

SUPPRESSION OF THE FAST BEAM-ION INSTABILITY BY TUNE SPREAD IN THE ELECTRON BEAM DUE TO BEAM-BEAM EFFECTS

Gennady Stupakov*, SLAC National Accelerator Laboratory, Menlo Park, CA, USA

Abstract

The fast beam-ion instability (FII) is caused by the interaction of an electron bunch train with the residual gas ions. The ion oscillations in the potential well of the electron beam have an inherent frequency spread due to the nonlinear profile of the potential. However, this frequency spread and associated with it Landau damping typically is not strong enough to suppress the instability. In this work, we develop a model of FII which takes into account the frequency spread in the electron beam due to the beam-beam interaction in an electron-ion collider. We show that with a large enough beam-beam parameter the fast ion instability can be suppressed. We estimate the strength of this effect for the parameters of the eRHIC electron-ion collider.

INTRODUCTION

A fast beam-ion instability (FII) which is caused by the interaction of a single electron bunch train with the residual gas ions, has been proposed and studied theoretically in Refs. [1, 2]. The instability mechanism is the same in linacs and storage rings assuming that the ions are cleared in one turn. The ions generated by the head of the bunch train oscillate in the transverse direction and resonantly interact with the betatron oscillations of the subsequent bunches, causing the growth of an initial perturbation of the beam.

An important element that has to be included into the treatment of the instability is the frequency spread in the ion population due to the nonlinearity of the potential well for the trapped ions [2], as well as the spatial variation of the ion frequency along the beam path [3]. This frequency spread introduces the mechanism of Landau damping, or decoherence, but does not completely suppress the instability — it only makes it somewhat slower. In this work, we study another source of the decoherence in the fast ion instability originating from the tune spread in the electron beam. Such a tune spread may be due to the beam-beam collisions in a lepton or electron-ion collider.

For an analytical study we adopt a model that treats the bunch train as a continuous beam. This model is applicable if the distance between the bunches l_b is smaller than the betatron wavelength, $l_b \ll c/\omega_\beta$, and is also smaller than the ion oscillation wavelength, $l_b \ll c/\omega_i$. We assume a one-dimensional model that treats only vertical linear oscillation of the centroids of the beam and the ions. We treat the electron tune spread using the method developed in Ref. [4].

EQUATIONS OF MOTION

We use the notation $\tilde{y}_e(s, t|\omega_\beta)$ for the vertical offset of an electron in the beam that is characterized by the betatron

* stupakov@slac.stanford.edu

frequency ω_β , at time t and longitudinal position s . The distance s is measured from the injection point at $t = 0$. The equation for \tilde{y}_e , including the interaction with the ion background, is derived in Refs. [1, 2],

$$\begin{aligned} & \left(\frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^2 \tilde{y}_e(s, t|\omega_\beta) + \frac{\omega_\beta^2}{c^2} \tilde{y}_e(s, t|\omega_\beta) \\ & = (ct - s)\kappa [\bar{y}_i(s, t) - \bar{y}_b(s, t)]. \end{aligned} \quad (1)$$

The left-hand side of this equation accounts for the free betatron oscillations of a moving beam (we assume $v_{\text{beam}} \approx c$). On the right hand side, we included the force acting on the beam from the ions whose centroid is offset by $\bar{y}_i(s, t)$. The centroid of the electron beam is denoted by $\bar{y}_b(s, t)$. In the linear theory, which is the subject of this work, the interaction force between the electron beam and ions is proportional to the relative displacement between the beam and ions centroids; it is also proportional to the ion density. Assuming a continuous electron beam with a uniform density per unit length, the ion density increases due to collisional ionization as $ct - s$ behind the head of the beam (it is equal to zero before the beam head arrives at the point s at time $t = s/c$). After separating the factor $ct - s$ on the right hand side of Eq. (1), the coefficient κ is

$$\kappa \equiv \frac{4\lambda_{ion}r_e}{3\gamma c\sigma_y(\sigma_x + \sigma_y)}, \quad (2)$$

where γ is the relativistic factor of the beam, r_e is the classical electron radius, $\sigma_{x,y}$ denote the horizontal and vertical rms-beam size respectively, and λ_{ion} is the number of ions per meter generated by the beam per unit time. Assuming a cross section for collisional ionization of about 2 Mbarns (corresponding to carbon monoxide and the electron energy ~ 10 GeV), we have

$$\lambda_{ion} [\text{m}^{-1}\text{s}^{-1}] \approx 1.8 \cdot 10^9 n_e [\text{m}^{-1}] p_{gas} [\text{torr}], \quad (3)$$

where n_e is the number of electrons in the beam per meter, and p_{gas} the residual gas pressure in torr.

We assume that there is a betatron frequency spread in the electron beam due to the beam-beam interaction which is described by the distribution function $f_e(\omega_\beta)$ normalized by unity, $\int f_e(\omega_\beta) d\omega_\beta = 1$. The betatron frequency spread is assumed small, so that the function $f_e(\omega_\beta)$ is localized around the central frequency $\omega_{\beta 0}$. The centroid offset is obtained through averaging $\tilde{y}_e(s, t|\omega_\beta)$ with the help of the distribution function,

$$\bar{y}_b(s, t) = \int f_e(\omega_\beta) \tilde{y}_e(s, t|\omega_\beta) d\omega_\beta. \quad (4)$$

To find the equation for ions, we will assume that they perform linear oscillations inside the beam with a frequency

ω_i . Furthermore, we will allow a continuous spectrum of ω_i given by a distribution function $f_i(\omega_i)$ normalized so that $\int f_i(\omega_i) d\omega_i = 1$. The distribution $f_i(\omega_i)$ is peaked around the frequency $\omega_i = \omega_{i0}$ corresponding to small vertical oscillations on the axis,

$$\omega_{i0} \equiv \left[\frac{4n_e r_p c^2}{3A\sigma_y(\sigma_x + \sigma_y)} \right]^{1/2}, \quad (5)$$

where A designates the atomic mass number of the ions, n_e the number of electrons in the beam per unit length, and r_p the classical proton radius ($r_p \approx 1.5 \cdot 10^{-16}$ cm). Typically, the frequency spread $\Delta\omega_i$ is not large; we assume $\Delta\omega_i \ll \omega_{i0}$.

We have to distinguish between the ions generated at different times t' because they will have an initial offset equal to the beam coordinate $\bar{y}_b(s, t')$. Let us denote by $\tilde{y}_i(s, t|t', \omega_i)$ the displacement, at time t and position s , of the ions generated at t' ($t' \leq t$) and oscillating with the frequency ω_i . We have an oscillator equation for \tilde{y}_i

$$\frac{\partial^2}{\partial t^2} \tilde{y}_i(s, t|t', \omega_i) + \omega_i^2 [\tilde{y}_i(s, t|t', \omega_i) - \bar{y}_b(s, t)] = 0, \quad (6)$$

with the initial condition

$$\tilde{y}_i(s, t'|t', \omega_i) = \bar{y}_b(s, t'), \quad \left. \frac{\partial \tilde{y}_i}{\partial t} \right|_{t=t'} = 0. \quad (7)$$

Finally, averaging the displacement of the ions produced at different times t' and having different frequencies ω_i gives the ion centroid $\bar{y}_i(s, t)$,

$$\bar{y}_i(s, t) = \frac{1}{t - s/c} \int_{s/c}^t dt' \int d\omega_i f_i(\omega_i) \tilde{y}_i(s, t|t', \omega_i). \quad (8)$$

Equations (1), (6)-(8) constitute a full set of equations governing the development of the instability.

AVERAGING EQUATIONS

Equation (6) can be easily integrated with the initial conditions (7) yielding

$$\tilde{y}_i(s, t|t', \omega_i) = \bar{y}_b(s, t) - \int_{t'}^t \frac{\partial \bar{y}_b(s, t'')}{\partial t''} \cos \omega_i(t - t'') dt''. \quad (9)$$

Now using Eq. (8) and (9) in Eq. (1) we find an integro-differential equation for \tilde{y}_e ,

$$\begin{aligned} & \left(\frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^2 \tilde{y}_e(s, t|\omega_\beta) + \frac{\omega_\beta^2}{c^2} \tilde{y}_e(s, t|\omega_\beta) \\ & = -\kappa \int_{s/c}^t (ct' - s) \frac{\partial \bar{y}_b(s, t')}{\partial t'} D_i(t - t') dt', \end{aligned} \quad (10)$$

where $D_i(t - t')$ denotes the ion decoherence function defined as

$$D_i(t - t') = \int d\omega_i \cos \omega_i(t - t') f_i(\omega_i). \quad (11)$$

This function represents the oscillation of the centroid of an ensemble of ions with a given frequency distribution $f_i(\omega_i)$ having an initial unit offset. If there is no frequency spread in the beam ($f_i = \delta(\omega_i - \omega_{i0})$) we have $D_i(t) = \cos(\omega_{i0}t)$.

Instead of t and s , it is convenient to transform to new independent variables z and s , where $z = ct - s$. The variable z measures the distance from the head of the beam train and for a fixed z the variable s plays a role of time. Denoting

$$\begin{aligned} y(s, z) & \equiv \bar{y}_b \left(s, \frac{1}{c}(s + z) \right), \\ y_e(s, z|\omega_\beta) & \equiv \tilde{y}_e \left(s, \frac{1}{c}(s + z)|\omega_\beta \right), \end{aligned} \quad (12)$$

Eq. (10) takes the form

$$\begin{aligned} & \frac{\partial^2}{\partial s^2} y_e(s, z|\omega_\beta) + \frac{\omega_\beta^2}{c^2} y_e(s, z|\omega_\beta) \\ & = -\kappa \int_0^z z' \frac{\partial y(s, z')}{\partial z'} D_i[(z - z')/c] dz'. \end{aligned} \quad (13)$$

We will assume that the parameter κ that defines the interaction between the beam and the ions is small,

$$c^2 \kappa l \ll \omega_{i0}^2, \quad \omega_\beta^2, \quad (14)$$

where l denotes the length of the bunch train. This inequality means that the instability develops on a time scale that is much larger than both the betatron period and the period of ion oscillations. Typically this inequality is easily satisfied. In such a situation, the most unstable solution of Eq. (13) can be represented as a wave propagating in the beam with a slowly varying amplitude and phase,

$$y_e(s, z|\omega_\beta) = \text{Re} A_e(s, z|\omega_\beta) e^{-i\omega_\beta s/c + i\omega_{i0} z/c}, \quad (15)$$

where the complex amplitude $A_e(s, z|\omega_\beta)$ is a ‘slow’ function of its variables,

$$\left| \frac{\partial \ln A_e}{\partial s} \right| \ll \frac{\omega_\beta}{c}, \quad \left| \frac{\partial \ln A_e}{\partial z} \right| \ll \frac{\omega_{i0}}{c}. \quad (16)$$

For a fixed z , the s -dependence of Eq. (15) describes a pure betatron oscillation, while, for a fixed s (that is in the frame co-moving with the ions) the z -dependent part implies oscillations (in time t) with the frequency ω_{i0} . Hence the wave resonantly couples the oscillations of ions and electrons. Note that from Eq. (4) it follows that for the average offset of the electron beam we have

$$y(s, z) = \text{Re} A(s, z) e^{-i\omega_\beta s/c + i\omega_{i0} z/c}, \quad (17)$$

with

$$A(s, z) = \int f_e(\omega_\beta) A_e(s, z|\omega_\beta) d\omega_\beta. \quad (18)$$

Substituting Eq. (15) into Eq. (13) and averaging it over the rapid oscillations with the frequencies ω_{i0} and ω_β , we

can re-formulate Eq. (13) so that it describes a slow evolution of the complex function A_e ,

$$\begin{aligned} & \frac{\partial A_e(s, z|\omega_\beta)}{\partial s} + \frac{i}{c}(\omega_\beta - \omega_{\beta 0})A_e(s, z|\omega_\beta) \\ &= \frac{\kappa\omega_{i0}}{4\omega_{\beta 0}} \int_0^z z' A(s, z') \hat{D}_i(z - z') dz', \end{aligned} \quad (19)$$

where the function $\hat{D}_i(z)$ is

$$\hat{D}_i(z) = \int d\omega_i f_i(\omega_i) e^{i(\omega_i - \omega_{i0})z/c}. \quad (20)$$

Eqs. (18) and (19) constitute a full set of equations that we need to solve.

We can make one more step and formulate an equation for the amplitude of the averaged offset A . For this, we integrate Eq. (19) over s ,

$$\begin{aligned} A_e(s, z|\omega_\beta) &= A_e(0, z|\omega_\beta) e^{-i(\omega_\beta - \omega_{\beta 0})s/c} \\ &+ \frac{\kappa\omega_{i0}}{4\omega_{\beta 0}} \int_0^s ds' e^{-i(\omega_\beta - \omega_{\beta 0})(s-s')/c} \\ &\times \int_0^z z' A(s', z') \hat{D}_i(z - z') dz'. \end{aligned} \quad (21)$$

We now average this equation with the distribution function $f_e(\omega_\beta)$. It is reasonable to assume that the initial offset $A_e(0, z|\omega_\beta)$ does not depend on ω_β , and write it as $A_0(z)$. We then obtain

$$\begin{aligned} A(s, z) &= A_0(z) \hat{D}_e(s) + \frac{\kappa\omega_{i0}}{4\omega_{\beta 0}} \int_0^s ds' \hat{D}_e(s - s') \\ &\times \int_0^z z' A(s', z') \hat{D}_i(z - z') dz', \end{aligned} \quad (22)$$

where

$$\hat{D}_e(s) = \int d\omega_\beta f_e(\omega_\beta) e^{-i(\omega_\beta - \omega_{\beta 0})s/c}. \quad (23)$$

Note that from Eq. (22) follows the initial condition for function A ,

$$A(s, 0) = A_0(0) \hat{D}_e(s). \quad (24)$$

We can also re-write Eq. (22) as an integro-differential equation

$$\begin{aligned} \frac{\partial}{\partial s} A(s, z) &= A_0(z) \hat{D}'_e(s) + \frac{\kappa\omega_{i0}}{4\omega_{\beta 0}} \int_0^z z' A(s, z') \hat{D}_i(z - z') dz' \hat{D}'_e(s) \\ &+ \frac{\kappa\omega_{i0}}{4\omega_{\beta 0}} \int_0^s ds' \hat{D}'_e(s - s') \int_0^z z' A(s', z') \hat{D}_i(z - z') dz', \end{aligned} \quad (25)$$

where the prime denotes the derivative with respect to s , and we have used $\hat{D}_e(0) = 1$. The first and the third terms on the right-hand side vanish for a constant \hat{D}_e that corresponds to the case of the zero electron tune spread, and in this limit we recover the result of Ref. [2].

SOLUTION OF FII EQUATIONS FOR SPECIAL CASES

In this section will show how to solve Eq. (22) for the case when one can neglect the ion decoherence, $\hat{D}_i(z) = 1$. In this case Eq. (22) reduces to

$$\begin{aligned} A(s, z) &= A_0(z) \hat{D}_e(s) + \frac{\kappa\omega_{i0}}{4\omega_{\beta 0}} \\ &\times \int_0^s ds' \hat{D}_e(s - s') \int_0^z z' A(s', z') dz'. \end{aligned} \quad (26)$$

We first make the Laplace transform with respect to the variable s , introducing the Laplace image $a(\varkappa, z)$,

$$a(\varkappa, z) = \int_0^\infty A(s, z) e^{-\varkappa s} ds. \quad (27)$$

Making the Laplace transform of Eq. (26) we find

$$a(\varkappa, z) = A_0(z) d(\varkappa) + \frac{\kappa\omega_{i0}}{4\omega_{\beta 0}} d(\varkappa) \int_0^z z' a(\varkappa, z') dz', \quad (28)$$

where

$$d(\varkappa) = \int_0^\infty \hat{D}_e(s) e^{-\varkappa s} ds. \quad (29)$$

We can solve Eq. (28) analytically for the special case when the initial amplitude of the beam offset, A_0 , does not depend on z , $A_0 = \text{const}$. Differentiating Eq. (28) with respect to z and solving the resulting differential equation with the initial condition $a(\varkappa, 0) = A_0 d(\varkappa)$ gives the following result

$$a(\varkappa, z) = A_0 d(\varkappa) e^{qd(\varkappa)z^2}, \quad (30)$$

with $q = \kappa\omega_{i0}/8\omega_{\beta 0}$. Making the inverse Laplace transform, we can find A ,

$$A(s, z) = A_0 \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} d\varkappa e^{-\varkappa s} d(\varkappa) e^{qd(\varkappa)z^2}. \quad (31)$$

In the case when the tune spread in the electron beam is so small that it can be neglected, we have $\hat{D}_e(s) = 1$ and $d(\varkappa) = 1/\varkappa$. We arrive at the integral

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\varkappa}{\varkappa} e^{-\varkappa s + qz^2/\varkappa} = I_0(2z\sqrt{qs}), \quad (32)$$

and $A = A_0 I_0(2z\sqrt{qs})$. This is the result of Refs. [1, 2] when the ion frequency spread is neglected.

Consider now a model electron decoherence function $\hat{D}_e(s) = e^{-ps}$ with $p > 0$, for which we have $d(\varkappa) = 1/(\varkappa + p)$. For the integral we have

$$\begin{aligned} A(s, z) &= \frac{A_0}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\varkappa}{\varkappa + p} e^{-\varkappa s + qz^2/(\varkappa + p)} \\ &= A_0 e^{-ps} I_0(2z\sqrt{qs}). \end{aligned} \quad (33)$$

Asymptotically, in the limit $s \rightarrow \infty$, we have $I_0(2z\sqrt{qs}) \propto \exp(2z\sqrt{qs})/\sqrt{2z\sqrt{qs}}$, and the exponential factor e^{-ps} overcomes the growing Bessel function, and hence, suppresses the instability.

DECOHERENCE DUE TO BEAM-BEAM COLLISIONS

The decoherence function $\hat{D}_e(s)$ for the case when the betatron tune spread is due to the beam-beam collisions at the interaction point in a collider was derived in Ref. [5]. Assuming round beams at the interaction point, the following expression for $\hat{D}_e(s)$ was obtained:

$$\hat{D}_e(s) = 4 \int_0^\infty \int_0^\infty da_1 da_2 \times \exp \left[-2(a_1 + a_2) + i(s/cT)\Delta v_y(a_1, a_2) \right], \quad (34)$$

where T is the revolution period in the ring and the tune shift $\Delta v(a_1, a_2)$ is given by the following formula [6],

$$\Delta v = \xi \int_0^1 du e^{-u(a_1+a_2)} I_0(a_2 u) [I_0(a_1 u) - I_1(a_1 u)].$$

Here a_1 and a_2 are the dimensionless amplitudes of the betatron oscillations, $I_n(z)$ is the modified Bessel function of the n -th order and ξ is the tune shift parameter, $\xi = N_p r_e / 4\pi\epsilon$ with N_p the number of particles in the bunch, r_e the classical electron radius and ϵ the normalized beam emittance. The tune shift Δv is positive due to the opposite signs of the charges of the colliding beams in an electron-ion collider. The plot of function $\hat{D}_e(s)$ is shown in Fig. 1.

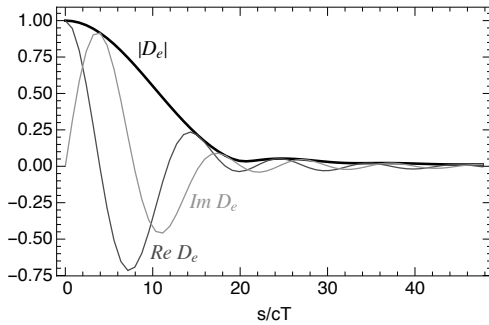


Figure 1: Plot of the real (black) and imaginary (magenta) parts of the function $\hat{D}_e(s)$. The black line is the absolute value $|\hat{D}_e(s)|$.

NUMERICAL SOLUTION OF EQ. (25)

Here we outline the numerical solution of (25) following the method proposed in Ref. [7]. We first normalize the variable z by the length of the bunch train l , $\zeta = z/l$, and then replace s in this equation by $\xi = s(\kappa\omega_{i0}l^2/4\omega_{\beta 0})$. Note that positions within the bunch train correspond to the interval $0 < \zeta < 1$. We then have

$$\begin{aligned} \frac{\partial}{\partial \xi} A(\xi, \zeta) &= A_0(\zeta) \hat{D}'_e(\xi) + \int_0^\zeta \zeta' A(\xi, \zeta') \hat{D}_i(\zeta - \zeta') d\zeta' \\ &+ \int_0^\xi d\xi' \hat{D}'_e(\xi - \xi') \int_0^\zeta \zeta' A(\xi', \zeta') \hat{D}_i(\zeta - \zeta') d\zeta', \end{aligned} \quad (35)$$

where \hat{D}'_e now denotes the derivative of \hat{D}_e with respect to ξ .

We first introduce a mesh in the unit interval $0 < \zeta < 1$, $\zeta_k = k\Delta$, $k = 1, \dots, n$, with $\Delta = 1/(n-1)$. The function $A(\xi, \zeta)$ is now represented on this mesh, $A(\xi, \zeta_k)$, and we use the trapezoidal integration rule to carry out the integration over ζ in Eq. (25),

$$\begin{aligned} &\int_0^\zeta \zeta' A(\xi', \zeta') \hat{D}_i(\zeta - \zeta') d\zeta' \\ &= \frac{\Delta}{2} [\zeta_1 A(\xi', \zeta_1) + 2\zeta_2 A(\xi', \zeta_2) + \dots \\ &+ 2\zeta_{n-1} A(\xi', \zeta_{n-1}) + \zeta_n A(\xi', \zeta_n)]. \end{aligned} \quad (36)$$

We will use the notation $\mathcal{A}(\xi)$ for the vector $A(\xi, \zeta_k)$ and denote the discretized integration (36) by the operator T ,

$$\int_0^\zeta \zeta' A(\xi', \zeta') \hat{D}_i(\zeta - \zeta') d\zeta' \rightarrow T \cdot \mathcal{A}. \quad (37)$$

Then Eq. (35) can be written as

$$\mathcal{A}'(\xi) = \mathcal{A}_0 \hat{D}'_e(\xi) + T \cdot \mathcal{A}(\xi) + \int_0^\xi d\mu \mathcal{D}(\xi, \mu) T \cdot \mathcal{A}(\mu), \quad (38)$$

where to simplify the notation we introduced $\mathcal{D}(\xi, \mu) \equiv \hat{D}'_e(\xi - \mu)$.

Introducing the step h in variable ξ we denote by \mathcal{A}_k the value of \mathcal{A} at $\xi_k = h(k-1)$. Likewise \mathcal{A}'_k denotes the value of $\partial \mathcal{A} / \partial \xi|_{\xi=\xi_k}$. Again, using the trapezoidal method of integration, we find

$$\begin{aligned} \mathcal{A}'_k &= \mathcal{A}_0 \hat{D}'_e(\xi_k) + T \cdot \mathcal{A}_k + \frac{h}{2} [\mathcal{D}(\xi_k, \xi_1) T \cdot \mathcal{A}_1 \\ &+ 2\mathcal{D}(\xi_k, \xi_2) T \cdot \mathcal{A}_2 + \dots + 2\mathcal{D}(\xi_k, \xi_{k-1}) T \cdot \mathcal{A}_{k-1} \\ &+ 2\mathcal{D}(\xi_k, \xi_k) T \cdot \mathcal{A}_k]. \end{aligned} \quad (39)$$

To advance the step from $\xi = \xi_k$ to $\xi = \xi_k + h$ we first integrate (38) to obtain

$$\begin{aligned} \mathcal{A}_{k+1} &= \mathcal{A}_k + \mathcal{A}_0 [\hat{D}_e(\xi_{k+1}) - \hat{D}_e(\xi_k)] + \int_{\xi_k}^{\xi_{k+1}} d\xi T \cdot \mathcal{A}(\xi) \\ &+ \int_{\xi_k}^{\xi_{k+1}} d\xi \int_0^\xi d\mu \mathcal{D}(\xi, \mu) T \cdot \mathcal{A}(\mu) \\ &= \mathcal{A}_k + \mathcal{A}_0 [\hat{D}_e(\xi_{k+1}) - \hat{D}_e(\xi_k)] + I_1 + I_2. \end{aligned} \quad (40)$$

For I_1 we use the trapezoidal rule together with the approximation $\mathcal{A}_{k+1} = \mathcal{A}_k + h\mathcal{A}'_k$ to yield

$$I_1 = \frac{h}{2} [T \cdot \mathcal{A}_k + T \cdot (\mathcal{A}_k + h\mathcal{A}'_k)]. \quad (41)$$

For I_2 we use the trapezoidal rule for the outer integral to obtain

$$\begin{aligned} I_2 &= \frac{h}{2} \left[\int_0^{\xi_k} d\mu \mathcal{D}(\xi_k, \mu) T \cdot \mathcal{A}(\mu) \right. \\ &\left. + \int_0^{\xi_{k+1}} d\mu \mathcal{D}(\xi_{k+1}, \mu) T \cdot \mathcal{A}(\mu) \right]. \end{aligned} \quad (42)$$

We then use the trapezoidal rule for the inner integral and again use the approximation $\mathcal{A}_{k+1} = \mathcal{A}_k + h\mathcal{A}'_k$ to obtain

$$\begin{aligned}
 I_2 = & \frac{h^2}{4} [\mathcal{D}(\xi_k, \xi_1)T \cdot \mathcal{A}_1 + 2\mathcal{D}(\xi_k, \xi_2)T \cdot \mathcal{A}_2 \dots \\
 & + 2\mathcal{D}(\xi_k, \xi_{k-1})T \cdot \mathcal{A}_{k-1} + \mathcal{D}(\xi_k, \xi_k)T \cdot \mathcal{A}_k] \\
 & + \frac{h^2}{4} [\mathcal{D}(\xi_{k+1}, \xi_1)T \cdot \mathcal{A}_1 + 2\mathcal{D}(\xi_{k+1}, \xi_2)T \cdot \mathcal{A}_2 \dots \\
 & + 2\mathcal{D}(\xi_{k+1}, \xi_k)T \cdot \mathcal{A}_k + \mathcal{D}(\xi_{k+1}, \xi_{k+1})T \cdot (\mathcal{A}_k + h\mathcal{A}'_k)].
 \end{aligned} \quad (43)$$

Eqs. (40), (41) and (43) finalize the one step advance in the numerical solution of Eq. (25).

FII AT ERHIC

In this section we will analyze the fast ion instability for the electron storage ring of the proposed Electron Ion Collider at BNL, eRHIC [8]. The parameters of eRHIC electron beam relevant for the fast ion instability are summarized in Table 1. The nominal tune shift for the electron beam is $\xi = 0.1$.

Electron beam energy	10 GeV
Vertical beam emittance, ϵ_y	4.9 nm
Horizontal beam emittance, ϵ_x	20 nm
Residual gas pressure, p	0.75 nTorr
Averaged beta function, β_x, β_y	18 m
Vertical betatron tune, ν_y	31.06
Number of electron bunches, N_b	567
Length of the bunch train, l_b	3451 m
Atomic mass number for ions, A	28
Number of electrons per unit length, n_e	$5.6 \times 10^{10} \text{ m}^{-1}$

Table 1: Parameters of the eRHIC collider relevant for FII.

Using the beam emittance and the value for the averaged beta functions we find the characteristic beam sizes in the vertical and horizontal directions, $\sigma_y = 0.3 \text{ mm}$ and $\sigma_x = 0.6 \text{ mm}$. From Eq. (5) we obtain the ion frequency $\omega_{i0} = 4.5 \times 10^7 \text{ s}^{-1}$ and for the vertical betatron frequency we calculate $\omega_\beta = 1.5 \times 10^7 \text{ s}^{-1}$. From Eq. (3) it follows that $\lambda = 7.5 \times 10^{10} \text{ m}^{-1}\text{s}^{-1}$ and the characteristic time of FII [2] is

$$\tau = \frac{4\omega_{\beta 0}}{\kappa\omega_{i0}cI^2} = 2.1 \mu\text{s}. \quad (44)$$

In our calculations we assume the worst case scenario when all residual gas pressure is due to the carbon monoxide (lighter ions are usually less trapped inside the beam).

Note that the parameter $c/\omega_\beta \approx 20 \text{ m}$ is several times larger than the distance between the electron bunches, $l_b = 6.1 \text{ m}$, but $c/\omega_{i0} = 6.5 \text{ m}$ is comparable with l_b , which means that, for the parameters of eRHIC, the model of continuous electron beam that we use in this paper is actually at the edge of its applicability range.

We first simulated the fast ion instability for eRHIC parameters neglecting the electron decoherence by taking into

account only ion decoherence effects. This case is described by Eq. (25) with $\hat{D}_e(s) = 1$. For the ion decoherence function $\hat{D}_i(z)$, following Ref. [2], we took

$$\hat{D}_i(z) = \left(1 + \frac{i}{4c}\omega_{i0}z\right)^{-1/2}. \quad (45)$$

Fig. 2 shows the plot of the FII amplitude obtained by numerical solution of Eq. (25) for this case at different locations along the bunch train. This plot shows that the amplitude A

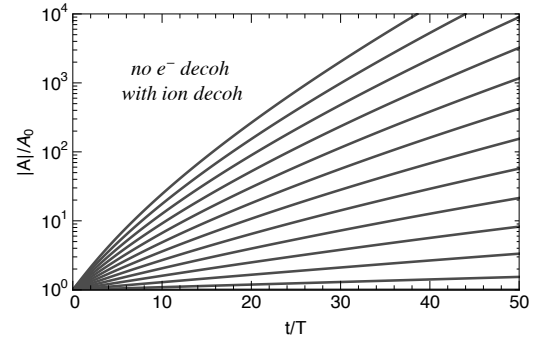


Figure 2: Amplitude A normalized by its initial value A_0 at 13 equidistant positions in the bunch train as a function of time measured in the revolution periods T in the ring. Electron decoherence effects are neglected. The amplitude grows faster with increase of the distance from the head of the train.

at the end of the electron bunch train grows more than four orders of magnitude after 50 revolution periods in the ring.

Using Fig. 2 as a reference case, we then simulated FII with account of the electron beam decoherence. This case is described by Eq. (25) with the electron decoherence function (34). For the ion decoherence function we again used Eq. (45). The result is shown in Fig. 3. One can see that

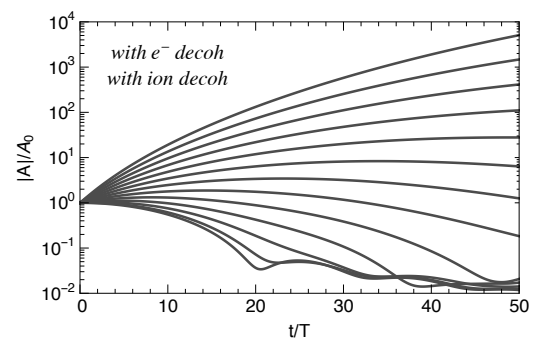


Figure 3: Amplitude A normalized by its initial value A_0 at 13 equidistant positions in the bunch train as a function of time measured in the revolution periods T in the ring. Electron decoherence effects are taken into account.

the electron decoherence suppresses the instability in the

region close to the head of the bunch train (the lowest 4-5 lines in the plot corresponding to positions near the head). However, the amplitude A still grows to unacceptably large values at the tail of the train. Simulations show that if the numerical solution is continued to even larger values of t , the amplitude A starts to decrease, but at its maximum at intermediate times, it is amplified by many orders of magnitude relative to the initial value. We conclude that while electron decoherence effects do provide some stabilization effect, it is not sufficient, for the nominal parameters of Table 1 (and 100% carbon monoxide residual gas), to fully suppress FII.

Finally, we repeated the previous simulation, but with 3 times smaller residual gas pressure, $p = 0.25$ nTorr. The result is shown in Fig. 4. This case can be characterized as a

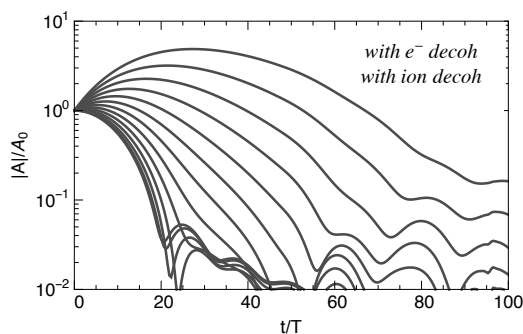


Figure 4: Amplitude A normalized by its initial value A_0 at 13 equidistant positions in the bunch train as a function of time measured in the revolution periods T in the ring. Electron decoherence effects are taken into account. The residual gas pressure is $p = 0.25$ nTorr.

stable one: after some increase in the tail of the bunch train the amplitude is suppressed through the electron decoherence effects to the values below the initial value A_0 .

SUMMARY

In this paper, we extended the theoretical analysis of Ref. [2] of the fast ion instability to include decoherence effects due to the tune spread in the electron beam. Specifically, we calculated the electron decoherence function for the case when the tune spread is caused by the beam-beam collisions in an electron-ion collider. We derived an equation that governs the evolution of the amplitude of the transverse oscillations in the beam, and numerically solved it for the nominal parameters of the eRHIC collider. We found that while the electron decoherence weakens the instability, it does not fully suppress it for the nominal parameters of eRHIC. The instability however is suppressed for three times smaller residual gas pressure.

We note that our model of continuous electron beam is not fully applicable for eRHIC parameter because the distance between the electron bunches is of the same order as the parameter c/ω_{i0} .

Qualitatively similar results have been recently obtained by M. Blaskiewicz [9] who used computer simulations of FII for eRHIC.

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