VLASOV EIGENFUNCTION ANALYSIS OF SPACE-CHARGE AND BEAM-BEAM EFFECTS*

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Abstract

Space-charge and beam-beam interaction affect both incoherent and coherent motion of particles potentially leading to instabilities and deterioration of the beam parameters. An overview of these phenomena will be given with an emphasis on the observable spectral characteristics and the mitigation methods of their harmful effects.

INTRODUCTION

With the advent of supercomputers with thousands of cores massive tracking simulations became the main method for studying intensity-related effects such as beam-beam and space-charge effects. Still, other methods can be useful to get insight in the observed phenomena and for preliminary study of mitigation strategies.

This report is devoted to the eigenvalue analysis of the Vlasov equation which fills an important niche between analytical calculations and tracking simulations providing the insight of the former and accuracy of the latter.

In the context of the beam-beam interaction this approach was successfully used in Ref. [1] and further developed by the author of this report [2, 3].

An extension of the method on the coherent modes in space-charge dominated beam will be discussed here for the case of a Gaussian beam in a parabolic RF well.

COHERENT BEAM-BEAM MODES

Traditionally the nonlinearity of the unperturbed motion is taken into account as the amplitude dependence of incoherent tunes entering the dispersion relation. However, in the case of strong-strong beam-beam interaction the deformation of the charge distribution of the beams also should be taken into account [1].

Let us emphasize that perturbation is considered infinitesimal, it is the unperturbed single particle oscillations that are nonlinear. Linearizing w.r.t. the perturbation the Liouville equation with self-consistent electromagnetic forces we come to the Vlasov equation.

For a Gaussian unperturbed distribution F_0 the Vlasov equation for perturbation F_1 in the action-angle variables $\underline{I} = (I_x, I_y, I_s)$, $\underline{\psi} = (\psi_x, \psi_y, \psi_s)$ has the form

$$i\frac{\partial}{\partial\theta}F_1^{(k)} = \hat{A}F_1^{(k)} = -i\underline{Q}^{(k)}\frac{\partial}{\partial\underline{\psi}}F_1^{(k)} - i\frac{r_p N_{3-k}}{2\pi\gamma}F_0\underline{\varepsilon}^{-1}\cdot\frac{\partial}{\partial\underline{\psi}}\int G^{(k)}F_1^{(3-k)}d^3I'\,d^3\psi' \quad (1)$$

where θ is the generalized azimuth, k=1,2 is the beam number, N_k is the number of particles per bunch, $\underline{Q}^{(k)} = (Q_x^{(k)},\ Q_y^{(k)}, -Q_s^{(k)})$, $\underline{\varepsilon}^{-1} = (\varepsilon_x^{-1},\ \varepsilon_y^{-1},\ \varepsilon_s^{-1})$ being the incoherent tunes and emittances in all three planes.

By performing Fourier expansion in the angle variables we obtain from Eq. (1) a system of equations (generally coupled) for the Fourier components of F_1. Outside of the resonances the coupling of the modes can be neglected so that they can be treated independently.

Van Kampen modes

Yokoya et al. [1] showed that the spectrum of operator $\hat{A}$ – let us call it the Vlasov operator – includes continuum covering the range of single particle tunes and possibly some discrete values lying outside the continuum. In particular, they found that out of phase dipole oscillations (π-mode) of two round beams with equal sizes, intensities and bare lattice tunes have the tuneshift 1.214 times the maximum (by absolute value) incoherent tuneshift, ξ, raising question of stability of this mode.

In-phase dipole oscillations (Σ-mode) also have mixed spectrum: a discrete value corresponding to a rigid bunch oscillations unaffected by the beam-beam interaction and a continuum covering the same range of incoherent tunes.

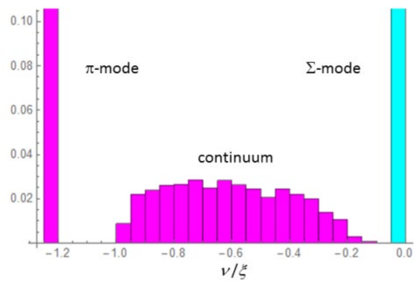

Figure 1: Spectrum of dipole beam-beam oscillations of particles with the same charge sign.

A crude picture of the spectrum of oscillations excited by a dipole kick is presented in Fig. 1 with the discrete mode peaks cut for better continuum visibility. Out of the Σ-modes only the discrete one can be seen, the continuum Σ-modes – being orthogonal to it – cannot be excited by a dipole kick.

The continuum modes – despite the coincidence of their spectrum with single particle tunes range – are truly coherent modes involving all particles in the bunch. However, they have a δ-function like singularity which does not permit to use smooth basis functions in the action variable space [2].

The physical significance of the continuum modes is

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that they describe Landau damping as it was shown by Van Kampen in the case of plasma oscillations [4].

The Vlasov eigenfunction method not only correctly predicts the spectrum, but allowed to understand why the coherent beam-beam modes were not always seen in practice. First of all, for the strong-strong regime to occur the parameters of the colliding beams should be close, in particular the weak/strong intensity ratio should exceed 0.65 for round beams [2]. Another natural mechanism that may be also at play is Landau damping by the synchrotron sidebands of incoherent tunes [3].

LHC example

A number of cures were proposed to suppress the discrete modes or move them inside the continuum, among them a split in bare lattice tunes between the two rings and a difference in phase advances separating two main IPs in each ring. Also, the effect of the long-range collisions can be minimized with alternating crossing: horizontally at one IP and vertically at the other.

In LHC the difference between IP1 → IP5 phase advances that particles see in ring 1 and ring 2 are 0.54 π horizontally and -0.18 π vertically. The alternating crossing is also implemented with 28 long-range collisions around each IP at ≈ 9.5σ average separation.

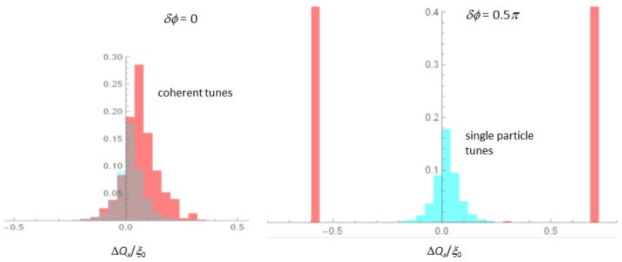

Figure 2: End-of-squeeze coherent beam-beam spectra (burgundy) and single particle tune distribution (cyan) at indicated values of the phase advance difference. The discrete mode peaks are cut.

Analysis showed that these two measures fight each other as far as it concerns the coherent modes. Figure 2 shows the end-of-squeeze single particle tune distribution and spectra of coherent oscillations with and without phase advance difference in units of the head-on beam-beam parameter ξ_0. The neighbouring long-range interactions were lumped at each IP.

Without phase advance difference there is no discrete modes, while with the difference as large as in the horizontal case there are two peaks of coherent oscillations well separated from incoherent tunes.

This result suggests that the horizontal oscillations are more prone to the end-of-squeeze instability. It can be damped by the transverse feedback.

SPACE-CHARGE MODES

The main distinction of the Vlasov eigenfunction method is treating the beams as transversely soft. It can be dubbed – making provision for the head-tail modes not discussed yet – as the soft-slice approach while the traditional approach is to consider the longitudinal slices transversely rigid.

In the case of a coasting beam with space-charge the coherent transverse oscillations are similar to the beam-beam Σ-mode. The only observable mode is "rigid-slice" mode with tune not shifted by the direct space-charge. In absence of other tuneshifts it is not Landau damped.

For treatment of bunched beam modes it is important that the transverse space-charge force is longitudinally local. In a sufficiently long bunch the locality can be described by δ-function. Mathematically this makes the use of action-angle variables in the longitudinal plane cumbersome. Instead we have to stay with coordinate and momentum.

Introducing normalized variables $\tau = z/\sigma_z$, $\upsilon = (p - p_0)/\sigma_p$, $J_x = I_x / \varepsilon_x$ we will search for the perturbed distribution function in form

$$F_1 = e^{i\psi_x - J_x/2 - (\tau^2+\upsilon^2)/2} f(J_x,\tau,\upsilon;\theta)/(2\pi)^2\varepsilon_x\varepsilon_z + c.c. \quad (2)$$

and introduce the Vlasov operator

$$\begin{aligned} &\hat{A}_0 f = -iQ_s\left(\upsilon\frac{\partial}{\partial\tau} - \tau\frac{\partial}{\partial\upsilon}\right)f + Q_x^{(\mathrm{ext})} f + \\ &e^{-\tau^2/2}\left[Q_x^{(SC)}(J_x)f + \frac{1}{\sqrt{2\pi}}\hat{G}\int_{-\infty}^{\infty} e^{-\upsilon'^2/2} f(J_x,\tau,\upsilon';)d\upsilon'\right] \end{aligned} \quad (3)$$

where in the case of horizontal oscillations in flat beam (see Ref.[1] for details)

$$\begin{aligned} &Q_x^{(SC)} = -Q_{\mathrm{SC}}\cdot(1-e^{-J_x})/J_x, \\ &\hat{G}f = Q_{\mathrm{SC}}\int_0^{\infty} e^{-(J_x+J_x')/2}\sqrt{J_x^{<}/J_x^{>}}\, f(J_x')dJ_x' \end{aligned} \quad (4)$$

Longitudinally – in absence of external impedances at least – operator (3) is well behaved and allows expansion in smooth basis functions. For a Gaussian bunch we can use the set

$$\Phi_k(\tau) = \sqrt{\frac{2k+1}{\sqrt{2\pi}}}P_k[\mathrm{erf}(\frac{\tau}{\sqrt{2}})], \quad k = 0,1,...,\infty \quad (5)$$

where $P_k(u)$ are the Legendre polynomials, which is orthonormal with weight w=exp(-τ^2/2). Figure 3 shows a few basis functions which look like the head-tail waveforms found in Ref. [5] for the rigid slice model.

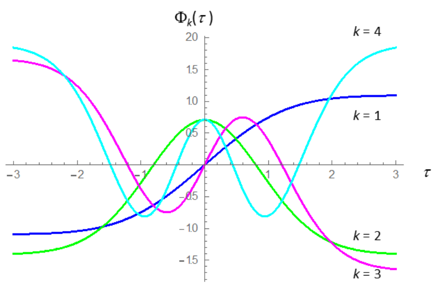

Figure 3: Longitudinal basis functions.

Using this basis the Vlasov operator eigenfunctions can be sought as

$$\underline{v}_m(J_x,\tau,\upsilon)=\sum_{k,l} a_{m;k,l}(J_x)\Phi_k(\upsilon)\Phi_l(\tau) \qquad (6)$$

The corresponding eigenvalues will be denoted as λ_m.

Landau Damping of the Head-Tail Modes

Just like it the case of the coherent beam-beam oscillations [1] the spectrum may contain a discrete set as well as continuum, the latter covering the range of incoherent tunes. A qualitative necessary condition of stability is absence of discrete spectrum.

This condition can be visualized with the help of spectral coefficients describing the projection of eigenmodes on a pickup and their excitation by a dipole kick varying longitudinally as $\Phi_l(\tau)$

$$c_{m;l}=\int_0^\infty \mathcal{R}(J_x)a_{m;0,l}(J_x)dJ_x \qquad (7)$$

where $\mathcal{R}(J_x)=\sqrt{J_x}e^{-J_x/2}$ is function describing horizontal "rigid-slice" motion.

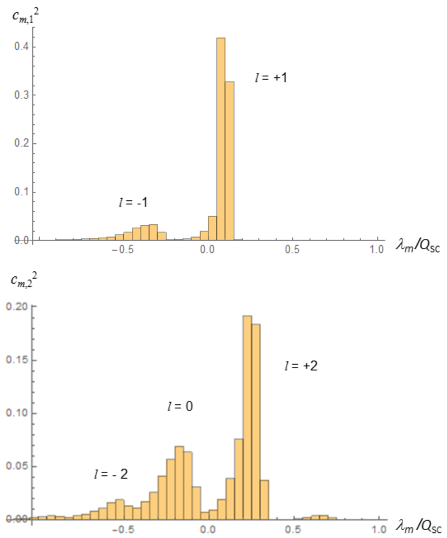

Figure 4: Spectral density of head-tail modes projection on l_0=1 and l_0=2 basis functions for Q_s = 0.2·Q_{SC}, Q_{SC} being the maximum absolute value of the SC tuneshift.

As Figure 4 shows other head-tail modes of the same parity have projection on the given basis function*, not only $l = l_0$.

The peak positions and the Landau damping rate estimated from the width of the peaks coincide almost exactly with Ref. [6] results obtained by tracking.

Since the $l \neq 0$ modes are intrinsically damped the main concern is Landau damping of the $l = 0$ mode. The possibility of employing an electron lens for this purpose was discussed at this Workshop [7].

TMCI with Strong Space Charge

The main mechanism of the single bunch transverse instability is TMCI. There was a long-standing question whether a strong space charge can suppress the TMCI (see e.g. Refs. [5, 8] and references therein).

To address the issue the following term originating from an external impedance is added to the Vlasov operator:

$$\hat{A}f=\hat{A}_0 f+i\sqrt{\frac{1}{2}\beta_x J_x}e^{-J_x/2}\frac{\delta p_x}{2\pi} \qquad (8)$$

with β_x being taken at the impedance location. The kick is produced by the wake function $W_\perp$

$$\frac{\delta p_x}{2\pi}=-\frac{e^2N}{mc^2\beta_0^2\gamma_0}\int_{-\infty}^{\infty}W_\perp(\tau-\tau')\langle x(\tau')\rangle\frac{1}{\sqrt{2\pi}}e^{-\tau'^2/2}d\tau' \qquad (9)$$

where N is the full number of particles in the bunch, β_0 is the average velocity in units of the speed of light c and γ_0 is the relativistic mass factor.

The causality requires $W_\perp(\tau)=0$ for $\tau>0$ breaking reciprocity and leading to the emergence of complex eigenvalues.

Vanishing TMCI threshold

Analysis in the simplest case of a step-like wake [9] showed that the growth rates go down with increase in the ratio Q_{SC}/Q_s but the threshold in $|N{\cdot}W_\perp|$ does not go to infinity as was suggested on the basis of the rigid-slice model (see e.g. [8]) but on the contrary goes to zero†.

Qualitatively the same behaviour was observed with a resonator wake. Figure 5 shows obtained by the described method growth rates for the SPS Q26 lattice and beam parameters and Rs=10MΩ/m (CERN units).

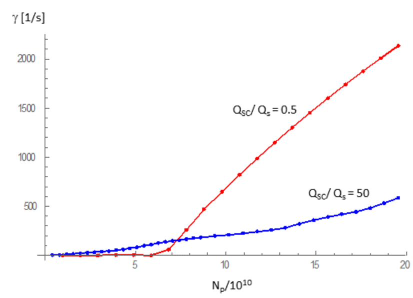

Figure 5: TMCI growth rate at the indicated values of the space charge strength.

At large ratio Q_{SC}/Q_s there is no well-defined threshold so that instead of "vanishing TMCI" we have a vanishing TMCI threshold. Similar results were independently obtained by tracking simulations [10, 11].

Tail-to-head feedback

We come to a conclusion that additional (w.r.t. the rigid-slice model) degree of freedom introduces

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* It is interesting that in projection on the l_0=2 basis function we see the l=0 continuum modes, while in projection on the l_0=0 function only the discrete peak at zero tuneshift can be seen, just like in the case of the beam-beam Σ-mode.

† The main result of Ref. [8] is that a large number of longitudinal eigenfunctions should be taken into account. In the soft-slice approach developed here the convergence is better, especially with smooth wakes.

qualitatively new effects. To understand them let us look at the obtained solutions in more detail.

Figure 6 shows the dipole moment $d_x = \langle x \rangle \cdot \exp(-\tau^2/2)$ along the bunch at a number of close moments in time in the strong space charge case of Fig. 5. The oscillation amplitude grows significantly from head to tail as in the case of the convective instability [12]. However, there is an appreciable growth rate: Im $\nu/Q_s = 0.21$. This means that there is a feedback from tail to head which is absent in the rigid-slice model.

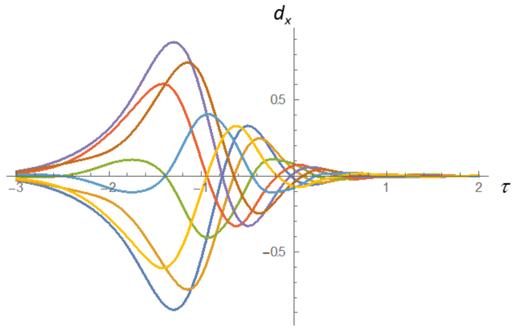

Figure 6: Dipole moment of the most unstable mode at $Q_{SC}/Q_s = 50$. The head of the bunch is at $\tau > 0$.

Figure 7 shows the eigenfunction of the mode presented in Fig. 6 as a function of the normalized action J_x at positions in the head (blue), center (green) and the tail (red) of the bunch. For comparison the dashed line shows function $\mathcal{R}(J_x)$ which describes the rigid-slice mode and was rescaled for convenience.

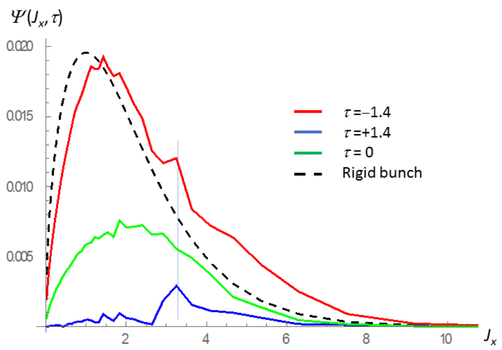

Figure 7: Eigenfunction of the most unstable mode at $Q_{SC}/Q_s = 50$ at the indicated positions inside the bunch.

The point to note here is that particles with larger unperturbed amplitudes participate stronger in coherent oscillations. This is especially noticeable at the head of the bunch. Of course it is not the unperturbed amplitude *per se* that matters but the reduced space-charge tuneshift.

It appears that particles with J_x marked in Fig. 7 by vertical line transfer perturbation from tail to head making the instability absolute. This mechanism was revealed by A. Burov and called by him the core-halo instability [13].

Practical conclusion

The fact that TMCI with strong space-charge requires for development the participation of large-amplitude particles with significantly reduced space-charge tuneshift suggests a cure: transverse KV distribution which equalizes the tuneshifts within a slice. Still – due to the longitudinal modulation – there will remain a tunespread to suppress other types of transverse instabilities.

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