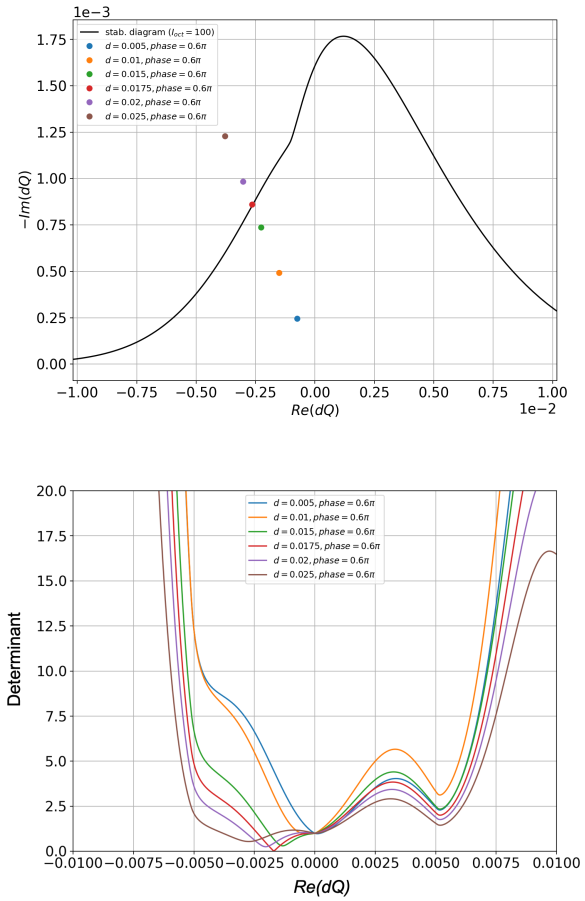


Figure 1: (Top) Unperturbed modes (color dots) for an ideal bunch-by-bunch damper at different transverse damper gains, at a given phase (here a phase of zero means a perfectly resistive damper, and phase of $\pi$ is a perfectly resistive anti-damper), at $Q' = 0$. The standard stability diagram for a Gaussian transverse distribution is also shown (black curve). (Bottom) Determinant of Eq. (53) plotted vs. real tune shift (for an imaginary tune shift of zero).

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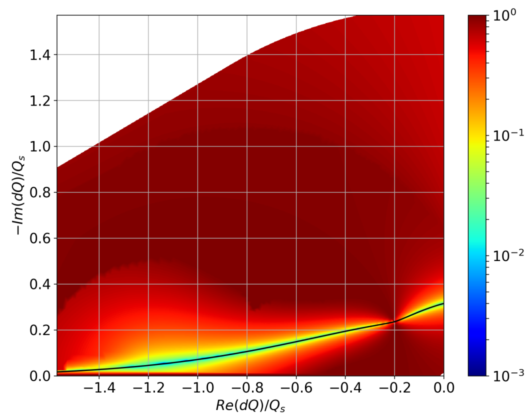


Figure 2: Minimum (color) of the 1D curve obtained when the determinant of Eq. (53) is plotted vs. real tune shift as in Fig. 1 (bottom). The x and y axes represent the coherent tune shift in the absence of tune spread, for an ideal bunch-by-bunch damper, with $Q' = 0$. The black curve represents the standard stability diagram theory (for a Gaussian distribution).

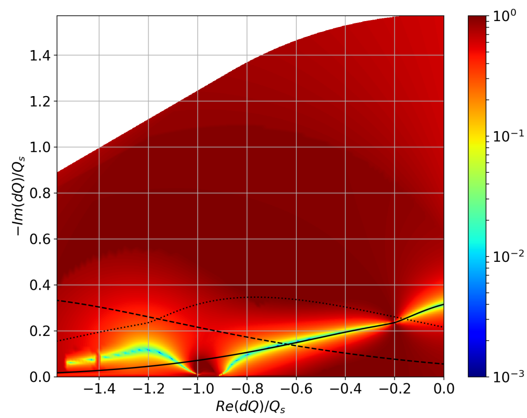


Figure 3: Minimum (color) of the 1D curve obtained when the determinant of Eq. (53) is plotted vs. real tune shift as in Fig. 1 (bottom). The x and y axes represent the coherent tune shift in the absence of tune spread, for an ideal bunch-by-bunch damper, with $Q' = 5$. The black curves represent the standard stability diagram theory (for a Gaussian distribution), replicated periodically every $Q_s$.

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Table 1: Parameters used.

| | |
|---|---|
| Number of bunches $M$ | 1 |
| Charge of each particle $q$ | $e$ |
| RMS bunch length $\sigma_z$ | 7.49 cm |
| Synchrotron tune $Q_s$ | $5.09096 \cdot 10^{-3}$ |
| Machine physical radius | 4242.89 m |
| Revolution frequency $\frac{\omega_0}{2\pi}$ | 11.2455kHz |
| Unperturbed tune $Q_{y0}$ | 62.31 |
| Direct detuning $a_{yy}$ | $7.67 \cdot 10^5$ |
| Indirect detuning $a_{xy}$ | $-5.54 \cdot 10^5$ |
| Normalized emittances $\varepsilon_x = \varepsilon_y$ | $1\mu$m |
| Relativistic $\gamma$ | 479.6 |
| Slippage factor $\eta$ | $3.43 \cdot 10^{-4}$ |


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