Chapter B

Precision calculations in the Standard Model

1 $\alpha_{\mathrm{QED,\,eff}}(s)$ for precision physics at the FCC-ee/ILC

Contribution* by: F. Jegerlehner [fjeger@physik.hu-berlin.de]

Discovering the 'physics behind precision' at future linear or circular colliders (ILC or FCC projects) requires improved SM predictions based on more precise input parameters. I will review the role of $\alpha_{\mathrm{QED,\,eff}}$ at future collider energies and report on possible progress based on results from low-energy machines.

1.1 $\alpha(M_Z^2)$ in precision physics (precision physics limitations)

Uncertainties of hadronic contributions to the effective fine structure constant $\alpha \equiv \alpha_{\mathrm{QED}}$ are a problem for electroweak (EW) precision physics. Presently, we have α, G_μ, and M_Z as the most precise input parameters, which, together with the top Yukawa coupling y_t, the Higgs self-coupling λ, and the strong interaction coupling α_s allow us to make precision predictions for the particle reaction cross-sections encompassed by the Standard Model (SM). The cross-section data unfolded form detector and photon radiation resolution effects are often conveniently representable in terms of so-called pseudo-observables, such as $\sin^2\Theta_f$, v_f, a_f, M_W, Γ_Z, $\Gamma_W, \ldots$, as illustrated in Fig. B.1.1.

Because of the large 6% relative correction between α in the classical limit and the effective value $\alpha(M_Z^2)$ at the Z mass scale, where 50% of the shift is due to non-perturbative hadronic effects, one is losing about a factor of five orders of magnitude in precision. Nevertheless, for the vector boson Z and W, top quark, and Higgs boson precision physics possible at future e^+e^- colliders, the best effective input parameters are given by $\alpha(M_Z), G_\mu$, and M_Z. The effective $\alpha(s)$ at a process scale $\sqrt{s}$ is given in terms of the photon vacuum polarisation (VP) self-energy correction $\Delta\alpha(s)$ by

$$\alpha(s) = \frac{\alpha}{1-\Delta\alpha(s)}\,; \qquad \Delta\alpha(s) = \Delta\alpha_{\mathrm{lep}}(s) + \Delta\alpha^{(5)}_{\mathrm{had}}(s) + \Delta\alpha_{\mathrm{top}}(s)\,. \tag{1.1}$$

To be included are the perturbative lepton and top quark contributions, in addition to the non-perturbative hadronic VP shift $\Delta\alpha^{(5)}_{\mathrm{had}}(s)$ from the five light quarks and the hadrons they form.

---

*This contribution should be cited as:
F. Jegerlehner, $\alpha_{\mathrm{QED,\,eff}}(s)$ for precision physics at the FCC-ee/ILC, DOI: 10.23731/CYRM-2020-003.9, in: Theory for the FCC-ee, Eds. A. Blondel, J. Gluza, S. Jadach, P. Janot and T. Riemann,
CERN Yellow Reports: Monographs, CERN-2020-003, DOI: 10.23731/CYRM-2020-003, p. 9.

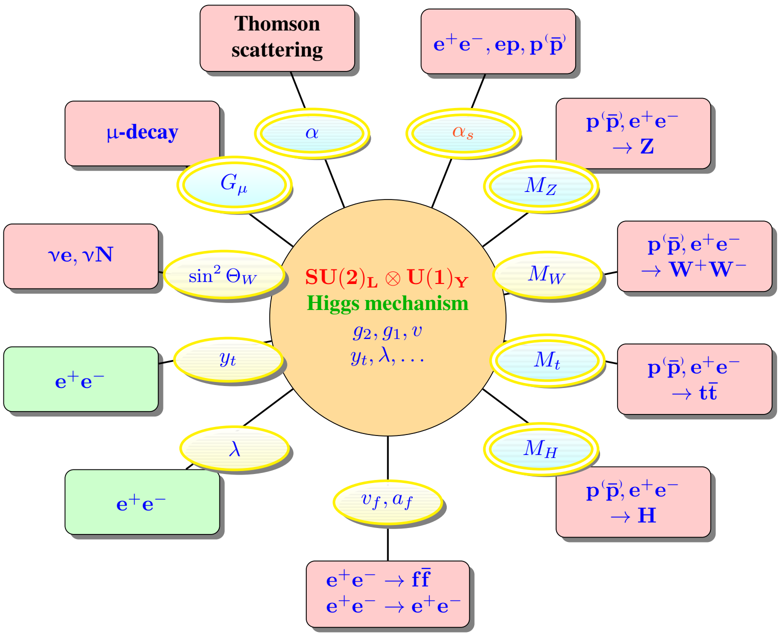


Fig. B.1.1: Many precisely measurable pseudo-observables associated with scattering-, production-, and decay processes are interrelated and predictable in terms of a few independent input parameters.

The current accuracies of the corresponding SM input parameter are:

$$\begin{array}{lccccl}
\frac{\delta\alpha}{\alpha} & \sim & 3.6 & \times & 10^{-9}\,, & \\
\frac{\delta G_\mu}{G_\mu} & \sim & 8.6 & \times & 10^{-6}\,, & \\
\frac{\delta M_Z}{M_Z} & \sim & 2.4 & \times & 10^{-5}\,, & \\
\frac{\delta\alpha(M_Z)}{\alpha(M_Z)} & \sim & 0.9 \div 1.6 & \times & 10^{-4} & (\text{present : lost } 10^5 \text{ in precision!})\,, \\
\frac{\delta\alpha(M_Z)}{\alpha(M_Z)} & \sim & 5 & \times & 10^{-5} & (\text{FCC-ee/ILC requirement})\,.
\end{array} \tag{1.2}$$

We further note that $\delta M_W/M_W \sim 1.5\times10^{-4}$, $\delta M_H/M_H \sim 1.3\times10^{-3}$, $\delta M_t/M_t \sim 2.3\times10^{-3}$, at present. Evidently, $\alpha(M_Z)$ is the least precise among the basic input parameters $\alpha(M_Z)$, G_μ, and M_Z, and requires a major effort of improvement. As an example, one of the most precisely measured derived observables, the leptonic weak mixing parameter $\sin^2\Theta_{\ell\,\mathrm{eff}} = (1-v_\ell/a_\ell)/4 = 0.231\,48 \pm 0.000\,17$ and also the related W mass $M_W = 80.379 \pm 0.012$ GeV are affected by the present hadronic uncdertainty $\delta\Delta\alpha(M_Z) = 0.000\,20$ in predictions by $\delta\sin^2\Theta_{\ell\,\mathrm{eff}} = 0.000\,07$ and $\delta M_W/M_W \sim 4.3\times10^{-5}$, respectively.

Here, one has to keep in mind that, besides $\Delta\alpha$, there is a second substantial leading one-loop correction, which enters the neutral to charged current effective Fermi-couplings ratio $\rho = G_{NC}(0)/G_{CC}(0) = 1+\Delta\rho$, where $\Delta\rho = 3\sqrt{2}M_t^2\,G_\mu/16\pi^2$ is quadratic in the top quark mass. The mentioned $\delta M_t/M_t$ uncertainty affects the M_W and $\sin^2\Theta_{\ell\,\mathrm{eff}}$ predictions, as given by

$$\frac{\delta M_W}{M_W} \sim M_W^2/(2M_W^2 - M_Z^2)\cdot\Delta\rho\,\frac{\delta M_t}{M_t} \sim 1.3\times10^{-2}\,\frac{\delta M_t}{M_t} \simeq 3.0\times10^{-5}\,, \tag{1.3}$$

$$\frac{\delta \sin^2\Theta_{\mathrm{f}}}{\sin^2\Theta_{\mathrm{f}}} \sim \frac{2\cos^2\Theta_{\mathrm{f}}}{\cos^2\Theta_{\mathrm{f}}-\sin^2\Theta_{\mathrm{f}}}\,\Delta\rho\,\frac{\delta M_{\mathrm{t}}}{M_{\mathrm{t}}} \sim 2.7\times 10^{-2}\,\frac{\delta M_{\mathrm{t}}}{M_{\mathrm{t}}} \simeq 6.2\times 10^{-5}\,, \tag{1.4}$$

which are comparable to the current uncertainties from $\delta\Delta\alpha$. Thus, an improvement of δM_{t} by a factor of five appears to be as important as an improvement of $\alpha(M_{\mathrm{Z}})$. We are reminded that the dependence on M_{H} is very much weaker because of the custodial symmetry, which implies the absence of M_{H}^2 corrections, such that only relatively weak $\log M_H$ effects are remaining.

The input parameter uncertainties affect most future precision tests and may obscure new physics searches! To reduce hadronic uncertainties for perturbative QCD (pQCD) contributions, last but not least, it is also very crucial to improve the precision of QCD parameters α_{s}, m_{c}, m_{b}, m_{t}, which is also a big challenge for lattice QCD.

1.1.1 The relevance of $\alpha(M_{\mathrm{Z}}^2)$

Understanding precisely even the simplest four-fermion, vector boson, and Higgs boson production and decay processes, requires very precise input parameters.

Unlike in QED and QCD in the SM, a spontaneously broken non-Abelian gauge theory, there are intricate parameter inter-dependences, all masses are related to couplings, and only six quantities (besides $\mathrm{f}\neq\mathrm{t}$ fermion masses and mixing parameters), α, G_μ, and M_{Z}, in addition to the QCD coupling α_{s}, the top quark Yukawa coupling y, and the Higgs boson self-coupling λ_{H}, are independent. The effective $\alpha(M_{\mathrm{Z}}^2)$ exhibits large hadronic correction that affects prediction-like versions of the weak mixing parameter via

$$\sin^2\Theta_i\,\cos^2\Theta_i = \frac{\pi\,\alpha}{\sqrt{2}\,G_\mu\,M_{\mathrm{Z}}^2}\,\frac{1}{1-\Delta r_i}\,; \qquad \Delta r_i = \Delta r_i(\alpha, G_\mu, M_{\mathrm{Z}}, m_{\mathrm{H}}, m_{\mathrm{f}\neq\mathrm{t}}, m_{\mathrm{t}})\,, \tag{1.5}$$

with quantum corrections from gauge-boson self-energies and vertex and box corrections, where Δr_i depends on the definition of $\sin^2\Theta_i$. The various definitions coincide at tree level and hence only differ by quantum effects. From the weak gauge-boson masses, the electroweak gauge couplings, and the neutral current couplings of the charged fermions, we obtain

$$\sin^2\Theta_{\mathrm{W}} = 1-\frac{M_{\mathrm{W}}^2}{M_{\mathrm{Z}}^2}\,, \tag{1.6}$$

$$\sin^2\Theta_{\mathrm{g}} = e^2/g^2 = \frac{\pi\alpha}{\sqrt{2}\,G_\mu\,M_{\mathrm{W}}^2}\,, \tag{1.7}$$

$$\sin^2\Theta_{\mathrm{f}} = \frac{1}{4|Q_{\mathrm{f}}|}\left(1-\frac{v_{\mathrm{f}}}{a_{\mathrm{f}}}\right)\,,\quad \mathrm{f}\neq\nu\,, \tag{1.8}$$

for the most important cases and the general form of Δr_i reads

$$\Delta r_i \;=\; \Delta\alpha - f_i(\sin^2\Theta_i)\,\Delta\rho + \Delta r_{i\,\mathrm{reminder}}\,, \tag{1.9}$$

with a universal term $\Delta\alpha$, which affects the predictions of M_{W}, A_{LR}, $A_{\mathrm{FB}}^{\mathrm{f}}$, Γ_{f}, etc. The leading corrections are $\Delta\alpha(M_{\mathrm{Z}}^2) = \Pi'_\gamma(0) - \mathrm{Re}\,\Pi'_\gamma(M_{\mathrm{Z}}^2)$ from the running fine structure constant and

$$\Delta\rho = \frac{\Pi_{\mathrm{Z}}(0)}{M_{\mathrm{Z}}^2} - \frac{\Pi_{\mathrm{W}}(0)}{M_{\mathrm{W}}^2} + 2\,\frac{\sin\Theta_{\mathrm{W}}}{\cos\Theta_{\mathrm{W}}}\,\frac{\Pi_{\gamma\mathrm{Z}}(0)}{M_{\mathrm{Z}}^2},$$

which is proportional to $G_\mu\,M_{\mathrm{t}}^2$ and therefore large, dominated by the heavy top quark mass effect, or by the large top Yukawa coupling.

The uncertainty $\delta\Delta\alpha$ implies uncertainties $\delta M_{\rm W}$, $\delta \sin^2\Theta_i$ given by

$$\frac{\delta M_{\rm W}}{M_{\rm W}} \sim \frac{1}{2}\frac{\sin^2\Theta_{\rm W}}{\cos^2\Theta_{\rm W}-\sin^2\Theta_{\rm W}}\,\delta\Delta\alpha \sim 0.23\,\delta\Delta\alpha\,, \tag{1.10}$$

$$\frac{\delta \sin^2\Theta_{\rm f}}{\sin^2\Theta_{\rm f}} \sim \frac{\cos^2\Theta_{\rm f}}{\cos^2\Theta_{\rm f}-\sin^2\Theta_{\rm f}}\,\delta\Delta\alpha \sim 1.54\,\delta\Delta\alpha\,. \tag{1.11}$$

Also affected are the important relationships between couplings and masses, such as

$$\lambda = 3\,\sqrt{2}G_\mu\,M_{\rm H}^2\,(1+\delta_{\rm H}(\alpha,\dots))\;;\qquad y_{\rm t}^2 = 2\,\sqrt{2}G_\mu\,M_{\rm t}^2\,(1+\delta_{\rm t}(\alpha,\dots)\,, \tag{1.12}$$

which currently offer the only way to determine λ and $y_{\rm t}$ via the experimentally accessible masses $M_{\rm H}$ and $M_{\rm t}$. Direct measurement of λ and $y_{\rm t}$ will probably be possible only at future lepton colliders, such as the FCC-ee.

The *parameter relationships* between very precisely measurable quantities provide stringent precision tests and, at high enough precision, would reveal the physics missing within the SM. Currently, the non-perturbative hadronic contribution $\Delta\alpha^{(5)}_{\rm had}(M_{\rm Z}^2)$ limits the precision predictions. Concerning the relevance of quantum corrections and their precision, one should keep in mind that a 30 SD disagreement between some SM prediction and experiment is obtained when subleading SM corrections are neglected, and only the leading corrections $\Delta\alpha(M_{\rm Z}^2)$ and $\Delta\rho$ in Eq. (1.9) are accounted for.

Calculate, for example, the W and Z mass from $\alpha(M_{\rm Z})$, G_μ and $\sin^2\Theta_{\ell\,\rm eff}$: first $\sin^2\Theta_{\rm W} = 1 - M_{\rm W}^2/M_{\rm Z}^2$ is related to $\sin^2\theta_{\ell,\rm eff}(M_{\rm Z})$ via

$$\sin^2\theta_{\ell,\rm eff}(M_{\rm Z}) = \left(1+\frac{\cos^2\Theta_{\rm W}}{\sin^2\Theta_{\rm W}}\,\Delta\rho\right)\sin^2\Theta_{\rm W}\,,$$

where the leading top quark mass square correction is

$$\Delta\rho = \frac{3\,M_{\rm t}^2\,\sqrt{2}G_\mu}{16\,\pi^2}\;;\qquad M_{\rm t} = 173\pm 0.4\,{\rm GeV}\,.$$

The iterative solution with input $\sin^2\theta_{\ell,\rm eff}(M_{\rm Z}) = 0.23148$ is $\sin^2\Theta_{\rm W} = 0.224\,26$ while $1 - M_{\rm W}^2/M_{\rm Z}^2 = 0.222\,63$ is what one gets using PDG:

$$M_{\rm W}^{\rm exp} = 80.379\pm 0.012\,{\rm GeV}\;;\qquad M_{\rm Z}^{\rm exp} = 91.1876\pm 0.0021\,{\rm GeV}\,.$$

Predicting, then, the masses, we have

$$M_{\rm W} = \frac{A_0}{\sin^2\Theta_{\rm W}}\;;\qquad A_0 = \sqrt{\frac{\pi\alpha}{\sqrt{2}G_\mu}}\;;\qquad M_{\rm Z} = \frac{M_{\rm W}}{\cos\Theta_{\rm W}}$$

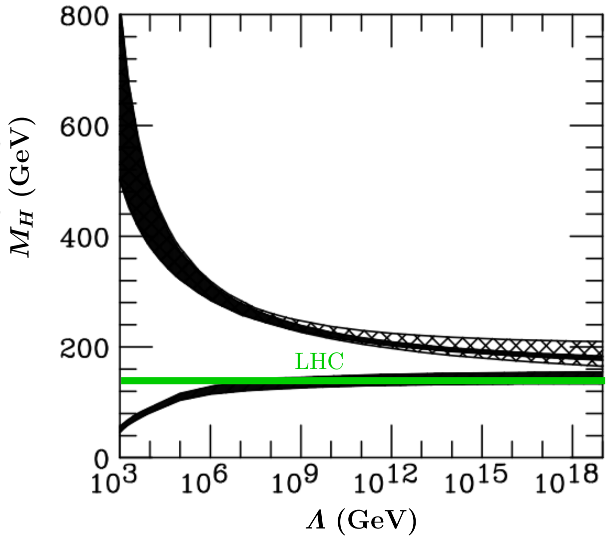
where, including photon VP correction $\alpha^{-1}(M_{\rm Z}) = 128.953\pm 0.016$. For the W and Z masses, we then get

$$M_{\rm W}^{\rm the} = 81.1636\pm 0.0346\,{\rm GeV}\;;\qquad M_{\rm Z}^{\rm the} = 92.1484\pm 0.0264\,{\rm GeV}\,.$$

This gives the following SD values:

$${\rm W}:\;\; 23\,\sigma\;;\qquad {\rm Z}:\;\; 36\,\sigma$$

Uncertainties from $\sin^2\theta$, $\alpha(M_{\rm Z})$, and $M_{\rm t}$, as well as experimental uncertainties, are added in quadrature. The result is, of course, scheme-dependent, but illustrates well the sensitivity to taking into account the proper radiative corrections. Actually, including full one-loop and leading two-loop corrections reduces the disagreement below the 2σ level.



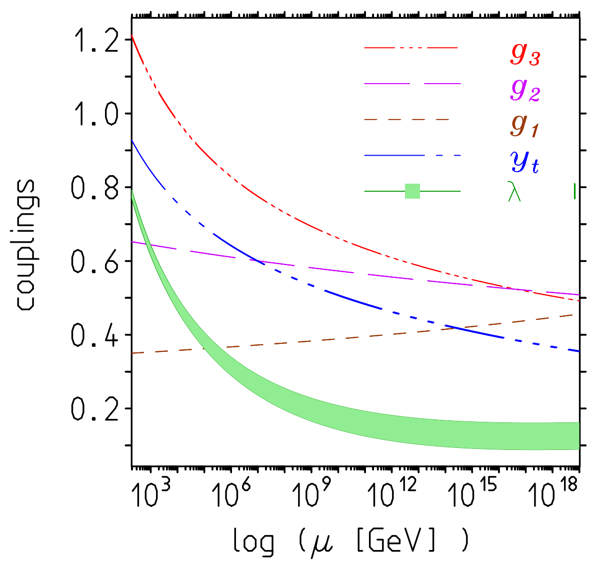


Fig. B.1.2: Left: Plot by Riesselmann and Hambye in 1996, the first two-loop analysis after knowing M_t from CDF [2]. Right: the SM dimensionless couplings in the $\overline{\mathrm{MS}}$ scheme as a function of the renormalization scale for $M_\mathrm{H} = 124$–$126\,\mathrm{GeV}$, which were obtained in Refs. [1, 3–5].

1.2 The ultimate motivation for high-precision SM parameters

After the ATLAS and CMS Higgs discovery at the LHC, the Higgs vacuum stability issue is one of the most interesting to be clarified at future $\mathrm{e^+e^-}$ facilities. Much more surprising than the discovery of its true existence is the fact that the Higgs boson turned out to exhibit a mass very close to what has been expected from vacuum stability extending up to the Planck scale Λ_{Pl} (see Fig. B.1.2). There appears to be a very tricky conspiracy with other couplings to achieve this 'purpose'. Related is the question of whether the SM allows us to extrapolate up to the Planck scale. Thus, the central issue for the future is the very delicate 'acting together' between SM couplings, which makes the precision determination of SM parameters more important than ever. Therefore, higher-precision SM parameters $g', g, g_\mathrm{s}, y_\mathrm{t}$, and λ are mandatory for progress in this direction. Actually, the vacuum stability is controversial at present at the 1.5σ level between a metastable and a stable EW vacuum, which depends on whether λ stays positive up to Λ_{Pl} or not. This is illustrated in Fig. B.1.3. If the SM extrapolates stable to Λ_{Pl}, obviously the resulting effective parameters affect early cosmology, Higgs inflation, Higgs reheating, etc. [1]. The sharp dependence of the Higgs vacuum stability on the SM input parameters, as well as on possible SM extensions and the vastly different scenarios that can result as a consequence of minor shifts in parameter space, makes the stable vacuum case a particularly interesting one and it could reveal the Higgs particle as 'the master of the Universe'. After all, it is commonly accepted that dark energy provided by some scalar field is the 'stuff' shaping the Universe both at very early (inflation) as well as at late times (accelerated expansion).

It is highly conceivable that perturbation expansion works up to the Planck scale without a Landau pole or other singularities and that the Higgs potential remains (meta)stable! The discovery of the Higgs boson has supplied us, for the first time, with the complete set of SM parameters and, for the peculiar SM configuration, revealed that all SM couplings, with the exception of the hypercharge g_1, are decreasing with energy. Very surprisingly, this implies

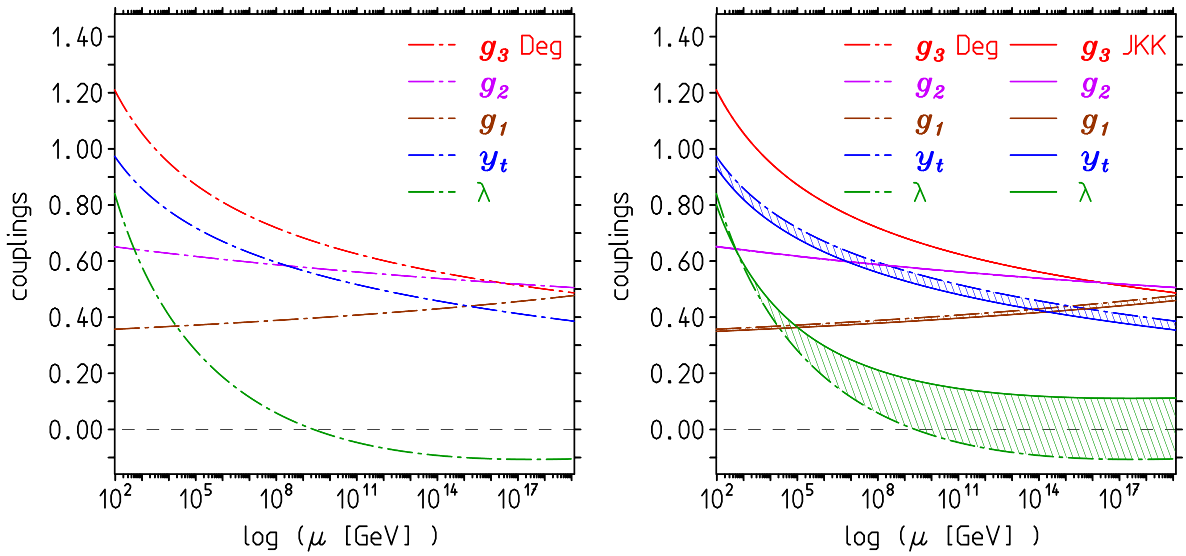


Fig. B.1.3: Left: Shaposhnikov *et al.* and Degrassi *et al.* matching [6, 7]. Right: The shaded bands show the difference in the SM parameter extrapolation using the central values of the $\overline{\mathrm{MS}}$ parameters obtained from differences in the matching procedures.

that perturbative SM predictions improve at higher energies. More specifically, the pattern now looks as follows: the gauge coupling related to $U(1)_{\mathrm{Y}}$ is screening (IR-free), the couplings associated with $SU(2)_{\mathrm{L}}$ and $SU(3)_{\mathrm{c}}$ are antiscreening (UV-free). Thus g_1, g_2, and g_3 behave as expected (standard wisdom). By contrast, the top Yukawa coupling y_{t} and Higgs self-coupling λ, while screening if stand-alone (IR-free, like QED), as part of the SM are transmuted from IR-free to UV-free. The SM reveals an amazing parameter conspiracy, which reminds us of phenomena often observed in condensed matter systems: *"There is a sudden rapid passage to a totally new and more comprehensive type of order or organisation, with quite new emergent properties"* [8], i.e., there must be reasons that couplings are as they are. This manifests itself in the QCD dominance within the renormalization group (RG) of the top Yukawa coupling, which requires $g_3 > 3\,y_{\mathrm{t}}/4$, and in the top Yukawa dominance within the RG of the Higgs boson coupling, which requires $\lambda < 3\,(\sqrt{5}-1)\,y_{\mathrm{t}}^2/2$ in the gaugeless $(g_1, g_2 = 0)$ limit. Under focus is the Higgs self-coupling. Does it stay positive $\lambda > 0$ up to Λ_{Pl}? A zero-valued λ would be an essential singularity. The key problem concerns the precise size of the top Yukawa coupling y_{t}, which decides the stability of our world! The metastability vs. stability controversy will be decided by obtaining more precise input parameters and by better-established EW matching conditions. Most important in this context is the direct measurement of y_{t} and λ at future $\mathrm{e}^+\mathrm{e}^-$ colliders, but also the important role that the running gauge couplings are playing requires substantial progress in obtaining more precise hadronic cross-sections in order to reduce hadronic uncertainties in $\alpha(M_{\mathrm{Z}})$ and $\alpha_2(M_{\mathrm{Z}})$. This is a big challenge for low-energy hadron facilities. Complementary, progress in lattice QCD simulations of two-point correlators will be important to pin down hadronic effects from first principles. Such improvement in SM precision physics could open a new gateway to precision cosmology of the early Universe!

1.3 *R* data evaluation of $\alpha(M_{\mathrm{Z}}^2)$

What we need is a precise calculation of the hadronic photon vacuum polarisation function. The non-perturbative hadronic piece from the five light quarks $\Delta\alpha^{(5)}_{\mathrm{had}}(s) = -(\Pi'_\gamma(s) - \Pi'_\gamma(0))^{(5)}_{\mathrm{had}}$

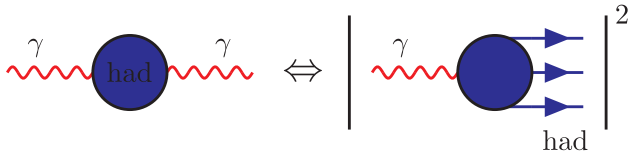


Fig. B.1.4: The master equation (1.13), relating $\Pi'^{\,\mathrm{had}}_{\gamma}(q^2)$ and $\sigma^{\mathrm{had}}_{\mathrm{tot}}(q^2)$, is based on analyticity and the optical theorem.

can be evaluated in terms of $\sigma(\mathrm{e^+e^-} \to \text{hadrons})$ data via the dispersion integral

$$\Delta\alpha^{(5)}_{\mathrm{had}}(s) = -\frac{\alpha\, s}{3\pi}\left(\mathcal{P}\!\!\int_{m^2_{\pi_0}}^{E^2_{\mathrm{cut}}} \mathrm{d}s' \,\frac{R^{\mathrm{data}}_{\gamma}(s')}{s'(s'-s)} + \mathcal{P}\!\!\int_{E^2_{\mathrm{cut}}}^{\infty} \mathrm{d}s' \,\frac{R^{\mathrm{pQCD}}_{\gamma}(s')}{s'(s'-s)} \right), \tag{1.13}$$

where $R_\gamma(s) \equiv \sigma^{(0)}(\mathrm{e^+e^-} \to \gamma^* \to \text{hadrons})/(4\pi\alpha^2/3s)$ measures the hadronic cross-section in units of the tree-level $\mathrm{e^+e^-} \to \mu^+\mu^-$ cross-section sufficiently above the muon pair production threshold ($s \gg 4m^2_\mu$). The master equation (Eq. (1.13)) is based on analyticity and the optical theorem, as shown in Fig. B.1.4.

A compilation of the available R data is shown in Fig. B.1.5 for the low-energy $\pi\pi$ channel and in Fig. B.1.6 for $R(s)$ above the ρ resonance peak. Since the mid 1990s [9], enormous progress has been achieved, also because the new initial-state radiation (ISR) radiative return approach† provided good statistics data from ϕ and B meson factories (see Refs. [10–53]). Still, an issue in hadronic vacuum polarisation (HVP) is the region 1.2–2 GeV, where we have a test ground for exclusive (more than 30 channels) versus inclusive R measurements, where data taking or data analysis is ongoing with CMD-3 and SND detectors (scan) and BaBar and BESIII detector data (radiative return). The region still contributes about 50% to the uncertainty of the hadronic contribution to the muon $g-2$, as we may learn from Fig. B.1.9, in the next section. Above 2 GeV, fairly accurate BES II data [49–51] are available. Recently, a new inclusive determination of $R_\gamma(s)$ in the range 1.84–3.72 GeV has been obtained with the KEDR detector at Novosibirsk [52, 53] (see Fig. B.1.7). At present, the results from the direct and the Adler function improved approach, to be discussed in Section 1.4, reads

$$\begin{aligned}
\Delta\alpha^{(5)}_{\mathrm{hadrons}}(M^2_Z) &= 0.027\,756 \pm 0.000\,157 && \\
&\ 0.027563 \pm 0.000120 && \text{Adler} \\
\alpha^{-1}(M^2_Z) &= 128.916 \pm 0.022 && \\
&\ 128.953 \pm 0.016 && \text{Adler}
\end{aligned} \tag{1.14}$$

In Fig. B.1.8, we show the effective fine structure constant as a function of the c.m. energy $E = \sqrt{s}$, for the time-like and space-like regions. The question now is: what are the possible improvements?

1. Evidently, a direct improvement of the dispersion integral involves reducing the uncertainty of $R(s)$ to 1% up to above the Υ resonances; probably, nobody will do that. One may rely on pQCD above 1.8 GeV and refer to quark–hadron duality, as in Ref. [57]. Then experimental input above 1.8 GeV is not required. But then we are left with questions

†This was pioneered by the KLOE Collaboration, followed by BaBar and BESIII experiments.

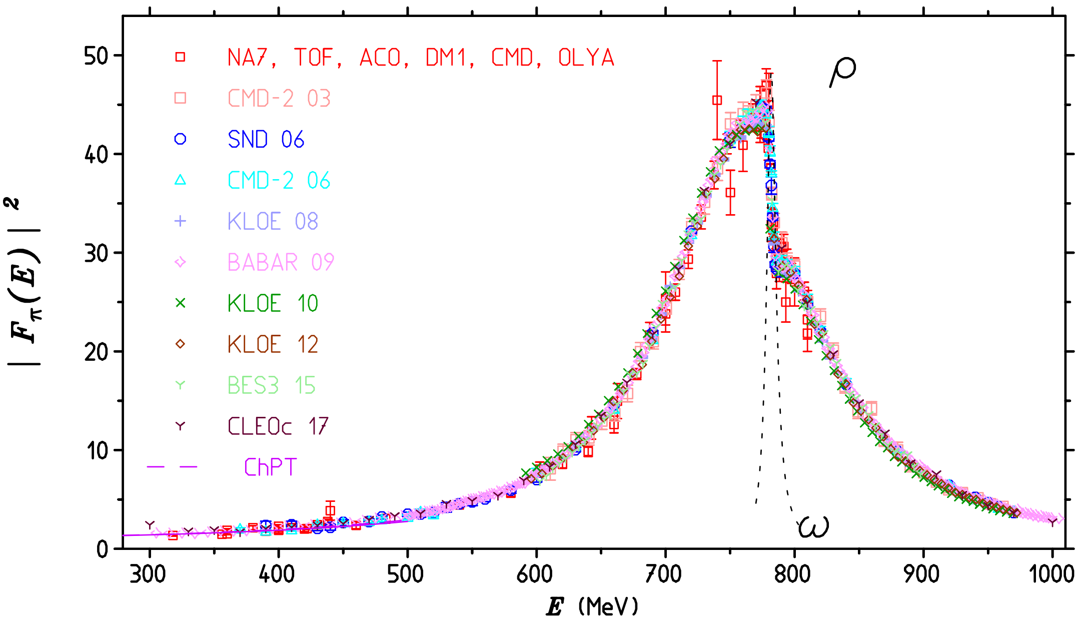


Fig. B.1.5: The low-energy tail of R is provided by $\pi^+\pi^-$ production data. Shown is a compilation of the modulus square of the pion form factor in the ρ meson region. The corresponding $R(s)$ is given by $R(s) = \frac{1}{4}\,\beta_\pi^3\,|F_\pi^{(0)}(s)|^2\,,\ \beta_\pi = (1-4m_\pi^2/s)^{1/2}$ is the pion velocity ($s=E^2$). Data from CMD-2, SND, KLOE, BaBar, BESIII, and CLEOc [10–24] besides some older sets.

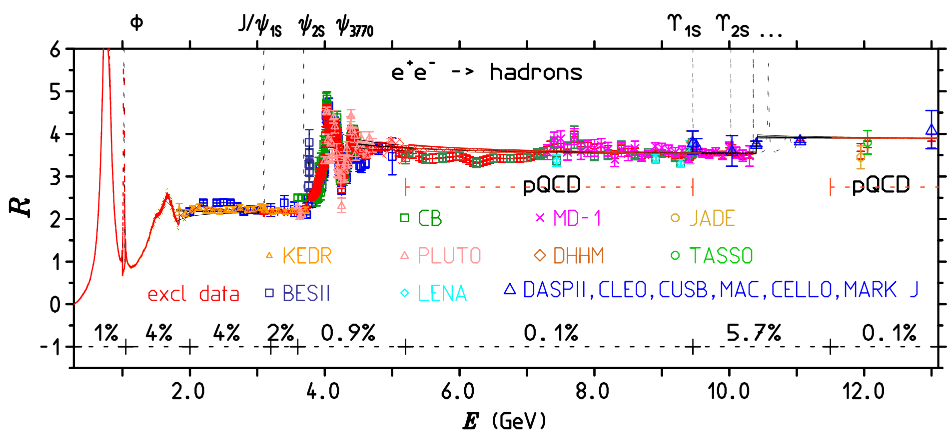


Fig. B.1.6: The compilation of $R(s)$ data utilised in the evaluation of $\Delta\alpha_{\rm had}$. The bottom line shows the relative systematic uncertainties within the split regions. Different regions are assumed to have uncorrelated systematics. Data from Refs. [25–53] and others. We apply pQCD from 5.2 GeV to 9.46 GeV and above 11.5 GeV using the code of Ref. [54].

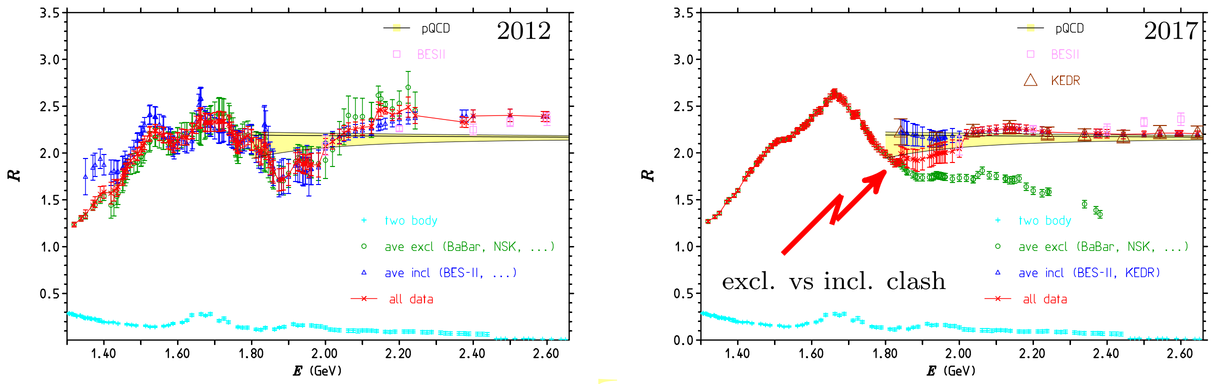


Fig. B.1.7: Illustrating progress by BaBar and NSK exclusive channel data vs. new inclusive data by KEDR. Why is the point at 1.84 GeV so high?

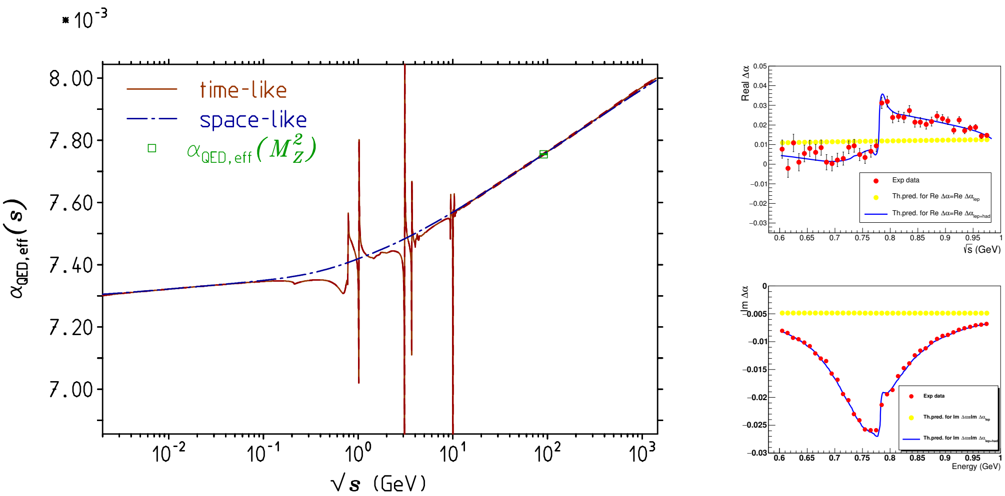


Fig. B.1.8: Left: The effective $\alpha(s)$ at time-like vs. space-like momentum transfer, showing quark–hadron duality at work. In the time-like region, the effective charge varies dramatically near resonances but agrees quite well on average with the space-like version. Locally, it is ill-defined near OZI suppressed meson decays $J/\psi, \psi_1, \Upsilon_{1,2,3}$ where Dyson series of self-energy insertions do not converge (see Section 5 of Ref. [55]). Right: A first experimental determination of the effective charge in the ρ resonance region by KLOE-2 [56], which demonstrates the pronounced variation of the vacuum polarisation (charge screening) across a resonance.

about where precisely to assume thresholds and what are the mass effects near thresholds. Commonly, pQCD is applied, taking into account uncertainties in α_s only. This certainly does not provide a result that can be fully trusted, although the R data integral in this range is much less precise at present. The problem is that, in this theory-driven approach, 70% of $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$ comes from pQCD. Thereby, one has to assume that, in the time-like region above 1.8 GeV, pQCD, on average, is as precise as the usually adopted $\overline{\mathrm{MS}}$ parametrization suggests. Locally, pQCD does not work near thresholds and resonances, obviously.

Table B.1.1: $\Delta\alpha_{\rm had}^{(5)}(M_Z)$ in terms of e^+e^- data and pQCD. The last two columns list the relative accuracy and the percentage contribution of the total. The systematic uncertainties (syst) are assumed to be independent among the different energy ranges listed in the table.

Final state	Range (GeV)	$\Delta\alpha_{\rm had}^{(5)} \times 10^4$ (stat) (syst) [tot]	Rel (%)	Abs (%)
ρ	(0.28, 1.05)	34.14 (0.03) (0.28) [0.28]	0.8	3.1
ω	(0.42, 0.81)	3.10 (0.03) (0.06) [0.07]	2.1	0.2
ϕ	(1.00, 1.04)	4.76 (0.04) (0.05) [0.06]	1.4	0.2
J/ψ		12.38 (0.60) (0.67) [0.90]	7.2	31.9
Υ		1.30 (0.05) (0.07) [0.09]	6.9	0.3
Had	(1.05, 2.00)	16.91 (0.04) (0.82) [0.82]	4.9	26.7
Had	(2.00, 3.20)	15.34 (0.08) (0.61) [0.62]	4.0	15.1
Had	(3.20, 3.60)	4.98 (0.03) (0.09) [0.10]	1.9	0.4
Had	(3.60, 5.20)	16.84 (0.12) (0.21) [0.25]	0.0	2.4
pQCD	(5.20, 9.46)	33.84 (0.12) (0.25) [0.03]	0.1	0.0
Had	(9.46,11.50)	11.12 (0.07) (0.69) [0.69]	6.2	19.1
pQCD	(11.50, 0.00)	123.29 (0.00) (0.05) [0.05]	0.0	0.1
Data	(0.3,∞)	120.85 (0.63) (1.46) [1.58]	1.0	0.0
Total		277.99 (0.63) (1.46) [1.59]	0.6	100.0

2. The more promising approach discussed in the following relies on the Euclidean split method (Adler function controlled pQCD), which only requires improved R measurements in the exclusive region from 1 to 2 GeV. Here, NSK, BESIII, and Belle II can top what BaBar has achieved. However, in this rearrangement, a substantially more precise calculation of the pQCD Adler function is as important. Required is an essentially exact massive four-loop result, which is equivalent to sufficiently high-order low- and high-energy expansions, of which a few terms are available already [58].

Because of the high sensitivity to the precise charm and bottom quark values, one also needs better parameters m_c and m_b besides α_s. Here one can profit from activities going on anyway and the FCC-ee and ILC projects pose further strong motivation to attempt to reach higher precision for QCD parameters.

1.3.1 $\Delta\alpha_{\rm had}(M_Z^2)$ *results from ranges*

Table B.1.1 shows the contributions and uncertainties to $\Delta\alpha_{\rm had}^{(5)}(M_Z)$ for $M_Z = 91.1876$ GeV in units 10^{-4} from different regions. Typically, depending on cuts applied, the direct evaluation of the dispersion integral of R yields 43% from data and 57% from perturbative QCD. Here, pQCD is used between 5.2 GeV and 9.5 GeV and above 11.5 GeV. Systematic uncertainties are taken to be correlated within the different ranges, but taken as independent between the different ranges.

In Fig. B.1.9, we illustrate the relevance of different energy ranges by comparing the hadronic contribution to the muon $g-2$ with that to the hadronic shift of the effective charge at M_Z. The point is that the new muon $g-2$ experiments strongly motivate efforts the measure $R(s)$ in the low-energy region more precisely. From Fig. B.1.9, we learn that low-energy data alone are not able to substantially improve a direct evaluation of the dispersion integral

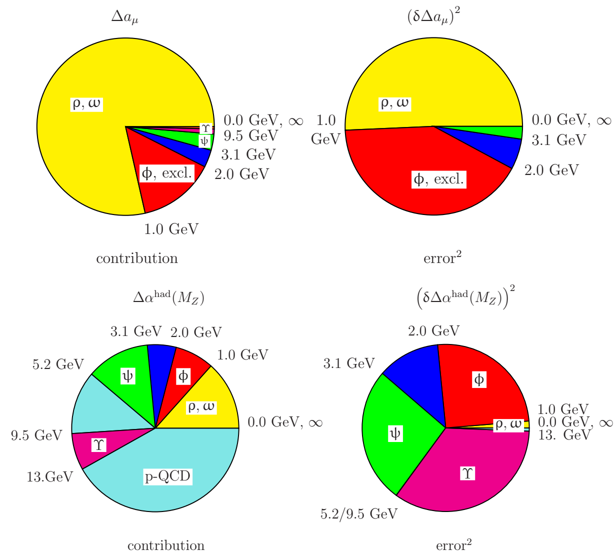


Fig. B.1.9: A comparison of the weights and square uncertainties between a_μ^{had} and $\Delta\alpha^{(5)}_{\mathrm{had}}\,(M_Z^2)$ of contributions from different regions. It reveals the importance of the different energy regions. In contrast to the low-energy dominated a_μ^{had}, $\Delta\alpha^{(5)}_{\mathrm{had}}\,(M_Z^2)$ is sensitive to data from much higher energies.

(Eq. (1.13)). Therefore, to achieve the required factor of five improvement, alternative methods to determine $\Delta\alpha^{(5)}_{\mathrm{had}}(s)$ at high energies must be developed.

1.4 Reducing uncertainties via the Euclidean split trick: Adler function controlled pQCD

As we learn from Fig. B.1.6, it is difficult, if not impossible, to tell at what precision pQCD can replace data. This especially concerns resonance and threshold effects and to what extent quark–hadron duality can be made precise. This is much simpler to accommodate by comparison in the Euclidean (space-like) region, as suggested by Adler [59] a long time ago and successfully tested [60]. As the data pool has been improving greatly, the 'experimental' Adler function is now known with remarkable precision. Actually, on the experimental side, new more precise measurements of $R(s)$ are being made, primarily in the low-energy range. On the theory side, pQCD calculations for Euclidean two-point current correlators are expected to be pushed further. Advances are also expected from lattice QCD, which can also produce data for the Adler function. As suggested in Refs. [61–63], in the Euclidean region, a split into a non-perturbative and a pQCD part is self-evident. One may write

$$\alpha\left(M_Z^2\right) = \alpha^{\mathrm{data}}\left(-M_0^2\right) + \left[\alpha\left(-M_Z^2\right) - \alpha\left(-M_0^2\right)\right]^{\mathrm{pQCD}} + \left[\alpha\left(M_Z^2\right) - \alpha\left(-M_Z^2\right)\right]^{\mathrm{pQCD}}, \quad (1.15)$$

where the space-like offset M_0 is chosen such that pQCD is well under control for $-s < -M_0^2$. The non-perturbative offset $\alpha^{\mathrm{data}}(-M_0^2)$ may be obtained by integrating $R(s)$ data, by choosing $s = -M_0^2$ in Eq. (1.13).

The crucial point is that the contribution from different energy ranges to $\alpha^{\mathrm{data}}(-M_0^2)$ is very different from those to $\alpha^{\mathrm{data}}(M_Z^2)$. Table B.1.1 now is replaced by Table B.1.2, where $\alpha^{\mathrm{data}}(-M_0^2)$ is listed for $M_0 = 2\,\mathrm{GeV}$ in units 10^{-4}. Here 94% results using data and only

Table B.1.2: $\Delta\alpha^{(5)}_{\rm had}(-M_0^2)$ at $M_0 = 2\,$GeV in terms of e^+e^- data and pQCD. Labels as in Table B.1.1.

Final state	Range (GeV)	$\Delta\alpha^{(5)}_{\rm had}(-M_0^2) \times 10^4$ (stat) (syst) [tot]	Rel (%)	Abs (%)
ρ	(0.28, 1.05)	29.97 (0.03) (0.24) [0.24]	0.8	14.3
ω	(0.42, 0.81)	2.69 (0.02) (0.05) [0.06]	2.1	0.8
ϕ	(1.00, 1.04)	3.78 (0.03) (0.04) [0.05]	1.4	0.6
J/ψ		3.21 (0.15) (0.15) [0.21]	6.7	11.2
ϒ		0.05 (0.00) (0.00) [0.00]	6.8	0.0
Had	(1.05, 2.00)	10.56 (0.02) (0.48) [0.48]	4.6	56.9
Had	(2.00, 3.20)	6.06 (0.03) (0.25) [0.25]	4.2	15.7
Had	(3.20, 3.60)	1.31 (0.01) (0.02) [0.03]	1.9	0.2
Had	(3.60, 5.20)	2.90 (0.02) (0.02) [0.03]	0.0	0.2
pQCD	(5.20, 9.46)	2.66 (0.02) (0.02) [0.00]	0.1	0.0
Had	(9.46, 11.50)	0.39 (0.00) (0.02) [0.02]	5.7	0.1
pQCD	(1.50, 0.00)	0.90 (0.00) (0.00) [0.00]	0.0	0.0
Data	(0.3, ∞)	60.92 (0.16) (0.62) [0.64]	1.0	0.0
Total		64.47 (0.16) (0.62) [0.64]	1.0	100.0

6% pQCD, applied again between 5.2 GeV and 9.5 GeV and above 11.5 GeV. Of $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ 22% data, 78% pQCD! The split point, M_0, may be shifted to optimise the uncertainty contributed from the pQCD part and the data based offset value. A reliable estimate of the latter is mandatory and we have also crosschecked its evaluation using the phenomenological effective Lagrangian global fit approach [64, 65], specifically, within the broken hidden local symmetry implementation.

In Fig. B.1.10, we illustrate the relevance of different energy ranges by comparing the hadronic shift of the effective charge as evaluated at space-like low-energy scale $M_0 = 2$ GeV with those at the time-like M_Z scale. The crucial point is that the profile of the offset α at M_0 much more closely resembles the profile found for the hadronic contribution to a_μ and improving $a_\mu^{\rm had}$ automatically leads to an improvement of $\Delta\alpha^{(5)}_{\rm had}(-M_0^2)$; this is the profit gained from the Euclidean split trick.

What does this have to do with the Adler function? (i) The Adler function is the monitor to control the applicability of pQCD and (ii) the pQCD part $[\alpha(-M_Z^2)-\alpha(-M_0^2)]^{\rm pQCD}$ is favourably calculated by integrating the Adler function $D(Q^2)$. The small remainder $[\alpha(M_Z^2)-\alpha(-M_Z^2)]^{\rm pQCD}$ can be obtained in terms of the VP function $\Pi'_\gamma(s)$. In fact, the Adler function is the ideal monitor for comparing theory and data. The Adler function is defined as the derivative of the VP function:

$$D(-s) \doteq \frac{3\pi}{\alpha}\, s\, \frac{\mathrm{d}}{\mathrm{d}s}\Delta\alpha_{\rm had}(s) = -\left(12\pi^2\right)\, s\, \frac{\mathrm{d}\Pi'_\gamma(s)}{\mathrm{d}s} \tag{1.16}$$

and can be evaluated in terms of e^+e^- annihilation data by the dispersion integral

$$D(Q^2) = Q^2\,\left(\int\limits_{4m_\pi^2}^{E^2_{\rm cut}} \mathrm{d}s\,\frac{R(s)^{\rm data}}{(s+Q^2)^2} + \int_{E^2_{\rm cut}}^{\infty} \mathrm{d}s\,\frac{R^{\rm pQCD}(s)}{(s+Q^2)^2}\right)\,. \tag{1.17}$$

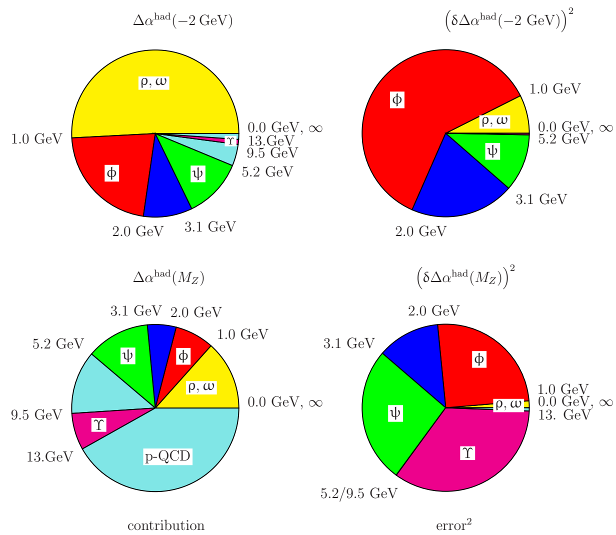


Fig. B.1.10: Contributions and square errors from e^+e^- data ranges and from pQCD to $\Delta\alpha^{(5)}_{\mathrm{had}}(-M_0^2)$ vs. $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$.

It is a finite object not subject to renormalization and it tends to a constant in the high-energy limit, where it is perfectly perturbative. Comparing the direct $R(s)$-based and the $D(Q^2)$-based methods

pQCD $\leftrightarrow R(s)$	pQCD $\leftrightarrow D(Q^2)$
Very difficult to obtain in theory	Smooth simple function in *Euclidean* region

we note that in the time-like approach pQCD only works well in 'perturbative windows' roughly in the ranges 3.00–3.73 GeV, 5.00–10.52 GeV and 11.50 GeV to ∞ [54], while in the space-like approach pQCD works well for $Q > 2.0$ GeV, a clear advantage.

In Fig. B.1.11, the 'experimental' Adler function is confronted with theory (pQCD + NP). Note that, in contrast to most xfR plots, like Fig. B.1.6, showing statistical errors only, in Fig. B.1.11. the total error is displayed as the shaded band. We see that while one-loop and two-loop predictions clearly fail to follow the data band, a full massive three-loop QCD prediction in the gauge-invariant background field MOM scheme [66] reproduces the experimental Adler function surprisingly well. This has been worked out [60] by Padé improvement of the moment expansions provided in Refs. [67–69]. Figure B.1.11 also shows that non-perturbative (NP) contributions from the quark and gluon condensates [70, 71][‡] start to contribute substantially only at energies where pQCD fails to converge because one is approaching the Landau pole in $\overline{\mathrm{MS}}$ parametrized QCD. Strong coupling constant freezing, as in analytic perturbation theory, advocated in Ref. [72] or similar schemes, is not actually able to improve the agreement in the low-energy regime. Coupling constant freezing also contradicts lattice QCD results [73].

From the three terms of Eq. (1.15), we already know the low-energy offset $\Delta\alpha_{\mathrm{had}}(-M_0^2)$ for $M_0 = 2.0$ GeV. We obtain the second term by integrating the pQCD predicted Adler function

$$\Delta_1 = \Delta\alpha_{\mathrm{had}}\left(-M_Z^2\right) - \Delta\alpha_{\mathrm{had}}\left(-M_0^2\right) = \frac{\alpha}{3\pi}\int_{M_0^2}^{M_Z^2} \mathrm{d}Q'^2 \frac{D\left(Q'^2\right)}{Q'^2}\,, \tag{1.18}$$

[‡]These are evaluated by means of operator product expansions; the explicit expressions may be found in Ref. [60].

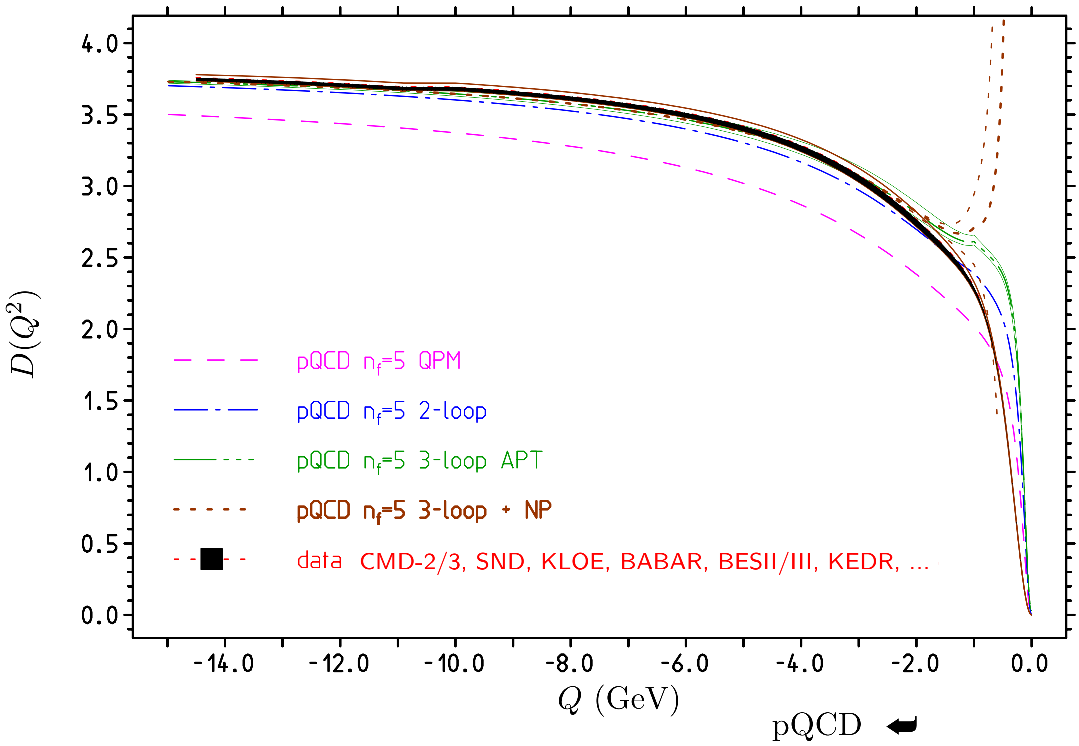


Fig. B.1.11: Monitoring pQCD vs. data: the pQCD prediction of $D(Q^2)$ works well down to $M_0 = 2.0\,\mathrm{GeV}$, provided full massive QCD at three-loop order or higher is employed.

based on a complete three-loop massive QCD analysis. The QCD parameters used are $\alpha_s(M_Z) = 0.1189(20)$, $m_c(m_c) = 1.286(13)[M_c = 1.666(17)]\,\mathrm{GeV}$, $m_b(m_c) = 4.164(25)[M_b = 4.800(29)]\,\mathrm{GeV}$. The result obtained is

$$\Delta_1 = \Delta\alpha_{\rm had}\left(-M_Z^2\right) - \Delta\alpha_{\rm had}\left(-M_0^2\right) = 0.021\,074 \pm 0.000\,100\ .$$

This includes a shift $+0.000\,008$ from the massless four-loop contribution included in the high-energy tail. The error $\pm 0.000\,100$ will be added in quadrature. Up to three loops, all contributions have the same sign and are substantial. Four-loop and higher orders could still add up to non-negligible contributions. An error for missing higher-order terms is not included.

The remaining term concerns the link between the space-like and the time-like region at the Z boson mass scale and is given by the difference

$$\Delta_2 = \Delta\alpha^{(5)}_{\rm had}\left(M_Z^2\right) - \Delta\alpha^{(5)}_{\rm had}\left(-M_Z^2\right) = 0.000\,045 \pm 0.000\,002\,,$$

which can be calculated in pQCD. It accounts for the $\mathrm{i}\pi$-terms from the logs $\ln(-q^2/\mu^2) = \ln(|q^2/\mu^2|) + \mathrm{i}\pi$. Since the term is small, we can also get it from direct data integration based on our data compilation. We obtain $\Delta\alpha_{\rm had}\left(-M_Z^2\right) = 276.44 \pm 0.64 \pm 1.78$ and $\Delta\alpha_{\rm had}\left(+M_Z^2\right) = 276.84 \pm 0.64 \pm 1.90$, and taking into account that errors are almost 100% correlated, we have $\Delta\alpha_{\rm had}(M_Z^2) - \Delta\alpha_{\rm had}(-M_Z^2) = 0.40 \pm 0.12$ less precise but in agreement with the pQCD result. We then have

$$\Delta\alpha^{(5)}_{\rm had}\left(-M_0^2\right)^{\rm data} = 0.006\,409 \pm 0.000\,063$$
$$\Delta\alpha^{(5)}_{\rm had}\left(-M_Z^2\right) = 0.027\,483 \pm 0.000\,118$$

$$\Delta\alpha^{(5)}_{\mathrm{had}}\left(M_Z^2\right) = 0.027\,523 \pm 0.000\,119 \;.$$

To get $\alpha^{-1}(M_Z^2)$, we also have to include the leptonic piece [74]

$$\Delta\alpha_{\mathrm{lep}}\left(M_Z^2\right) \simeq 0.031\,419\,187\,418\,, \tag{1.19}$$

and the top quark contribution. A very heavy top quark decouples as

$$\Delta\alpha_{\mathrm{top}} \simeq -\frac{\alpha}{3\pi}\frac{4}{15}\frac{s}{m_t^2} \to 0$$

when $m_t \gg s$. At $s = M_Z^2$, the top quark contributes

$$\Delta\alpha_{\mathrm{top}}\left(M_Z^2\right) = -0.76 \times 10^{-4} \;. \tag{1.20}$$

Collecting terms, this leads to the result presented in Eq. (1.14). One should note that the Adler function controlled Euclidean data vs. pQCD split approach is only moderately more pQCD-driven than the time-like approach adopted by Davier *et al.* [57] and others, as follows from the collection of results shown in Fig. B.1.12. The point is that the Adler function driven method only uses pQCD where reliable predictions are possible and direct cross checks against lattice QCD data may be carried out. Similarly, possible future direct measurements of $\alpha(-Q^2)$ in μ-e scattering [75] can provide Euclidean HVP data, in particular, also for the offset $\Delta\alpha_{\mathrm{had}}(-M_0^2)$.

1.5 Prospects for future improvements

The new muon $g-2$ experiments at Fermilab and at JPARC in Japan (expected to go into operation later) trigger the continuation of $e^+e^- \to$ hadrons cross-section measurements in the low-energy region by CMD-3 and SND at BINP Novosibirsk, by BES III at IHEP Beijing and soon by Belle II at KEK Tsukuba. This automatically helps to improve $\Delta\alpha(-M_0^2)$ and hence $\alpha(M_Z^2)$ via the Adler function controlled split-trick approach. Equally important are the results from lattice QCD, which come closer to being competitive with the data-driven dispersive method.

The improvement by a factor of five to ten in this case largely relies on improving the QCD prediction of the two-point vector correlator above the 2 GeV scale, which is a well-defined and comparably simple task. The mandatory pQCD improvements required are as follows.

(a) Four-loop massive pQCD calculation of Adler function. In practice, this requires the calculation of a sufficient number of terms in the low- and high-momentum series expansions, such that an accurate Padé improvement is possible.
(b) m_c, m_b improvements by sum rule or lattice QCD evaluation.
(c) Improved α_s in low Q^2 region above the τ mass.

Note that the direct dispersion relations (DR) approach requires precise data up to much higher energies or a heavy reliance on the pQCD calculation of the time-like $R(s)$! The virtues of the Adler function approach are obvious:

(a) no problems with physical threshold and resonances;
(b) pQCD is used only where we can check it to work accurately (Euclidean $Q \gtrsim 2.0$ GeV);
(c) no manipulation of data, no assumptions about global or local duality;

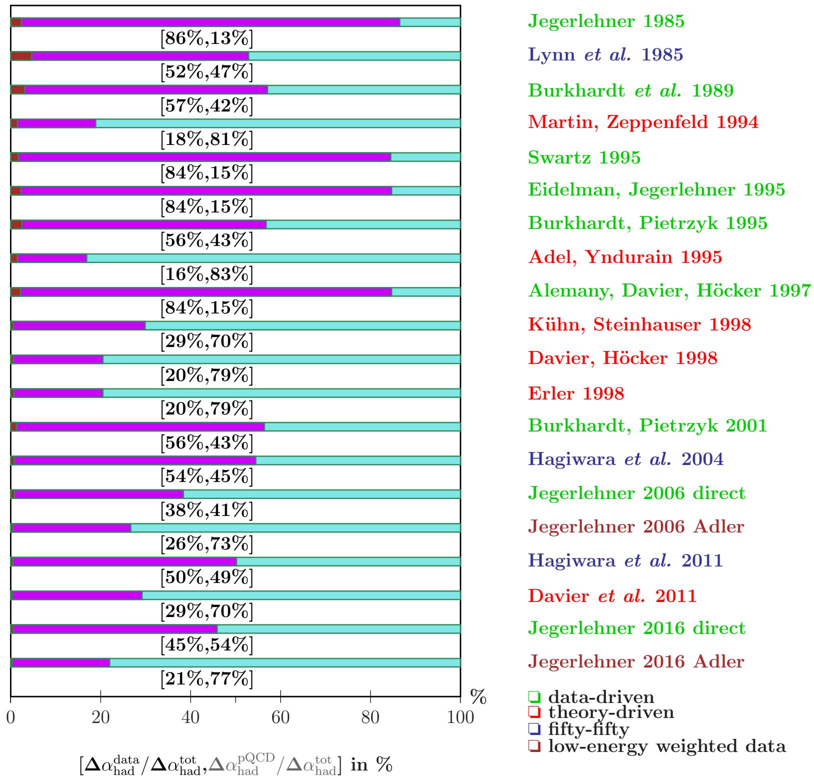


Fig. B.1.12: How much pQCD? Here a history of results by different authors. It shows that the Adler function controlled approach to $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ is barely more pQCD-driven than many of the standard evaluations. The pQCD piece is 70% in Davier *et al.* [57] and 77% in our Adler-driven case, with an important difference: in the Adler-controlled case, the major part of 71% is based on pQCD in the space-like region and only 6% contributing to the non-perturbative offset value is evaluated in the time-like region, while in the standard theory-driven, as well as in the more data-driven approaches, pQCD is applied in the time-like region, where it is much harder to be tested against data.

(d) the non-perturbative 'remainder' $\Delta\alpha(-M_0^2)$ is mainly sensitive to low-energy data;
(e) $\Delta\alpha(-M_0^2)$ would be directly accessible in a MUonE experiment (project) [75] or in lattice QCD.

In the direct approach, e.g., Davier *et al.* [57] use pQCD above 1.8 GeV, which means that no error reduction follows from remeasuring cross-sections above 1.8 GeV. Also, there is no proof that pQCD is valid at 0.04% precision as adopted. This is a general problem when utilising pQCD at time-like momenta exhibiting non-perturbative features.

What we can achieve is illustrated in Fig. B.1.13 and Table B.1.3. Our analysis shows that the Adler function inspired method is competitive with Patrick Janot's [76] direct near-Z pole determination via a measurement of the forward–backward asymmetry $A_{\rm FB}^{\mu\mu}$ in $e^+e^- \to \mu^+\mu^-$. The modulus square of the sum of the two tree-level diagrams has three terms: the Z exchange alone, $\mathcal{Z} \propto (M_Z^2 G_\mu)^2$, the γ–Z interference, $\mathcal{I} \propto \alpha(s)\, M_Z^2 G_\mu$, and the γ-exchange only, $\mathcal{G} \propto \alpha^2(s)$. The interference term determines the forward–backward (FB) asymmetry, which is linear in $\alpha(s)$; v denotes the vector $Z\mu\mu$ coupling that depends on $\sin^2\Theta_{\ell\,\rm eff}(s)$, while a denotes the axial $Z\mu\mu$ coupling that is sensitive to the ρ-parameter (strong M_t dependence). In extracting $\alpha(M_Z^2)$,

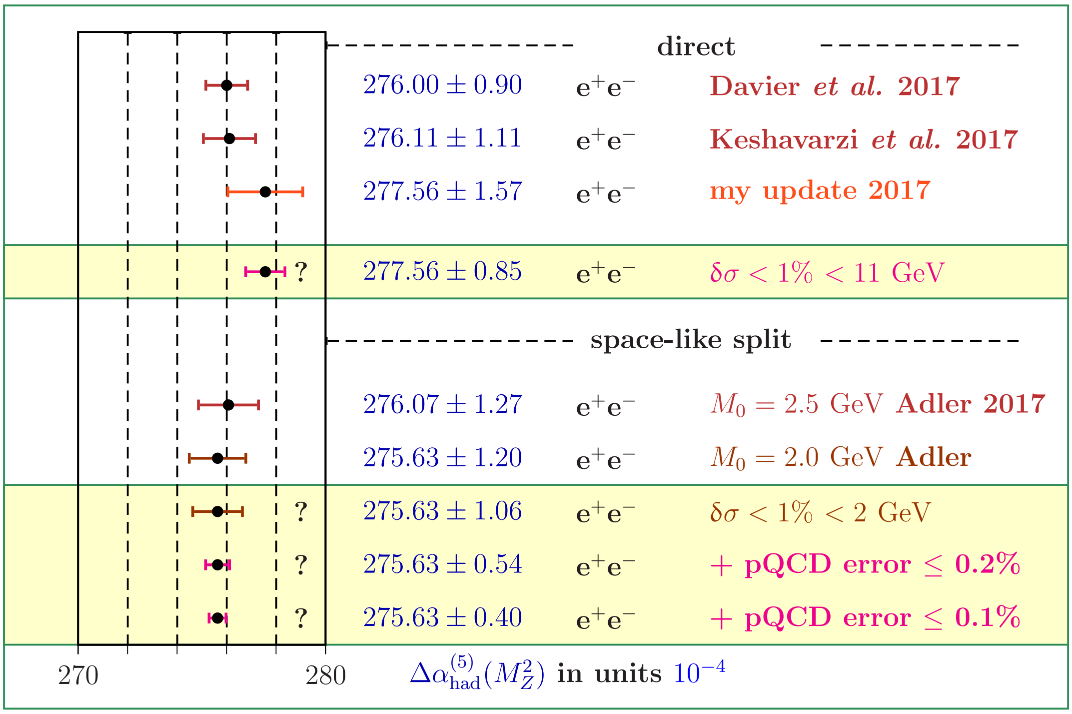


Fig. B.1.13: Comparison of possible improvements. My 'direct' analysis is data-driven, adopting pQCD in the window 5.2–9.5 GeV and above 11.5 GeV . The Adler-driven results under 'space-like split' show the current status for the two offset energies, $M_0 = 2.5$ GeV and 2 GeV. The improvement potential is displayed for three options: reducing the error of the data offset by a factor of two, improving pQCD to a 0.2% precision Adler function in addition and the same by improving pQCD to a 0.1% precision Adler function. The direct results are from Refs. [57,77,78].

one is using the v and a couplings as measured at the Z peak directly. At tree level, one then has

$$A_{\rm FB}^{\mu\mu} = A_{\rm FB,0}^{\mu\mu} + \frac{3\,a^2}{4\,v^2}\,\frac{\mathcal{I}}{\mathcal{Z}+\mathcal{G}}\;; \qquad A_{\rm FB,0}^{\mu\mu} = \frac{3}{4}\,\frac{4v^2a^2}{(v^2+a^2)^2}\,, \tag{1.21}$$

where

$$\mathcal{G} = \frac{c_\gamma^2}{s}\,, \qquad \mathcal{I} = \frac{2c_\gamma c_{\rm Z}\,v^2\,(s-M_{\rm Z}^2)}{\left(s-M_{\rm Z}^2\right)^2 + M_{\rm Z}^2\Gamma_{\rm Z}^2}\,, \qquad \mathcal{Z} = \frac{c_{\rm Z}^2\,(v^2+a^2)\,s}{\left(s-M_{\rm Z}^2\right)^2 + M_{\rm Z}^2\Gamma_{\rm Z}^2}$$

$$c_\gamma = \sqrt{\frac{4\pi}{3}}\,\alpha(s)\,, \qquad c_{\rm Z} = \sqrt{\frac{4\pi}{3}\,\frac{M_{\rm Z}^2}{2\pi}\,\frac{G_\mu}{\sqrt{2}}}\,, \qquad v = \left(1-4\,\sin^2\Theta_\ell\right)\,a\,, \qquad a = -\frac{1}{2}\,.$$

Note that $M_{\rm Z}^2 G_\mu = M_{\rm W}^2 G_\mu/\cos^2\Theta_{\rm W} = \pi\,(\alpha_2(s))/\sqrt{2}(\cos^2\Theta_{\rm g}(s))$ and $\sin^2\Theta_{\rm g}(s) = \alpha(s)/\alpha_2(s)$. i.e., all parameters vary more or less with energy, depending on the renormalization scheme utilised. The challenges for this direct measurement are precise radiative corrections (see Refs. [79, 80] and references therein) and the required dedicated off-Z peak running. Short accounts of the methods proposed for improving $\alpha(M_{\rm Z}^2)$ may be found in Sections 8 and 9 of Ref. [81].

The Adler function based method is much cheaper, I think, and does not depend on understanding the Z peak region with unprecedented precision. Another very crucial point may

Table B.1.3: Precision in $\alpha(M_Z^2)$

Present	Direct	1.7×10^{-4}	
	Adler	1.2×10^{-4}	
Future	Adler QCD 0.2%	5.4×10^{-5}	
	Adler QCD 0.1%	3.9×10^{-5}	
Future	Via $A_{\mathrm{FB}}^{\mu\mu}$ off-Z	$3 \quad \times 10^{-5}$	[76]

Table B.1.4: Possible achievements for the FCC-ee project

$\sqrt{s}$	$\sqrt{t}$ (GeV)	1996 [83, 84]	Present	FCC-ee expected [82]
M_Z	3.5	0.040%	0.013%	0.6×10^{-4}
350 GeV	13		1.2×10^{-4}	2.4×10^{-4}

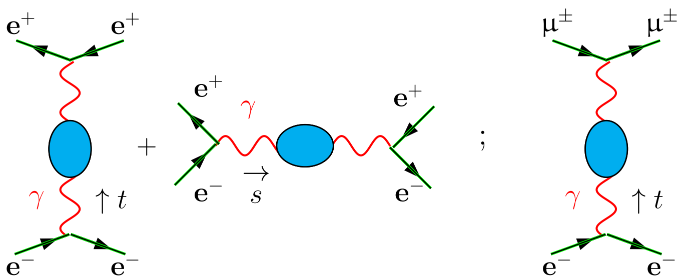


Fig. B.1.14: t-channel dominated QED processes. Left: VP dressed tree-level Bhabha scattering at small scattering angles. Right: the leading VP effect in μe scattering.

be that the dispersive method and the Adler function modified version provide the effective $\alpha(s)$ for arbitrary c.m. energies, not at $s = M_Z^2$ only; although, given a very precise $\alpha(M_Z^2)$, one can reliably calculate $\alpha(s) - \alpha(M_Z^2)$ via pQCD for values of s in the perturbative regime, i.e., especially going to higher energies. In any case, the requirements specified here that must be satisfied in order to reach a factor of five improvement appears to be achievable.

1.6 The need for a space-like effective $\alpha(t)$

As a normalization in measurements of cross-sections in e^+e^- collider experiments, small-angle Bhabha scattering is the standard choice. This reference process is dominated by the t-channel diagram of the Bhabha scattering process shown in the left of Fig. B.1.14. In small-angle Bhabha scattering, we have $\delta_{\mathrm{HVP}}\sigma/\sigma = 2\,\delta\alpha(\bar{t})/\alpha(\bar{t})$, and for the FCC-ee luminometer $\sqrt{\bar{t}} \simeq 3.5\,\mathrm{GeV}$ near the Z peak and $\simeq 13\,\mathrm{GeV}$ at $350\,\mathrm{GeV}$ [82]. The progress achieved after LEP times is displayed in Fig. B.1.15. What can be achieved for the FCC-ee project is listed in Table B.1.4. The estimates are based on expected improvements possible for $\Delta\alpha_{\mathrm{had}}(-Q^2)$ in the appropriate energy ranges, centred at $\sqrt{\bar{t}}$.

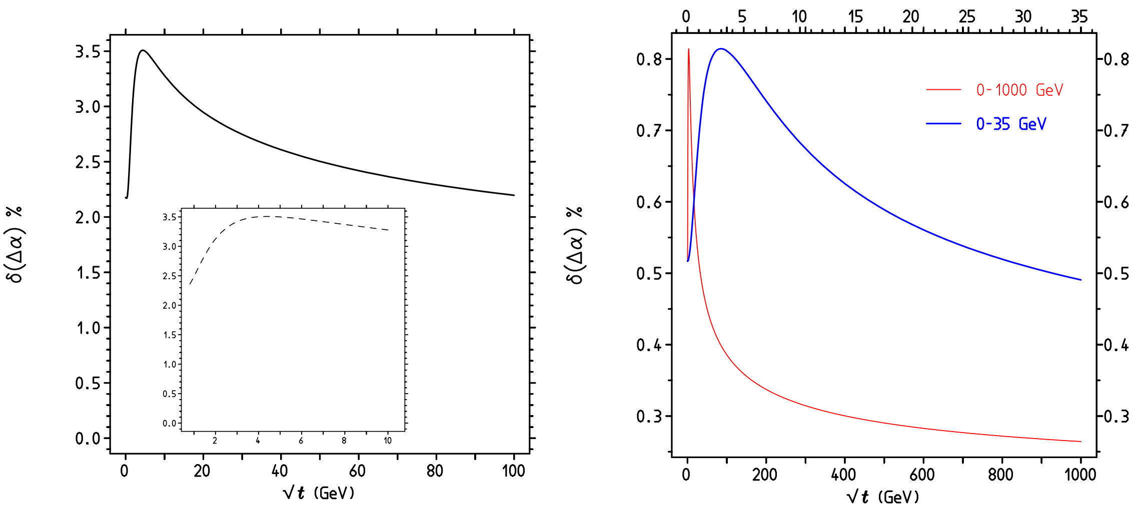


Fig. B.1.15: Hadronic uncertainty $\delta\Delta\alpha_{\mathrm{had}}(\sqrt{t})$. The progress since LEP times, from 1996 (left) to now (right) is remarkable. A great deal of much more precise low-energy data, $\pi\pi$, etc., are now available.

1.6.1 A new project: measuring the low-energy $\alpha(t)$ directly

The possible direct measurement of $\Delta\alpha_{\mathrm{had}}(-Q^2)$ follows a very different strategy of evaluating the HVP contribution to the muon $g-2$. There is no VP subtraction issue, there is no exclusive channel separation and recombination, no issue of combining data from very different experiments and controlling correlations. Even a 1% level measurement can provide invaluable independent information. The recent proposal [75] to measure $\alpha(-Q^2)$ via μ^-e^--scattering (see right part of Fig. B.1.15) in the MUonE projects at CERN is very important for future precision physics. It is based on a cross-section measurement

$$\frac{\mathrm{d}\sigma^{\mathrm{unpol.}}_{\mu^-e^-\to\mu^-e^-}}{\mathrm{d}t} = 4\pi\,\alpha(t)^2\,\frac{1}{\lambda(s,m_e^2,m_\mu^2)}\left\{\frac{\left(s-m_\mu^2-m_e^2\right)^2}{t^2}+\frac{s}{t}+\frac{1}{2}\right\}\,. \tag{1.22}$$

The primary goal of the project concerns the determination of a_μ^{had} in an alternative way

$$a_\mu^{\mathrm{had}} = \frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}x\,(1-x)\,\Delta\alpha_{\mathrm{had}}\left(-Q^2(x)\right)\,, \tag{1.23}$$

where $Q^2(x)\equiv x^2m_\mu^2/(1-x)$ is the space-like square momentum transfer and

$$\Delta\alpha_{\mathrm{had}}(-Q^2) = \frac{\alpha}{\alpha(-Q^2)} + \Delta\alpha_{\mathrm{lep}}(-Q^2) - 1 \tag{1.24}$$

directly compares with lattice QCD data and the offset $\alpha(-M_0^2)$ discussed before. We propose to determine, very accurately, $\Delta\alpha_{\mathrm{had}}\,(-Q^2)$ at $Q\approx 2.5\,\mathrm{GeV}$ by this method (one single number!) as the non-perturbative part of $\Delta\alpha_{\mathrm{had}}\,(M_Z^2)$, as needed in the 'Adler function approach'. It would also be of direct use for a precise small-angle Bhabha luminometer! Because of the high precision required, accurate radiative corrections are mandatory and corresponding calculations are in progress [85–88].

1.7 Conclusions

Reducing the muon $g-2$ prediction uncertainty remains the key issue of high-precision physics and strongly motivates more precise measurements of low-energy $e^+e^- \to$ hadrons cross-sections. Progress is expected from Novosibirsk (VEPP 2000/CMD3,SND), Beijing (BEPCI-I/BESIII), and Tsukuba (SuperKEKB/BelleII). This helps to improve $\alpha(t)$ in the region relevant for small-angle Bhabha scattering and in calculating $\alpha(s)$ at FCC-ee/ILC energies using the Euclidean split-trick method. The latter method requires pQCD prediction of the Adler function to improve by a factor of two. This also means that we need improved parameters, in particular, m_c and m_b.

One question remains to be asked. Are presently estimated and essentially agreed-on evaluations of $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ in terms of R data reliable? One has to keep in mind that the handling of systematic errors is rather an art than a science. Therefore, alternative methods are very important and, fortunately, are under consideration.

Patrick Janot's approach is certainly an important alternative method, directly accessing $\alpha(M_Z^2)$ with very different systematics. This is a challenging project.

Another interesting option is an improved radiative return measurement of $\sigma(e^+e^- \to$ hadrons) at the GigaZ, allowing for directly improved dispersion integral input, which would include all resonances and thresholds in one experiment!

In any case, on paper, $e^-\mu^+ \to e^-\mu^+$ appears to be the ideal process to perform an unambiguous measurement of $\alpha(-Q^2)$, which determines the leading-order (LO) HVP to a_μ, as well as the non-perturbative part of $\alpha(s)$!

Lattice QCD results are very close to becoming competitive here as well. Thus, in the end, we will have alternatives, allowing for important improvements and crosschecks.

The improvement obtained by reducing the experimental error to 1% in the range from ϕ to 3 GeV would allow one to choose a higher cut point, e.g., for $\sqrt{M_0} = 3.0\,\text{GeV}$. One can then balance the importance of data against pQCD differently. This would provide further important consolidation of results. For a 3 GeV cut, one gets $\Delta\alpha_{\rm had}(-M_0^2) = 82.21 \pm 0.88[0.38]$ in 10^{-4}. The QCD contribution is then smaller, as well as safer, because the mass effects that are responsible for the larger uncertainty of the pQCD prediction are also substantially reduced. Taking the view that a massive four-loop QCD calculation is a challenge, the possibility of optimising the choice of split scale M_0 would be very useful. Therefore, the ILC/FCC-ee community should actively support these activities as an integral part of the e^+e^--collider precision physics programme!

1.8 Addendum: the coupling α_2, M_W, and $\sin^2\Theta_f$

Besides α, the $SU(2)$ gauge coupling $\alpha_2 = g^2/(4\pi)$ is also running and thereby affected by non-perturbative hadronic effects [78,89,90]. Related with the $U_Y(1) \otimes SU_L(2)$ gauge couplings is the running of the weak mixing parameter $\sin^2\Theta_f$, which is actually defined by the ratio α/α_2. In Refs. [78, 89, 90], the hadronic effects have been evaluated by means of DRs in terms of e^+e^- data with appropriate flavour separation and reweighting. Commonly, a much simpler approach is adopted in studies of the running of $\sin^2\Theta_f$, namely using pQCD with effective quark masses [91–94], which have been determined elsewhere.

Given $g \equiv g_2$ and the Higgs vacuum expectation value (VEV) v, then

$$M_{\rm W}^2 = \frac{g^2\,v^2}{4} = \frac{\pi\,\alpha_2}{\sqrt{2}\,G_\mu}\,.$$

The running $\sin^2\Theta_{\rm f}(s)$ relates electromagnetic to weak neutral channel mixing at the LEP scale to low-energy $\nu_{\rm e}$e scattering as

$$\sin^2\Theta_{\rm lep}(M_{\rm Z}^2) = \left\{\frac{1-\Delta\alpha_2}{1-\Delta\alpha} + \Delta_{\nu_\mu{\rm e,vertex+box}} + \Delta\kappa_{\rm e,vertex}\right\}\sin^2\Theta_{\nu_\mu{\rm e}}(0)\,. \tag{1.25}$$

The first correction from the running coupling ratio is largely compensated for by the ν_μ charge radius, which dominates the second term. The ratio $\sin^2\Theta_{\nu_\mu{\rm e}}/\sin^2\Theta_{\rm lep}$ is close to 1.002, independent of the top and Higgs masses. Note that errors in the ratio $(1-\Delta\alpha_2)/(1-\Delta\alpha)$ can be taken to be 100% correlated and thus largely cancel. A similar relation between $\sin^2\Theta_{\rm lep}(M_{\rm Z}^2)$ and the weak mixing angle appearing in polarised Møller scattering asymmetries has been worked out [91, 92]. It includes specific bosonic contribution $\Delta\kappa_{\rm b}(Q^2)$, such that

$$\kappa(s=-Q^2) = \frac{1-\Delta\alpha_2(s)}{1-\Delta\alpha(s)} + \Delta\kappa_{\rm b}(Q^2) - \Delta\kappa_{\rm b}(0)\,, \tag{1.26}$$

where, in our low-energy scheme, we require $\kappa(Q^2)=1$ at $Q^2=0$. Explicitly [91,92], at one-loop order

$$\begin{aligned}\Delta\kappa_{\rm b}(Q^2) = -\frac{\alpha}{2\pi\,s_{\rm W}}\Bigg\{&-\frac{42\,c_{\rm W}+1}{12}\,\ln c_{\rm W} + \frac{1}{18} - \left(\frac{r}{2}\,\ln\xi - 1\right)\left[(7-4z)\,c_{\rm W} + \frac{1}{6}\,(1+4z)\right]\\ &- z\left[\frac{3}{4} - z + \left(z-\frac{2}{3}\right)\,r\,\ln\xi + z\,(2-z)\,\ln^2\xi\right]\Bigg\}\,,\end{aligned} \tag{1.27}$$

$$\Delta\kappa_{\rm b}(0) = -\frac{\alpha}{2\pi\,s_{\rm W}}\left\{-\frac{42\,c_{\rm W}+1}{12}\,\ln c_{\rm W} + \frac{1}{18} + \frac{6\,c_{\rm W}+7}{18}\right\}\,, \tag{1.28}$$

with $z = M_{\rm W}^2/Q^2$, $r=\sqrt{1+4z}$, $\xi = (r+1)/(r-1)$, $s_{\rm W} = \sin^2\Theta_{\rm W}$, and $c_{\rm W} = \cos^2\Theta_{\rm W}$. Results obtained in Refs. [91, 92] based on one-loop perturbation theory using light quark masses $m_{\rm u} = m_{\rm d} = m_{\rm s} = 100\,$MeV are compared with results obtained in our non-perturbative approach in Fig. B.1.16.

How can we evaluate the leading non-perturbative hadronic corrections to α_2? As in the case of α, they are related to quark-loop contributions to gauge-boson self-energies (SE) $\gamma\gamma$, γZ, ZZ, and WW, in particular, those involving the photon, which exhibit large leading logarithms. To disentangle the leading corrections, decompose the self-energy functions as follows ($s_\Theta^2 = e^2/g^2$; $c_\Theta^2 = 1 - s_\Theta^2$):

$$\begin{aligned}\Pi^{\gamma\gamma} &= e^2\;\hat{\Pi}^{\gamma\gamma}\,,\\ \Pi^{\rm Z\gamma} &= \tfrac{eg}{c_\Theta}\;\hat{\Pi}_V^{3\gamma} - \tfrac{e^2\,s_\Theta}{c_\Theta}\;\hat{\Pi}_V^{\gamma\gamma}\,,\\ \Pi^{\rm ZZ} &= \tfrac{g^2}{c_\Theta^2}\;\hat{\Pi}_{V-A}^{33} - 2\,\tfrac{e^2}{c_\Theta^2}\;\hat{\Pi}_V^{3\gamma} + \tfrac{e^2\,s_\Theta^2}{c_\Theta^2}\;\hat{\Pi}_V^{\gamma\gamma}\,,\\ \Pi^{\rm WW} &= g^2\;\hat{\Pi}_{V-A}^{+-}\,.\end{aligned} \tag{1.29}$$

With $\hat{\Pi}(s) = \hat{\Pi}(0) + s\,\Pi'(s)$, we find the leading hadronic corrections

$$\Delta\alpha^{(5)}_{\rm had}(s) = -e^2\,\left[{\rm Re}\,\Pi'^{\gamma\gamma}(s) - \Pi'^{\gamma\gamma}(0)\right]\,, \tag{1.30}$$

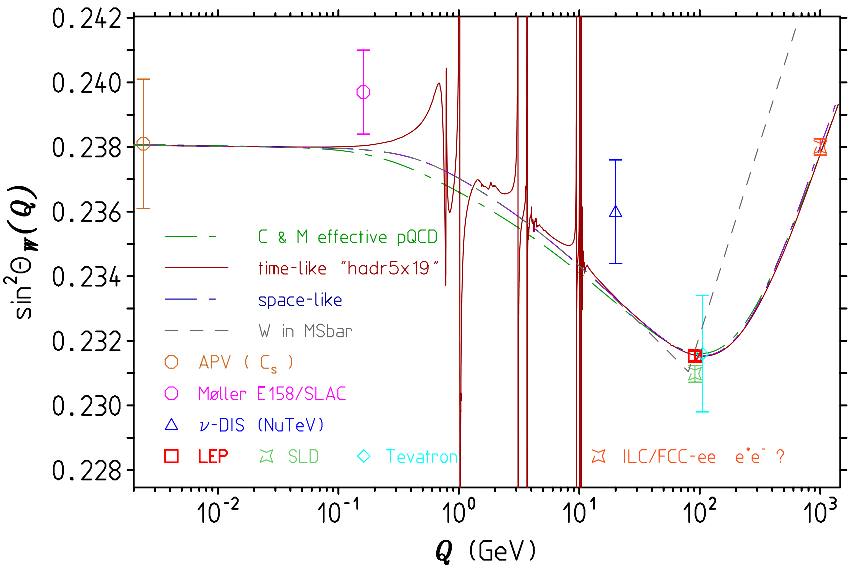


Fig. B.1.16: $\sin^2\Theta_W(Q)$ as a function of Q in the time-like and space-like region. Hadronic uncertainties are included but barely visible in this plot. Uncertainties from the input parameter $\sin^2\Theta_W(0) = 0.238\,22(100)$ or $\sin^2\Theta_W(M_Z^2) = 0.231\,53(16)$ are not shown. Note the substantial difference from applying pQCD with effective quark masses. Future FCC-ee/ILC measurements at 1 TeV would be sensitive to Z′, H^{--}, etc.

$$\Delta\alpha_{2\,\mathrm{had}}^{(5)}(s) = -\frac{e^2}{s_\Theta^2}\left[\mathrm{Re}\,\Pi'^{3\gamma}(s) - \Pi'^{3\gamma}(0)\right] , \tag{1.31}$$

which exhibit the leading hadronic non-perturbative parts, i.e., the ones involving the photon field via mixing. Besides $\Delta\alpha_{\mathrm{had}}^{(5)}(s)$, $\Delta\alpha_{2\,\mathrm{had}}^{(5)}(s)$ can also then be obtained in terms of e^+e^- data, together with isospin flavour separation of (u, d) and s components

$$\Pi_{\mathrm{ud}}^{3\gamma} = \frac{1}{2}\,\Pi_{\mathrm{ud}}^{\gamma\gamma} \; ; \qquad \Pi_{\mathrm{s}}^{3\gamma} = \frac{3}{4}\,\Pi_{\mathrm{s}}^{\gamma\gamma} \tag{1.32}$$

and for resonance contributions

$$\begin{aligned} \Pi^{\gamma\gamma} &= \Pi^{(\rho)} + \Pi^{(\omega)} + \Pi^{(\phi)} + \cdots \\ \Pi^{3\gamma} &= \frac{1}{2}\,\Pi^{(\rho)} + \frac{3}{4}\,\Pi^{(\phi)} + \cdots \end{aligned} \tag{1.33}$$

We are reminded that gauge-boson self-energies are potentially very sensitive to new physics (oblique corrections) and the discovery of what is missing in the SM may be obscured by non-perturbative hadronic effects. Therefore, it is important to reduce the related uncertainties. Interestingly, flavour separation assuming OZI violating terms to be small implies a perturbative reweighting, which, however, has been shown to disagree with lattice QCD results [95–98]! Indeed, the 'wrong' perturbative flavour weighting

$$\Pi_{\mathrm{ud}}^{3\gamma} = \frac{9}{20}\,\Pi_{\mathrm{ud}}^{\gamma\gamma} \; ; \qquad \Pi_{\mathrm{s}}^{3\gamma} = \frac{3}{4}\,\Pi_{\mathrm{s}}^{\gamma\gamma}$$

clearly mismatch lattice results, while the replacement $9/20 \Rightarrow 10/20$ is in good agreement. This also means that the OZI suppressed contributions should be at the 5% level and not negligibly

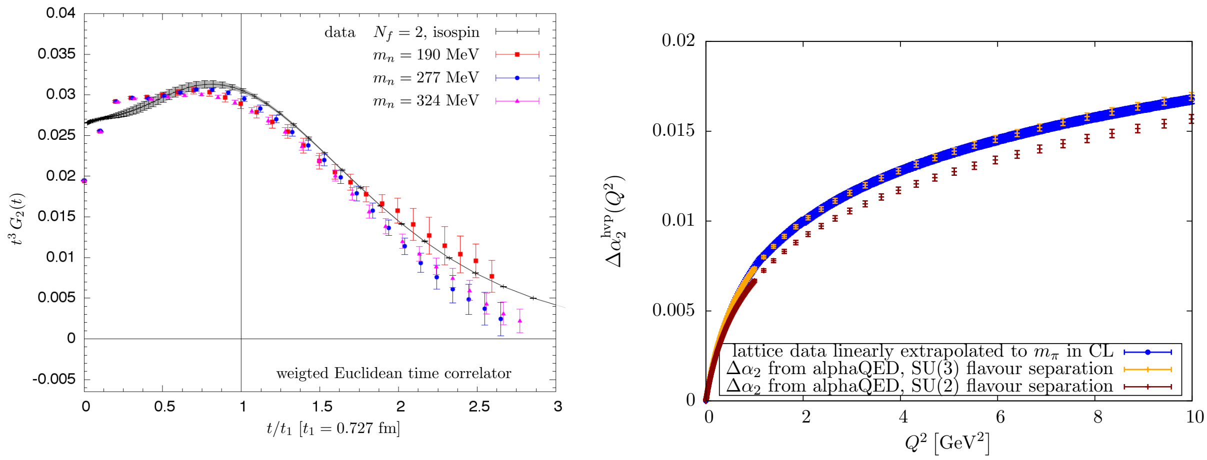


Fig. B.1.17: Testing flavour separation in lattice QCD. Left: a rough test by checking the Euclidean time correlators clearly favours the flavour separation of Eq. (1.33) [95–97], while the pQCD reweighting (not displayed) badly fails. Right: the renormalised photon self-energy at Euclidean Q^2 [98] is in good agreement with the flavour $SU(3)$ limit, while again it fails with the $SU(2)$ case, which coincides with perturbative reweighting.

small. Actually, if we assume flavour $SU(3)$ symmetry to be an acceptable approximation, we obtain

$$\Pi^{3\gamma}_{\mathrm{uds}} = \frac{1}{2}\,\Pi^{\gamma\gamma}_{\mathrm{uds}}\,,$$

which does not require any flavour separation in the uds-sector, i.e., up to the charm threshold at about 3.1 GeV. Figure B.1.17 shows a lattice QCD test of two flavour separation schemes. One, labelled '$SU(2)$', denotes the perturbative reweighting advocated in Refs. [91–94] and the other, labelled '$SU(3)$', represents that proposed in Ref. [89]. Lattice data clearly disprove pQCD reweighting for the uds-sector! This also shows that pQCD-type predictions based on effective quark masses cannot be accurate. This criticism also applies in cases where the effective quark masses have been obtained by fitting $\Delta\alpha^{(5)}_{\mathrm{had}}(s)$, even more so, when constituent quark masses are used.

The effective $SU(2)$ coupling $\Delta\alpha_2(E)$ in comparison with $\Delta\alpha_{\mathrm{QED}}(E)$ is displayed in Fig. B.1.18, and the updated $\sin^2\Theta_W(s)$ is shown in Fig. B.1.16 for time-like as well as for space-like momentum transfer. Note that $\sin^2\Theta_{\mathrm{W}}(0)/\sin^2\Theta_{\mathrm{W}}(M_Z^2) = 1.028\,76$; a 3% correction is established at 6.5σ. Except for the LEP and SLD points (which deviate by 1.8σ), all existing measurements are of rather limited accuracy, unfortunately. Upcoming experiments will substantially improve results at low space-like Q. We are reminded that $\sin^2\Theta_{\ell\,\mathrm{eff}}$, exhibiting a specific dependence on the gauge-boson self-energies, is an excellent monitor for new physics. At pre-LHC times, it was the predestined monitor for virtual Higgs particle effects and a corresponding limiter for the Higgs boson mass.

Acknowledgements

I would like to thank Janusz Gluza and the organising committee for the invitation to this workshop and for support. Thanks also to Maurice Benayoun, Simon Eidelman, and Graziano Venanzoni for collaboration and many useful discussions on the topics presented here.

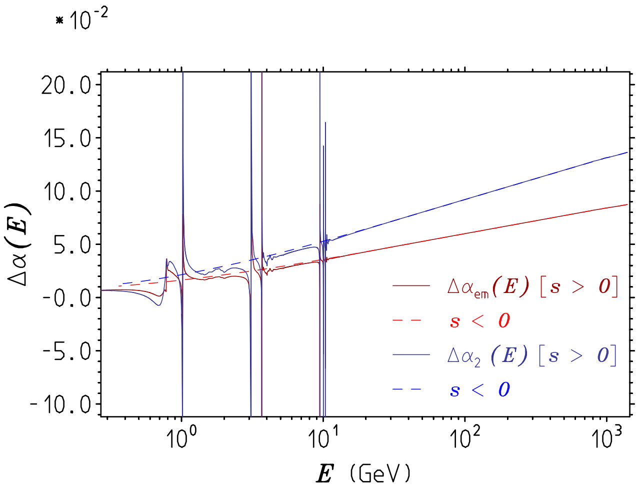


Fig. B.1.18: $\Delta\alpha_{\mathrm{QED}}(E)$ and $\Delta\alpha_2(E)$ as functions of energy E in the time-like and space-like domain. The smooth space-like correction (dashed line) agrees rather well with the non-resonant 'background' above the ϕ resonance (a kind of duality). In resonance regions, as expected, 'agreement' is observed in the mean, with huge local deviations.

References

[1] F. Jegerlehner, *Acta Phys. Pol.* **B45** (2014) 1167. arXiv:1304.7813, doi:10.5506/APhysPolB.45.1167

[2] T. Hambye and K. Riesselmann, *Phys. Rev.* **D55** (1997) 7255. arXiv:hep-ph/9610272, doi:10.1103/PhysRevD.55.7255

[3] F. Jegerlehner *et al.*, *Phys. Lett.* **B722** (2013) 123. arXiv:1212.4319, doi:10.1016/j.physletb.2013.04.012

[4] F. Jegerlehner, *Acta Phys. Pol.* **B45** (2014) 1215. arXiv:1402.3738, doi:10.5506/APhysPolB.45.1215

[5] F. Jegerlehner *et al.*, *J. Phys. Conf. Ser.* **608** (2015) 012074. arXiv:1412.4215, doi:10.1088/1742-6596/608/1/012074

[6] F. Bezrukov *et al.*, *JHEP* **10** (2012) 140, [,275(2012)]. arXiv:1205.2893, doi:10.1007/JHEP10(2012)140

[7] G. Degrassi *et al.*, *JHEP* **08** (2012) 098. arXiv:1205.6497, doi:10.1007/JHEP08(2012)098

[8] J.S. Mackenzie, *Int. J. Ethics* **4** (1893) 126. doi:10.1086/intejethi.4.1.2375721

[9] S. Eidelman and F. Jegerlehner, *Z. Phys.* **C67** (1995) 585. arXiv:hep-ph/9502298, doi:10.1007/BF01553984

[10] R.R. Akhmetshin *et al.*, *Phys. Lett.* **B578** (2004) 285. arXiv:hep-ex/0308008, doi:10.1016/j.physletb.2003.10.108

[11] V.M. Aul'chenko *et al.*, *JETP Lett.* **82** (2005) 743 [*Pisma Zh. Eksp. Teor. Fiz.* **82** (2005) 841.] arXiv:hep-ex/0603021, doi:10.1134/1.2175241

[12] V.M. Aul'chenko *et al.*, *JETP Lett.* **84** (2006) 413 [*Pisma Zh. Eksp. Teor. Fiz.* 84 (2006) 491]. arXiv:hep-ex/0610016, doi:10.1134/S0021364006200021

[13] R.R. Akhmetshin *et al.*, *Phys. Lett.* **B648** (2007) 28. arXiv:hep-ex/0610021, doi:10.1016/j.physletb.2007.01.073

[14] M.N. Achasov *et al.*, *J. Exp. Theor. Phys.* **103** (2006) 380 [*Zh. Eksp. Teor. Fiz.* 130 (2006) 437]. arXiv:hep-ex/0605013, doi:10.1134/S106377610609007X

[15] A. Aloisio *et al.*, *Phys. Lett.* **B606** (2005) 12. arXiv:hep-ex/0407048, doi:10.1016/j.physletb.2004.11.068

[16] F. Ambrosino *et al.*, *Phys. Lett.* **B670** (2009) 285. arXiv:0809.3950, doi:10.1016/j.physletb.2008.10.060

[17] F. Ambrosino *et al.*, *Phys. Lett.* **B700** (2011) 102. arXiv:1006.5313, doi:10.1016/j.physletb.2011.04.055

[18] D. Babusci *et al.*, *Phys. Lett.* **B720** (2013) 336. arXiv:1212.4524, doi:10.1016/j.physletb.2013.02.029

[19] A. Anastasi *et al.*, *JHEP* **03** (2018) 173. arXiv:1711.03085, doi:10.1007/JHEP03(2018)173

[20] G. Venanzoni, *EPJ Web Conf.* **166** (2018) 00021. arXiv:1705.10365, doi:10.1051/epjconf/201816600021

[21] B. Aubert *et al.*, *Phys. Rev. Lett.* **103** (2009) 231801. arXiv:0908.3589, doi:10.1103/PhysRevLett.103.231801

[22] J.P. Lees *et al.*, *Phys. Rev.* **D86** (2012) 032013. arXiv:1205.2228, doi:10.1103/PhysRevD.86.032013

[23] M. Ablikim *et al.*, *Phys. Lett.* **B753** (2016) 629. arXiv:1507.08188, doi:10.1016/j.physletb.2015.11.043

[24] T. Xiao *et al.*, *Phys. Rev.* **D97** (2018) 032012. arXiv:1712.04530, doi:10.1103/PhysRevD.97.032012

[25] B. Aubert *et al.*, *Phys. Rev.* **D70** (2004) 072004. arXiv:hep-ex/0408078, doi:10.1103/PhysRevD.70.072004

[26] B. Aubert *et al.*, *Phys. Rev.* **D71** (2005) 052001. arXiv:hep-ex/0502025, doi:10.1103/PhysRevD.71.052001

[27] B. Aubert *et al.*, *Phys. Rev.* **D73** (2006) 012005. arXiv:hep-ex/0512023, doi:10.1103/PhysRevD.73.012005

[28] B. Aubert *et al.*, *Phys. Rev.* **D73** (2006) 052003. arXiv:hep-ex/0602006, doi:10.1103/PhysRevD.73.052003

[29] B. Aubert *et al.*, *Phys. Rev.* **D76** (2007) 092005 [Erratum: **D77** (2008) 119902]. arXiv:0708.2461, doi:10.1103/PhysRevD.76.092005, doi:10.1103/PhysRevD.77.119902

[30] B. Aubert *et al.*, *Phys. Rev.* **D77** (2008) 092002. arXiv:0710.4451, doi:10.1103/PhysRevD.77.092002

[31] J.P. Lees *et al.*, *Phys. Rev.* **D85** (2012) 112009. arXiv:1201.5677, doi:10.1103/PhysRevD.85.112009

[32] J.P. Lees *et al.*, *Phys. Rev.* **D86** (2012) 012008. arXiv:1103.3001, doi:10.1103/PhysRevD.86.012008

[33] J.P. Lees *et al.*, *Phys. Rev.* **D87** (2013) 092005. arXiv:1302.0055, doi:10.1103/PhysRevD.87.092005

[34] J.P. Lees *et al.*, *Phys. Rev.* **D88** (2013) 032013. arXiv:1306.3600, doi:10.1103/PhysRevD.88.032013

[35] J.P. Lees *et al.*, *Phys. Rev.* **D89** (2014) 092002. arXiv:1403.7593, doi:10.1103/PhysRevD.89.092002

[36] J.P. Lees *et al.*, *Phys. Rev.* **D98** (2018) 112015. arXiv:1810.11962, doi:10.1103/PhysRevD.98.112015

[37] M. Davier, *Nucl. Part. Phys. Proc.* **260** (2015) 102. doi:10.1016/j.nuclphysbps.2015.02.021

[38] M. Davier *et al.*, in *The Standard Theory of Particle Physics*, Eds. L. Maiani and L. Rolandi (World Scientific, Singapore, 2016), Chap. 7, pp. 129–144. doi:10.1142/9789814733519_0007

[39] R.R. Akhmetshin *et al.*, *Phys. Lett.* **B723** (2013) 82. arXiv:1302.0053, doi:10.1016/j.physletb.2013.04.065

[40] R.R. Akhmetshin *et al.*, *Phys. Lett.* **B759** (2016) 634. arXiv:1507.08013, doi:10.1016/j.physletb.2016.04.048

[41] E.A. Kozyrev *et al.*, *Phys. Lett.* **B760** (2016) 314. arXiv:1604.02981, doi:10.1016/j.physletb.2016.07.003

[42] M.N. Achasov *et al.*, *Phys. Rev.* **D90** (2014) 112007. arXiv:1410.3188, doi:10.1103/PhysRevD.90.112007

[43] V.M. Aulchenko *et al.*, *Phys. Rev.* **D91** (2015) 052013. arXiv:1412.1971, doi:10.1103/PhysRevD.91.052013

[44] M.N. Achasov *et al.*, *Phys. Rev.* **D93** (2016) 092001. arXiv:1601.08061, doi:10.1103/PhysRevD.93.092001

[45] M.N. Achasov *et al.*, *Phys. Rev.* **D94** (2016) 032010. arXiv:1606.06481, doi:10.1103/PhysRevD.94.032010

[46] M.N. Achasov *et al.*, *Phys. Rev.* **D94** (2016) 092002. arXiv:1607.00371, doi:10.1103/PhysRevD.94.092002

[47] M.N. Achasov *et al.*, *Phys. Rev.* **D94** (2016) 112006. arXiv:1608.08757, doi:10.1103/PhysRevD.94.112006

[48] M.N. Achasov *et al.*, *Phys. Rev.* **D94** (2016) 112001. arXiv:1610.00235, doi:10.1103/PhysRevD.94.112001

[49] J.Z. Bai *et al.*, *Phys. Rev. Lett.* **84** (2000) 594. arXiv:hep-ex/9908046, doi:10.1103/PhysRevLett.84.594

[50] J.Z. Bai *et al.*, *Phys. Rev. Lett.* **88** (2002) 101802. arXiv:hep-ex/0102003, doi:10.1103/PhysRevLett.88.101802

[51] M. Ablikim *et al.*, *Phys. Lett.* **B677** (2009) 239. arXiv:0903.0900, doi:10.1016/j.physletb.2009.05.055

[52] V.V. Anashin *et al.*, *Phys. Lett.* **B753** (2016) 533. arXiv:1510.02667, doi:10.1016/j.physletb.2015.12.059

[53] V.V. Anashin *et al.*, *Phys. Lett.* **B770** (2017) 174. arXiv:1610.02827, doi:10.1016/j.physletb.2017.04.073

[54] R.V. Harlander and M. Steinhauser, *Comput. Phys. Commun.* **153** (2003) 244. arXiv:hep-ph/0212294, doi:10.1016/S0010-4655(03)00204-2

[55] F. Jegerlehner, *EPJ Web Conf.* **118** (2016) 01016. arXiv:1511.04473, doi:10.1051/epjconf/201611801016

[56] A. Anastasi *et al.*, *Phys. Lett.* **B767** (2017) 485. arXiv:1609.06631, doi:10.1016/j.physletb.2016.12.016

[57] M. Davier *et al.*, *Eur. Phys. J.* **C77** (2017) 827. arXiv:1706.09436, doi:10.1140/epjc/s10052-017-5161-6

[58] A. Maier and P. Marquard, *Phys. Rev.* **D97** (2018) 056016. arXiv:1710.03724, doi:10.1103/PhysRevD.97.056016

[59] S.L. Adler, *Phys. Rev.* **D10** (1974) 3714. doi:10.1103/PhysRevD.10.3714

[60] S. Eidelman *et al.*, *Phys. Lett.* **B454** (1999) 369. arXiv:hep-ph/9812521, doi:10.1016/S0370-2693(99)00389-5

[61] F. Jegerlehner, Hadronic effects in $(g-2)_\mu$ and $\alpha_{\mathrm{QED}}(M_Z^2)$: status and perspectives, Radiative Corrections: Application of Quantum Field Theory to Phenomenology. Proc. 4th Int. Symp., RADCOR'98, Barcelona, Spain, 1998, pp. 75–89, arXiv:hep-ph/9901386

[62] F. Jegerlehner, Hadronic vacuum polarisation effects in $\alpha_{\mathrm{em}}(M_Z^2)$, Electroweak Precision Data and the Higgs Mass. Proc. Workshop, Zeuthen, Germany, 2003, pp. 97–112, arXiv:hep-ph/0308117

[63] F. Jegerlehner, *Nucl. Phys. Proc. Suppl.* **181–182** (2008) 135. arXiv:0807.4206, doi:10.1016/j.nuclphysbps.2008.09.010

[64] M. Benayoun *et al.*, *Eur. Phys. J.* **C75** (2015) 613. arXiv:1507.02943, doi:10.1140/epjc/s10052-015-3830-x

[65] M. Benayoun *et al.*, *Eur. Phys. J.* **C80** (2020), 81 [Erratum: **C80** (2020), 224]. arXiv:1903.11034, doi:10.1140/epjc/s10052-020-7611-9, doi:10.1140/epjc/s10052-020-7724-1

[66] F. Jegerlehner and O.V. Tarasov, *Nucl. Phys.* **B549** (1999) 481. arXiv:hep-ph/9809485, doi:10.1016/S0550-3213(99)00141-8

[67] K.G. Chetyrkin *et al.*, *Nucl. Phys.* **B482** (1996) 213. arXiv:hep-ph/9606230, doi:10.1016/S0550-3213(96)00534-2

[68] K.G. Chetyrkin *et al.*, *Nucl. Phys.* **B503** (1997) 339. arXiv:hep-ph/9704222, doi:10.1016/S0550-3213(97)00383-0

[69] K.G. Chetyrkin *et al.*, *Nucl. Phys.* **B505** (1997) 40. arXiv:hep-ph/9705254, doi:10.1016/S0550-3213(97)00481-1

[70] M.A. Shifman *et al.*, *Nucl. Phys.* **B147** (1979) 385. doi:10.1016/0550-3213(79)90022-1

[71] M.A. Shifman *et al.*, *Nucl. Phys.* **B147** (1979) 448. doi:10.1016/0550-3213(79)90023-3

[72] D.V. Shirkov and I.L. Solovtsov, *Phys. Rev. Lett.* 79 (1997) 1209. arXiv:hep-ph/9704333, doi:10.1103/PhysRevLett.79.1209

[73] M. Bruno *et al.*, *Phys. Rev. Lett.* **119** (2017) 102001. arXiv:1706.03821, doi:10.1103/PhysRevLett.119.102001

[74] M. Steinhauser, *Phys. Lett.* **B429** (1998) 158. arXiv:hep-ph/9803313, doi:10.1016/S0370-2693(98)00503-6

[75] G. Abbiendi *et al.*, *Eur. Phys. J.* **C77** (2017) 139. arXiv:1609.08987, doi:10.1140/epjc/s10052-017-4633-z

[76] P. Janot, *JHEP* **02** (2016) 053 [Erratum: **11** (2017) 164]. arXiv:1512.05544, doi:10.1007/JHEP02(2016)053, doi:10.1007/JHEP11(2017)164

[77] A. Keshavarzi *et al.*, *Phys. Rev.* **D97** (2018) 114025. arXiv:1802.02995, doi:10.1103/PhysRevD.97.114025

[78] F. Jegerlehner, Variations on photon vacuum polarisation, , *EPJ Web Conf.* **218** (2019), 01003 arXiv:1711.06089 doi:10.1051/epjconf/201921801003

[79] A. Blondel *et al.*, Standard Model theory for the FCC-ee Tera-Z stage, CERN (CERN Yellow Rep. Monogr. 3, Geneva, Switzerland), arXiv:1809.01830, doi:10.23731/CYRM-2019-003

[80] I. Dubovyk *et al.*, *Phys. Lett.* **B783** (2018) 86. arXiv:1804.10236, doi:10.1016/j.physletb.2018.06.037

[81] P. Azzurri *et al.*, Physics behind precision, arXiv:1703.01626

[82] S. Jadach *et al.*, *Phys. Lett.* **B790** (2019) 314. arXiv:1812.01004, doi:10.1016/j.physletb.2019.01.012

[83] S. Jadach *et al.*, Event generators for Bhabha scattering, CERN Workshop on LEP2 Physics, (CERN, Geneva, Switzerland, 1995), pp. 229–298, arXiv:hep-ph/9602393

[84] A. Arbuzov *et al.*, *Phys. Lett.* **B383** (1996) 238. arXiv:hep-ph/9605239, doi:10.1016/0370-2693(96)00733-2

[85] P. Mastrolia *et al.*, *JHEP* **11** (2017) 198. arXiv:1709.07435, doi:10.1007/JHEP11(2017)198

[86] S. Di Vita *et al.*, *JHEP* **09** (2018) 016. arXiv:1806.08241, doi:10.1007/JHEP09(2018)016

[87] M. Fael, *JHEP* **02** (2019) 027. arXiv:1808.08233, doi:10.1007/JHEP02(2019)027

[88] M. Alacevich *et al.*, *JHEP* **02** (2019) 155. arXiv:1811.06743, doi:10.1007/JHEP02(2019)155

[89] F. Jegerlehner, *Z. Phys.* **C32** (1986) 195. doi:10.1007/BF01552495

[90] F. Jegerlehner, *Nuovo Cim.* **C034S1** (2011) 31. arXiv:1107.4683, doi:10.1393/ncc/i2011-11011-0

[91] A. Czarnecki and W.J. Marciano, *Phys. Rev.* **D53** (1996) 1066. arXiv:hep-ph/9507420, doi:10.1103/PhysRevD.53.1066

[92] A. Czarnecki and W.J. Marciano, *Int. J. Mod. Phys.* **A15** (2000) 2365. arXiv:hep-ph/0003049, doi:10.1142/S0217751X00002433

[93] J. Erler and M.J. Ramsey-Musolf, *Phys. Rev.* **D72** (2005) 073003. arXiv:hep-ph/0409169, doi:10.1103/PhysRevD.72.073003

[94] J. Erler and R. Ferro-Hernández, *JHEP* **03** (2018) 196. arXiv:1712.09146, doi:10.1007/JHEP03(2018)196

[95] D. Bernecker and H.B. Meyer, *Eur. Phys. J.* **A47** (2011) 148. arXiv:1107.4388, doi:10.1140/epja/i2011-11148-6

[96] A. Francis *et al.*, Vector correlator and scale determination in lattice QCD, PoS LATTICE 2013, Mainz 2013, paper 320. arXiv:1312.0035, doi:10.22323/1.187.0320

[97] M. Cè *et al.*, The leading hadronic contribution to the running of the Weinberg angle using covariant coordinate-space methods, 36th Ann. Int. Symp. Lattice Field Theory, 2018, East Lansing, MI, USA, 2018, paper 137. arXiv:1811.08669

[98] F. Burger *et al.*, *JHEP* **11** (2015) 215. arXiv:1505.03283, doi:10.1007/JHEP11(2015)215