7 Prospects for higher-order corrections to W pair production near threshold in the EFT approach

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The precise measurement of the mass of the W boson plays an essential role for precision tests of the Standard Model (SM) and indirect searches for new physics through global fits to electroweak observables. Cross-section measurements near the W pair production threshold at a possible future e^-e^+ collider promise to reduce the experimental uncertainty to the level of 3 MeV at an International Linear Collider (ILC) [1, 2], while a high-luminosity circular collider offers a potential improvement to 0.5 MeV in the case of the FCC-ee [3, 4] or 1 MeV at the CEPC [5]. At the point of highest sensitivity, an uncertainty in the cross-section measurement of 0.1% translates to an uncertainty of ∼1.5 MeV on M_W [3]. Therefore, a theoretical prediction for the cross-section with an accuracy of $\Delta\sigma \sim 0.01\%$ at threshold is required to fully exploit the potential of a future circular e^-e^+ collider. Theory predictions using the double-pole approximation (DPA) [6] at next-to-leading order (NLO) [7–11] successfully described LEP2 results with an accuracy of better than 1% above threshold. An extension of the DPA to NNLO appears to be appropriate for a future e^-e^+ collider operating above the W pair threshold, e.g., for the interpretation of anomalous triple-gauge-coupling measurements at $\sqrt{s} = 240$ GeV. However, the accuracy of the DPA at NLO degrades to 2–3% near the threshold. In this region, the combination of a full NLO calculation of four-fermion production [12, 13] with leading NNLO effects obtained using effective field theory (EFT) methods [14, 15] reduces the theory uncertainty of the total cross-section to below 0.3%; sufficient for the ILC target uncertainty but far above that of the FCC-ee. This raises the question of the methods required to reach a theory accuracy ∼0.01%. In this contribution, this issue is addressed from the EFT point of view. The discussion is limited to the total cross-section, where the EFT approach is best developed so far, although cuts on the W decay products can also be incorporated [15]. To reach the target accuracy, it will also be essential to have theoretical control of effects beyond the pure electroweak effects considered here. In particular, it is assumed that next-to-leading logarithmic corrections $(\alpha/\pi)^2 \ln(m_\mathrm{e}^2/s)$ from collinear initial-state photon radiation (ISR), which have been estimated to be $\lesssim 0.1\%$ [12], will be resummed to all orders. The QCD effects, which are particularly important for the fully hadronic decay modes, are only briefly considered. In Section 7.1, aspects of the EFT approach are reviewed from an updated perspective using insight into the factorisation of soft, hard, and Coulomb corrections [16]. The NLO and leading NNLO results are summarised and compared with the $\mathrm{NLO^{ee4f}}$ calculation [12]. In Section 7.2, the structure of the EFT expansion and calculations of subsets of corrections are used to estimate the magnitude of the NNLO and leading $\mathrm{N^3LO}$ corrections and to determine whether such calculations are sufficient to meet the FCC-ee target accuracy.

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7.1 Effective theory approach to W pair production

In the EFT approach to four-fermion production near the W pair production threshold [14], the cross-section is expanded simultaneously in the coupling, the W decay width, and the energy relative to the production threshold, which are taken to be of similar order and are denoted collectively by

$$\delta \sim v^2 \equiv \frac{(s-4M_{\mathrm{W}}^2)}{M_{\mathrm{W}}^2} \sim \frac{\Gamma_{\mathrm{W}}}{M_{\mathrm{W}}} \sim \alpha. \tag{7.1}$$

An NNLO$^{\mathrm{EFT}}$ calculation includes corrections up to $\mathcal{O}(\delta^n)$, whereas, as usual, NNLO refers to the $\mathcal{O}(\alpha^n)$ corrections. As discussed in Sections 7.1.1 and 7.1.2, non-resonant and Coulomb corrections lead to odd powers of v, so that the expansion proceeds in half-integer powers of δ. The current state of the art in the EFT is the calculation of the total cross-section for the semi-leptonic final state $\mu^-\bar{\nu}_\mu \mathrm{u}\bar{\mathrm{d}}$ up to NLO$^{\mathrm{EFT}}$ [14], which includes corrections of the order

$$\mathrm{NLO}^{\mathrm{EFT}}: v^2, \alpha, \alpha^2/v^2, \tag{7.2}$$

supplemented with the genuine $\mathcal{O}(\alpha^2, \alpha^3)$ corrections at the next order, $\delta^{3/2}$, in the δ-expansion [15],

$$\mathrm{N}^{3/2}\mathrm{LO}^{\mathrm{EFT}}: \quad \alpha v, \quad \alpha^2/v, \quad \alpha^3/v^3. \tag{7.3}$$

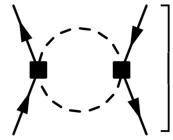
In the following, aspects of these results and the EFT method are reviewed that are useful for the estimate of NNLO$^{\mathrm{EFT}}$ corrections and the remaining uncertainty.

7.1.1 Expansion of the Born cross-section

The total cross-section $\mathrm{e}^-\mathrm{e}^+ \to 4\mathrm{f}$ can be obtained from the imaginary part of the forward-scattering amplitude $\mathrm{e}^-\mathrm{e}^+ \to \mathrm{e}^-\mathrm{e}^+$, where the Cutkosky cuts are restricted to those with four-fermion final states. Flavour-specific final states can be selected accordingly. The expansion of the forward-scattering amplitude in δ can be formulated in terms of an EFT [14, 17, 18], where the initial-state leptons are described by soft-collinear effective theory [19], and the W bosons by a non-relativistic EFT. Similarly to the DPA [6], the cross-section is decomposed into resonant and non-resonant contributions:

$$\sigma^{\mathrm{ee4f}}(s \approx 4M_{\mathrm{W}}^2) = \sigma_{\mathrm{res}}(s) + \sigma_{\text{non-res}}(s). \tag{7.4}$$

The EFT method enables computation of the Born cross-section as an expansion according to the counting (Eq. (7.1)), $\sigma^{\mathrm{ee4f}}_{\mathrm{Born}} = \sigma^{(0)}_{\mathrm{Born}} + \sigma^{(1/2)}_{\mathrm{Born}} + \ldots$ This is not necessary in practice since the full $\mathrm{e}^-\mathrm{e}^+ \to 4\mathrm{f}$ Born cross-section for arbitrary kinematics can be computed using automated Monte Carlo programs. However, the expansion serves as a test-case of the EFT method and provides useful input for estimating the accuracy of a future NNLO$^{\mathrm{EFT}}$ calculation. The leading-order resonant contribution to the cross-section is given by the imaginary part of a one-loop EFT diagram with non-relativistic W propagators, denoted by dashed lines,

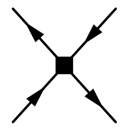
$$\sigma^{(0)}_{\mathrm{Born}}(s) = \sigma^{(0)}_{\mathrm{res}}(s) = \frac{1}{s}\mathrm{Im}\Big[$$

$$\Big] = \frac{\pi\alpha^2}{s_{\mathrm{W}}^4 s}\mathrm{Im}\left[-\sqrt{-\frac{\mathcal{E}_{\mathrm{W}}}{M_{\mathrm{W}}}}\,\right]. \tag{7.5}$$

Here, the complex energy variable $\mathcal{E}_{\mathrm{W}} \equiv \sqrt{s} - 2M_{\mathrm{W}} + \mathrm{i}\Gamma_{\mathrm{W}} \sim M_{\mathrm{W}}v^2$ has been introduced and $s_{\mathrm{W}} = \sin\theta_{\mathrm{W}}$ with the weak mixing angle θ_{W}. A specific final state is selected by multiplying Eq. (7.5) by the LO branching ratios,

$$\sigma^{(0)}_{\mathrm{f}_1\bar{\mathrm{f}}_2\mathrm{f}_3\bar{\mathrm{f}}_4} = \frac{\Gamma^{(0)}_{\mathrm{W}^-\to\mathrm{f}_1\bar{\mathrm{f}}_2}\Gamma^{(0)}_{\mathrm{W}^+\to\mathrm{f}_3\bar{\mathrm{f}}_4}}{\Gamma_{\mathrm{W}}^2}\sigma^{(0)}_{\mathrm{res}}. \tag{7.6}$$

The non-resonant contribution to the cross-section arises from local four-electron operators,

$$\sigma_{\text{non-res}}(s) = \frac{1}{s}\mathrm{Im}\left[\quad\right] = \frac{\alpha^3}{s_{\mathrm{W}}^6 s}\mathcal{K}, \tag{7.7}$$



where the dimensionless constant $\mathcal{K} = \mathcal{K}^{(0)} + \alpha\mathcal{K}^{(1)}/s_{\mathrm{W}}^2 + \ldots$ is computed from the forward-scattering amplitude in the full SM without self-energy resummation in the W propagators. The first contribution is of order α^3 and arises from cut two-loop diagrams corresponding to squared tree diagrams of the $\mathrm{e}^-\mathrm{e}^+ \to \mathrm{W}^\pm\mathrm{f}\bar{\mathrm{f}}$ processes. Hence, the leading non-resonant contribution $\sigma^{(1/2)}_{\text{non-res}} \equiv \sigma^{(1/2)}_{\text{Born}}$ is suppressed by $\alpha/v \sim \delta^{1/2}$ compared with the resonant LO cross-section (Eq. (7.5)). For the final state, $\mu^-\bar{\nu}_\mu \mathrm{u}\bar{\mathrm{d}}$, the explicit result is [14]*

$$\mathcal{K}^{(0)} = -4.25698. \tag{7.8}$$

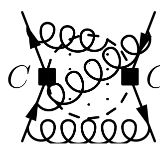
The $\mathcal{O}(v^2)$ corrections in Eq. (7.2) originate from higher-order terms in the EFT expansion of the resonant Born cross-section, $\sigma^{(1)}_{\text{Born}}$, and depend strongly on the centre-of-mass energy [14],

$$\sigma^{(1)}_{\text{Born}}(\sqrt{s} = 161\ \text{GeV}) = 8\% \times \sigma^{\text{ee4f}}_{\text{Born}}, \qquad \sigma^{(1)}_{\text{Born}}(\sqrt{s} = 170\ \text{GeV}) = -8\% \times \sigma^{\text{ee4f}}_{\text{Born}}. \tag{7.9}$$

7.1.2 Radiative corrections

Including radiative corrections, the resonant cross-section factorises into hard, soft, and Coulomb functions [16]. (This formula holds for the leading term in the expansion in v. Subleading terms result in a sum over Wilson coefficients and Green functions related to higher partial waves. In higher orders, there are also soft corrections to the Coulomb function analogous to ultrasoft QCD corrections in $\mathrm{t}\bar{\mathrm{t}}$ production [20].)

$$\sigma_{\text{res}}(s) = \mathrm{Im}\left[\quad\right] = \frac{4\pi^2\alpha^2}{sM_{\mathrm{W}}^2 s_{\mathrm{W}}^4}\mathrm{Im}\left[C^2\int \mathrm{d}\omega W(\omega) G_{\mathrm{C}}(0,0,\mathcal{E}_{\mathrm{W}}-\omega)\right]. \tag{7.10}$$



Here, curly lines depict soft photons with momenta $(q^0, \vec{q}) \sim (\delta, \delta)$, while dotted lines denote potential (Coulomb) photons with $(q^0, \vec{q}) \sim (\delta, \sqrt{\delta})$. The Wilson coefficient $C = 1 + \alpha C^{(1)}/2\pi\ldots$ is related to contributions of hard loop momenta $q \sim M_{\mathrm{W}}$ to the on-shell amplitudes $\mathrm{e}^-\mathrm{e}^+ \to \mathrm{W}^-\mathrm{W}^+$ evaluated at the production threshold. For the input parameters used in Ref. [14], the explicit value of the one-loop coefficient is

$$C^{(1)} = \mathrm{Re}\, c^{(1,\text{fin})}_{p,\text{LR}} = -10.076. \tag{7.11}$$

The function $W(\omega)$ includes soft-photon effects, which decouple from the W bosons [21, 22] for the total cross-section, since soft radiation is only sensitive to the total (i.e., vanishing) electric charge of the produced system. This function is the QED analogue of the soft function for Drell–Yan production near the partonic threshold [23, 24]. The leading Coulomb Green function at the origin,

$$G^{(0)}_{\mathrm{C}}(0,0;\mathcal{E}_{\mathrm{W}}) = -\frac{M_{\mathrm{W}}^2}{4\pi}\left\{\sqrt{-\frac{\mathcal{E}_{\mathrm{W}}}{M_{\mathrm{W}}}} + \frac{\alpha}{2}\ln\left(-\frac{\mathcal{E}_{\mathrm{W}}}{M_{\mathrm{W}}}\right) - \frac{\alpha^2\pi^2}{12}\sqrt{-\frac{M_{\mathrm{W}}}{\mathcal{E}_{\mathrm{W}}}} + \alpha^3\frac{\zeta(3)}{4}\frac{M_{\mathrm{W}}}{\mathcal{E}_{\mathrm{W}}} + \cdots\right\}, \tag{7.12}$$

*Equation (7.8) is obtained by setting $s = 4M_{\mathrm{W}}^2$ in Eq. (37) in Ref. [14], where an additional s-dependence of $\mathcal{K}$ has been kept.

sums Coulomb exchange and is known to all orders (see, e.g., Ref. [14]). At each order, the Coulomb corrections $\sim(\alpha/v)^n \sim \delta^{n/2}$ are parametrically enhanced over the remaining $\mathcal{O}(\alpha^n)$ corrections but do not have to be resummed to all orders, owing to the screening of the Coulomb singularity by Γ_W [25]. The convolution of the soft and Coulomb functions results in logarithms of $\mathcal{E}_\mathrm{W} \sim M_\mathrm{W} v^2$, which can be resummed in analogy to threshold resummation at hadron colliders [24,26,27]. However, for QED corrections, $\alpha \log v$ is not enhanced, so this resummation is formally not necessary.† Higher-order corrections to the non-resonant cross-section (Eq. (7.7)) only arise through hard corrections to $\mathcal{K}$, while loop corrections in the EFT vanish.

These ingredients provide results for massless initial-state electrons and could be used, in analogy to QCD predictions at hadron colliders, to define appropriate 'partonic' cross-sections that are convoluted with corresponding electron structure functions resumming large mass logarithms. Structure functions in such a scheme are known up to NNLO [29]. In the NLO$^\mathrm{EFT}$ calculation of Ref. [14], however, electron mass effects have been treated by including collinear corrections and matching to the commonly used resummed structure functions [30] by subtracting double-counting contributions.‡

A useful result [15] for computing a class of higher-order effects of the form α^{n+1}/v^n is obtained from Eq. (7.10) by combining the all-order Coulomb Green function with one-loop hard and soft corrections and matching to ISR structure functions, as in the NLO$^\mathrm{EFT}$ calculation:

$$\Delta\sigma^{\mathrm{C}\times[\mathrm{S+H}]_1}(s) = \frac{4\pi^2\alpha^2}{sM_\mathrm{W}^2 s_\mathrm{W}^4}\,\frac{\alpha}{\pi}\left\{\left(\frac{7}{2}+\frac{\pi^2}{4}+C^{(1)}\right)\mathrm{Im}\,G_\mathrm{C}(0,0;\mathcal{E}_\mathrm{W}).\right. \qquad (7.13)$$

Corrections of the same order, α^{n+1}/v^n, result from the NLO Green function [31] $G_\mathrm{C}^{(1)}$, which includes the $\mathcal{O}(\alpha)$ correction to the Coulomb potential. In the G_μ input parameter scheme, the $\mathcal{O}(\alpha^2/v)$ correction reads [15]

$$\Delta G_\mathrm{C}^{(1)}(0,0,\mathcal{E}_\mathrm{W}) = -\frac{M_\mathrm{W}^2}{4\pi}\frac{\alpha^2}{8\pi}\ln\left(-\frac{\mathcal{E}_\mathrm{W}}{M_\mathrm{W}}\right)\left\{-\frac{\beta_0}{2}\left[\ln\left(-\frac{\mathcal{E}_\mathrm{W}}{M_\mathrm{W}}\right)\right]+\Delta_{G_\mu}\right\}+\mathcal{O}(\alpha^3) \qquad (7.14)$$

with the QED beta function with five quark flavours, $\beta_0 = -4(\sum_{\mathrm{f}\neq\mathrm{t}} N_{C_\mathrm{f}} Q_\mathrm{f}^2)/3 = -80/9$, and where the scheme-dependent constant $\Delta_{G_\mu} = 61.634$ is related to the quantity

$$\delta_{\alpha(M_\mathrm{Z})\to G_\mu} = \frac{\alpha}{4\pi}\left(\Delta_{G_\mu} + 2\beta_0 \ln\left(\frac{2M_\mathrm{W}}{M_\mathrm{Z}}\right)\right)$$

used in Ref. [15]. Equations (7.13) and (7.14) are the basis for computing examples of leading N^{3}LO corrections in Section 7.2.

7.1.3 NLO^{EFT} result

The genuine radiative corrections at NLO$^\mathrm{EFT}$ can be obtained by expanding Eq. (7.13) to $\mathcal{O}(\alpha)$ relative to the leading order and adding the second-order Coulomb correction from Eq. (7.12).

† An initial study obtained NLL effects of 0.1% [28], so the relevance for the FCC-ee may have to be revisited.

‡ In the process of finalizing this report, we have noted that NLL contributions arising from the combination of numerator factors of m_e and integrals with negative powers of m_e have been inadvertently omitted in the computation of the collinear corrections. The expressions and numerical predictions in this report are preliminary results including the missing contributions. A more complete discussion will be given elsewhere.

A specific four-fermion final state is selected by multiplying the NLO correction with the LO branching ratios (Eq. (7.6)) and adding NLO decay corrections,

$$\Delta\sigma^{(1)}_{\rm decay} = \left(\frac{\Gamma^{(1,\rm ew)}_{\rm f_1\bar f_2}}{\Gamma^{(0)}_{\rm f_1\bar f_2}} + \frac{\Gamma^{(1,\rm ew)}_{\rm f_3\bar f_4}}{\Gamma^{(0)}_{\rm f_3\bar f_4}} \right) \sigma^{(0)}_{\rm res}, \tag{7.15}$$

with the one-loop electroweak corrections to the partial decay widths, $\Gamma^{(1,\rm ew)}_{\rm f_i\bar f_j}$. For hadronic decay modes, QCD corrections to the partial decay widths must also be included up to NNLO, using the counting $\alpha_{\rm s}^2 \sim \alpha$. In Table B.7.1, the $\mathcal{O}(\alpha)$-contributions of the NLO$^{\rm EFT}$ result are compared with the NLO$^{\rm ee4f}$ calculation in the full SM [12].[§] The differences are of the order

$$\Delta\sigma^{(1)}_{\rm 4f}(s) \equiv \sigma^{\rm ee4f}_{\rm NLO}(s) - \sigma^{(1)}_{\rm EFT}(s) = \sigma^{\rm ee4f}_{\rm Born}(s) \times (0.7 - 0.1)\% \tag{7.16}$$

for $\sqrt{s} = 161$–170 GeV. Near the threshold, the dominant source of this discrepancy is expected to be the $\mathcal{O}(\delta^{3/2})$ contribution from the $\mathcal{O}(\alpha)$ correction to the non-resonant cross-section (Eq. (7.7)), which has not been computed in the EFT approach.[¶] Attributing the difference at $\sqrt{s} = 161$ GeV to this correction, one obtains

$$\mathcal{K}^{(1)} \approx 1.4, \tag{7.17}$$

indicating that the $\mathcal{O}(\alpha)$ corrections to the non-resonant contribution (Eq. (7.8)) are moderate, $|\mathcal{K}^{(1)}/\mathcal{K}^{(0)}| \approx 0.3$. Above the threshold, $\mathcal{O}(\delta^{3/2})$ and $\mathcal{O}(\delta^2)$ corrections to the resonant cross-section are expected to be important; these arise from the combination of $\mathcal{O}(\alpha/v, \alpha)$ corrections in the EFT with $\mathcal{O}(v^2)$ kinematic corrections and from $\mathcal{O}(\alpha)$ corrections to the Wilson coefficients of subleading production operators. Naive estimates using the $\mathcal{O}(v^2)$ expansion of the Born amplitude and the first Coulomb correction,

$$\sigma^{(3/2)}_{\alpha v}(s) \sim |\sigma^{(1)}_{\rm Born}(s)| \sigma^{(1/2)}_{\rm C}(s)/\sigma^{(0)}_{\rm Born}(s), \qquad \sigma^{(2)}_{\alpha v^2}(s) \sim \frac{\alpha}{s^2_{\rm W}} |\sigma^{(1)}_{\rm Born}(s)|, \tag{7.18}$$

indicate that both corrections are $\sim 0.3\% \times \sigma^{\rm ee4f}_{\rm Born}$ at $\sqrt{s} = 170$ GeV, overestimating the discrepancy to the NLO$^{\rm ee4f}$ calculation. To assess the accuracy of the EFT expansion, it would be interesting to calculate these corrections exactly and investigate whether the difference to the NLO$^{\rm ee4f}$ calculation could be reduced, e.g., by resumming relativistic corrections to the W propagators.

7.1.4 Leading NNLO corrections

In Ref. [15], those $\mathcal{O}(\delta^{3/2})$ corrections according to Eq. (7.3) have been computed that originate from genuine NNLO corrections in the usual counting in α. These consist of several classes: (a) interference of one-loop Coulomb corrections with soft and hard corrections (Eq. (7.13)); (b) interference of one-loop Coulomb corrections with corrections to W decay, obtained from Eq. (7.15) by replacing the LO cross-section with the first Coulomb correction; (c) interference of one-loop Coulomb corrections with NLO corrections to residues of W propagators; and (d) radiative NLO corrections to the Coulomb potential (Eq. (7.14)). The third Coulomb correction

[§] Note that here the updated results in the erratum to Ref. [12] are used. The EFT results here and in Table B.7.2 differ from those of Refs. [14, 15] because of the corrected collinear contributions.

[¶] For $\rm e^-e^+ \to t\bar t$, a related calculation has been performed recently [32].

Table B.7.1: Comparison of the strict electroweak NLO results (without QCD corrections, second Coulomb correction and ISR resummation) in the EFT approach to the full NLO$^{\mathrm{ee4f}}$ calculation and the DPA implementation of Ref. [11].

$\sqrt{s}$ (GeV)	$\sigma(\mathrm{e^-e^+} \to \mu^-\bar{\nu}_\mu \mathrm{u}\bar{\mathrm{d}}\,\mathrm{X})$(fb) Born	NLO(EFT) [14]	ee4f [12]	DPA [12]
161	150.05(6)	107.34(6)	106.33(7)	103.15(7)
170	481.2(2)	379.03(2)	379.5(2)	376.9(2)

Table B.7.2: Leading $\mathcal{O}(\alpha^2)$ corrections [15] (second and third column) and contributions to leading $\mathcal{O}(\alpha^3)$ corrections from triple-Coulomb exchange [15] (fourth column), interference of double-Coulomb exchange with soft and hard corrections (Eq. (7.26)) (fifth column), and double-Coulomb exchange with the NLO Coulomb potential (Eq. (7.27)) (sixth column). The relative correction is given with respect to the Born cross-section without ISR improvement, as quoted in Ref. [15].

$\sqrt{s}$ (GeV)	$\sigma(\mathrm{e^-e^+} \to \mu^-\bar{\nu}_\mu \mathrm{u}\bar{\mathrm{d}}\,\mathrm{X})$(fb) $\mathcal{O}(\alpha^2/v^2)$	$\mathcal{O}(\alpha^2/v)$	$\mathcal{O}(\alpha^3/v^3)$	$\mathcal{O}(\alpha^3/v^2)\vert_{\mathrm{C_2\times[S+H]_1}}$	$\mathcal{O}(\alpha^3/v^2)\vert_{\mathrm{C_2^{NLO}}}$
158	0.151 [+0.245%]	0.061 [+0.099%]	3.82×10^{-3} [+0.006%]	-1.50×10^{-3} [−0.002%]	5.38×10^{-3} [+0.009%]
161	0.437 [+0.284%]	0.331 [+0.215%]	9.92×10^{-3} [+0.006%]	-0.433×10^{-2} [−0.003%]	1.52×10^{-2} [+0.010%]
164	0.399 [+0.132%]	1.038 [+0.342%]	2.84×10^{-3} [+0.001%]	-3.95×10^{-3} [−0.001%]	1.97×10^{-2} [+0.007%]
167	0.303 [+0.074%]	1.479 [+0.362%]	9.43×10^{-4} [+0.000%]	-3.00×10^{-3} [−0.001%]	1.77×10^{-2} [+0.004%]
170	0.246 [+0.051%]	1.734 [+0.360%]	4.39×10^{-4} [+0.000%]	-2.43×10^{-3} [−0.001%]	1.56×10^{-2} [+0.003%]

from Eq. (7.12) contributes at the same order, $\delta^{3/2}$. Care has been taken to avoid double-counting corrections already included in the NLO$^{\mathrm{ee4f}}$ calculation, so the two results can be added to obtain the current best prediction for the total cross-section near the threshold. The numerical results are reproduced in Table B.7.2, together with the second Coulomb correction included in the NLO$^{\mathrm{EFT}}$ calculation. The results show that the leading Coulomb-enhanced two-loop corrections are of the order of 0.3%. The uncertainty due to the remaining non-Coulomb-enhanced NNLO corrections was estimated to be below the ILC target accuracy of $\Delta M_{\mathrm{W}} = 3$ MeV [15] but not sufficient for the FCC-ee.

7.2 Estimate of NNLO$^{\mathrm{EFT}}$ corrections and beyond

In this section, the structure of the EFT expansion of the cross-section and the ingredients for higher-order corrections reviewed in Section 7.1 are used to estimate the possible effects of a future NNLO$^{\mathrm{EFT}}$ calculation. Owing to the counting (Eq. (7.1)), this also includes leading corrections beyond NNLO in the conventional perturbative expansion:

$$\mathrm{NNLO}^{\mathrm{EFT}}: \quad v^4, \quad \alpha v^2, \quad \alpha^2 \quad \alpha^3/v^2, \quad \alpha^4/v^4. \tag{7.19}$$

The contributions of $\mathcal{O}(v^4, \alpha v^2)$ in Eq. (7.19) arise from kinematic corrections to the Born and NLO cross-section in the full SM, as discussed in Sections 7.1.1 and 7.1.3, respectively. The genuine $\mathcal{O}(\alpha^2)$ corrections are estimated in Section 7.2.1. A representative subset of the $\mathcal{O}(\alpha^3/v^2)$ corrections is computed in Section 7.2.2 and serves as an estimate of effects beyond a conventional NNLO calculation. The quadruple-Coulomb correction α^4/v^4 follows from the expansion of the known Coulomb Green function and is smaller than 0.001% and therefore negligible. Counting $\alpha \sim \alpha_{\mathrm{s}}^2$, QCD corrections to W self-energies and decay widths up to

$$\alpha\alpha_{\mathrm{s}}^2, \quad \alpha_{\mathrm{s}}^4 \tag{7.20}$$

are also required. Currently, the required $\mathcal{O}(\alpha_{\mathrm{s}}^4)$ corrections for inclusive hadronic vector boson decays are known [33], while mixed QCD-EW corrections are known up to $\mathcal{O}(\alpha\alpha_{\mathrm{s}})$ [34]. The uncertainty of a future NNLO$^{\mathrm{EFT}}$ calculation can be estimated by considering the impact of corrections at the next order in the δ-expansion, i.e.,

$$\mathrm{N}^{5/2}\mathrm{LO}^{\mathrm{EFT}}: \quad \alpha v^3, \quad \alpha^2 v, \quad \alpha^3/v, \quad \alpha^4/v^3, \quad \alpha^5/v^5. \tag{7.21}$$

The contributions $\sim\alpha v^3$ are already included in the NLO$^{\mathrm{ee4f}}$ calculation. The fifth Coulomb correction $\sim\alpha^5/v^5$ is known but negligibly small. The corrections $\sim\alpha^4/v^3$ arise from the combination of $\mathcal{O}(\alpha)$ corrections with triple-Coulomb exchange and are also expected to be negligible, since the latter is <0.01%. Therefore, the dominant genuine radiative corrections beyond NNLO$^{\mathrm{EFT}}$ are expected to be of order α^3/v. These arise from a combination of single Coulomb exchange and various sources of $\mathcal{O}(\alpha^2)$ corrections and are estimated in Section 7.2.3. Further contributions from triple-Coulomb exchange combined with $\sim v^2$ kinematic corrections are again expected to be negligible. The $\mathcal{O}(\alpha^2)$ corrections to the non-resonant cross-section (Eq. (7.7)) also provide $\sim\alpha^3/v$ corrections relative to the LO cross-section, while corrections $\sim\alpha^2 v$ arise from a combination of single Coulomb exchange with kinematic corrections $\sim\alpha v^2$. Such non-resonant and kinematic corrections are estimated in Section 7.2.4. It is assumed throughout that large logarithms of m_{e} are absorbed in electron structure functions and only the uncertainty due to non-universal $\mathcal{O}(\alpha^2, \alpha^3)$ corrections is considered.

7.2.1 $\mathcal{O}(\alpha^2)$ corrections in the EFT

The most involved corrections of order α^2 in the EFT arise from hard two-loop corrections to the Wilson coefficients of production operators and to decay rates and from soft two-loop corrections to the forward-scattering amplitude. Additional corrections from higher-order potentials or the combination of double-Coulomb exchange with kinematic corrections $\sim v^2$ are anticipated to be subdominant. The soft corrections for massless initial-state electrons can be extracted from the two-loop Drell–Yan soft function [23, 24] and converted to the electron mass regulator scheme using the NNLO structure functions computed in Ref. [29]. We make no attempt here to estimate these soft corrections, which are formally of the same order as the hard corrections.

This is supported by the NLO result, where hard corrections alone provide a reasonable order-of-magnitude estimate and soft corrections contribute less than 50% of the NLO corrections for $\sqrt{s} = 158$–170 GeV. The contribution of the NNLO Wilson coefficient of the production operator to the cross-section reads

$$\sigma^{(2)}_{\text{hard}}(s) = \frac{\pi\alpha^2}{s_{\mathrm{W}}^4 s}\mathrm{Im}\left[-\sqrt{-\frac{\mathcal{E}_{\mathrm{W}}}{M_{\mathrm{W}}}}\left(\frac{\alpha}{\pi}\right)^2 C^{(2)}\right], \tag{7.22}$$

where the NNLO hard coefficient is defined in terms of the squared Wilson coefficient,

$$C^2 = 1 + \frac{\alpha}{\pi}C^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 C^{(2)} + \cdots$$

The computation of $C^{(2)}$ involves the two-loop amplitude for $\mathrm{e}^-\mathrm{e}^+ \to \mathrm{W}^-\mathrm{W}^+$, evaluated directly at the threshold. Such a computation is beyond the current state of the art, which includes two-loop EW corrections to three-point functions [35–37], but will presumably be feasible before the operation of the FCC-ee. A naive estimate of the NNLO coefficient in terms of the the one-loop result (Eq. (7.11)),

$$C^{(2)} \sim (C^{(1)})^2, \tag{7.23}$$

suggests an effect on the cross-section of

$$\Delta\sigma^{(2)}_{\text{hard}} \approx \sigma^{(0)}_{\text{res}} \times 0.06\%. \tag{7.24}$$

The NNLO corrections to W boson decay give rise to the correction

$$\Delta\sigma^{(2)}_{\text{decay}} = \left(\frac{\Gamma^{(2,\text{ew})}_{\mu^-\bar{\nu}_\mu}}{\Gamma^{(0)}_{\mu^-\bar{\nu}_\mu}} + \frac{\Gamma^{(2,\text{ew})}_{\mathrm{u}\bar{\mathrm{d}}}}{\Gamma^{(0)}_{\mathrm{u}\bar{\mathrm{d}}}} + \frac{\Gamma^{(1,\text{ew})}_{\mu^-\bar{\nu}_\mu}\Gamma^{(1,\text{ew})}_{\mathrm{u}\bar{\mathrm{d}}}}{\Gamma^{(0)}_{\mu^-\bar{\nu}_\mu}\Gamma^{(0,\text{ew})}_{\mathrm{u}\bar{\mathrm{d}}}}\right)\sigma^{(0)}_{\text{res}}. \tag{7.25}$$

The product of NLO corrections in the last term contributes a negligible 0.001% to the G_μ input parameter scheme. A naive estimate of the currently unknown $\mathcal{O}(\alpha^2)$ corrections to W decay suggests

$$\Gamma^{(2,\text{ew})}_{\mathrm{f}_i\bar{\mathrm{f}}_j} \approx \frac{\alpha}{s_{\mathrm{W}}^2}\Gamma^{(1,\text{ew})}_{\mathrm{f}_i\bar{\mathrm{f}}_j} \sim 0.01\% \times \Gamma^{(0)}_{\mathrm{f}_i\bar{\mathrm{f}}_j},$$

consistent with the size of the $\mathcal{O}(\alpha^2)$ corrections to Z decay [36,37]. The estimates given in this subsection indicate that the combined non-Coulomb-enhanced corrections of $\mathcal{O}(\alpha^2)$ are of the order of 0.1% and are therefore mandatory to reduce the uncertainty below $\Delta M_{\mathrm{W}} \lesssim 1.5$ MeV.

7.2.2 Corrections of $\mathcal{O}(\alpha^3/v^2)$

The corrections of $\mathcal{O}(\alpha^3/v^2)$ involve a double-Coulomb exchange in combination with an $\mathcal{O}(\alpha)$ correction and arise from similar sources to those of the $\mathcal{O}(\alpha^2/v)$ corrections discussed in Section 7.1.4. The subclass of contributions arising from the combination of double-Coulomb exchange with soft and hard corrections is obtained by inserting the $\mathcal{O}(\alpha^2)$ term in the expansion of the Coulomb Green function (Eq. (7.12)) into Eq. (7.13), resulting in the contribution to the cross-section

$$\Delta\sigma^{\mathrm{C}_2\times[\mathrm{S+H}]_1} = \frac{\alpha^2}{s_{\mathrm{W}}^4 s}\frac{\alpha^3\pi^2}{12}\mathrm{Im}\left\{\sqrt{-\frac{M_{\mathrm{W}}}{\mathcal{E}_{\mathrm{W}}}}\left[\left(\frac{7}{2} + \frac{\pi^2}{4} + C^{(1)}\right)\right]\right. \tag{7.26}$$

Corrections from the NLO Coulomb potential to double-Coulomb exchange can be obtained by expanding the expression for the NLO Coulomb Green function [31] quoted in Ref. [38] and using the result for the Coulomb potential in the G_μ input parameter scheme [15], resulting in

$$\Delta\sigma^{\mathrm{C}_2^{\mathrm{NLO}}}(s) = \frac{\alpha^2}{s_\mathrm{W}^4 s}\frac{\alpha^3}{24}\mathrm{Im}\left\{\sqrt{-\frac{M_\mathrm{W}}{\mathcal{E}_\mathrm{W}}}\left[\pi^2\left(-\beta_0\ln\left(-\frac{\mathcal{E}_\mathrm{W}}{M_\mathrm{W}}\right)+\Delta_{G_\mu}\right)-12\beta_0\zeta_3\right]\right\}. \tag{7.27}$$

The combination of double-Coulomb exchange with NLO corrections to W decay is obtained from Eq. (7.15) by replacing $\sigma^{(0)}_{\mathrm{res}}$ with the second Coulomb correction. The resulting effect is, at most, 0.002%. Further corrections arise from corrections to the propagator residues and can be computed with current methods, but are beyond the scope of the present simple estimates. At $\mathcal{O}(\alpha^2/v)$, the corresponding corrections are of a similar size to the mixed soft+hard Coulomb corrections [15]. Therefore, the predictions from Eqs. (7.26) and (7.27), which are shown in Table B.7.2 together with the known two- and three-loop corrections [15], are expected to be representative of the the $\mathcal{O}(\alpha^3/v^2)$ corrections. They are of a similar order as the third Coulomb correction, and individually of the order $\lesssim$0.01% near the threshold. The sum of all $\mathcal{O}(\alpha^3/v^2)$ corrections may, therefore, be of the order of 0.01%, indicating the need to go beyond a strict $\mathcal{O}(\alpha^2)$ calculation to reach the FCC-ee accuracy goal.

7.2.3 Radiative corrections of $\mathcal{O}(\alpha^3/v)$

Genuine three-loop corrections at $\mathcal{O}(\alpha^3/v)$ can arise from a combination of the first Coulomb correction and soft or hard $\mathcal{O}(\alpha^2)$ corrections, corrections from higher-order potentials to the Coulomb Green function or a combination of $\mathcal{O}(\alpha)$ hard or soft and potential corrections. One contribution in the latter class can be computed by inserting the NLO Green function (Eq. (7.14)) into the product with the $\mathcal{O}(\alpha)$ hard and soft corrections (Eq. (7.13)),

$$\hat{\sigma}^{\mathrm{C}^{\mathrm{NLO}}\times[\mathrm{S+H}]_1}(s) = \frac{\alpha^2}{s s_\mathrm{W}^4}\frac{\alpha^3}{8\pi}\mathrm{Im}\left\{\left(\frac{7}{2}+\frac{\pi^2}{4}+C^{(1)}\right)\left(-\frac{\beta_0}{2}\ln\left(-\frac{\mathcal{E}_W}{M_W}\right)+\Delta_{G_\mu}\right)\ln\left(-\frac{\mathcal{E}_W}{M_W}\right)\right\}. \tag{7.28}$$

The corrections to the cross-section for $\sqrt{s} = 161$–170 GeV are given by

$$\Delta\sigma^{\mathrm{C}^{\mathrm{NLO}}\times[\mathrm{S+H}]_1} = -0.001\% \times \sigma_{\mathrm{LO}}. \tag{7.29}$$

A further indication for the magnitude of corrections at this order can be obtained from the combination of the NNLO hard coefficient with the first Coulomb correction,

$$\Delta\sigma^{\mathrm{C}_1\times\mathrm{H}_2} = -\frac{\pi\alpha^2}{s_\mathrm{W}^4 s}\frac{\alpha^3 C^{(2)}}{2\pi}\mathrm{Im}\left[\ln\left(-\frac{\mathcal{E}_\mathrm{W}}{M_\mathrm{W}}\right)\right], \tag{7.30}$$

and using the estimate (Eq. (7.23)) for the hard two-loop coefficient, which results in

$$\Delta\sigma^{\mathrm{C}_1\times\mathrm{H}_2}(161\ \mathrm{GeV}) \approx 0.005\% \times \sigma_{\mathrm{LO}}, \qquad \Delta\sigma^{\mathrm{C}_1\times\mathrm{H}_2}(170\ \mathrm{GeV}) \approx 0.002\% \times \sigma_{\mathrm{LO}}. \tag{7.31}$$

These results indicate that the $\mathcal{O}(\alpha^3)$ corrections beyond NNLO$^{\mathrm{EFT}}$ are $\lesssim$0.01%. It is expected that the factorisation (Eq. (7.10)) and the N^{3}LO Coulomb Green function [39] enable the computation of all $\mathcal{O}(\alpha^3/v)$ corrections once the NNLO$^{\mathrm{EFT}}$ result is known, as for a related calculation for hadronic $\mathrm{t\bar{t}}$ production [40].

7.2.4 *Non-resonant and kinematic $\mathcal{O}(\alpha^2)$ corrections*

Kinematic $\mathcal{O}(\alpha^2 v)$ corrections and $\mathcal{O}(\alpha^2)$ corrections to the non-resonant cross-section in Eq. (7.21) would be included in a full $\mathrm{NNLO}^{\mathrm{ee4f}}$ calculation, which is far beyond current calculational methods. The comparison of the $\mathrm{NLO}^{\mathrm{EFT}}$ and $\mathrm{NLO}^{\mathrm{ee4f}}$ results in Section 7.1.3 indicate a well-behaved perturbative expansion of the non-resonant corrections (Eq. (7.7)), with coefficients $\mathcal{K}^{(i)}$ of order one. This suggests that the non-resonant and kinematic NNLO corrections are reasonably estimated by scaling the corresponding NLO corrections,

$$\Delta\sigma^{(2)}_{4\mathrm{f}}(s) = \sigma^{\mathrm{ee4f}}_{\mathrm{NNLO}}(s) - \sigma^{(2)}_{\mathrm{EFT}}(s) \approx \frac{\alpha}{s^2_{\mathrm{W}}}\left(\sigma^{\mathrm{ee4f}}_{\mathrm{NLO}}(s) - \sigma^{(1)}_{\mathrm{EFT}}(s)\right) = \sigma^{\mathrm{ee4f}}_{\mathrm{Born}}(s) \times 0.02\% \tag{7.32}$$

for $\sqrt{s} = 161$–170 GeV. Therefore, these effects must be under control to reach the desired accuracy for the FCC-ee. A calculation of the $\mathcal{O}(\alpha^2)$ non-resonant correction in the EFT involves a combination of $\mathcal{O}(\alpha^2)$ corrections to the processes $\mathrm{e}^-\mathrm{e}^+ \to \mathrm{W}^\pm \mathrm{f}\bar{\mathrm{f}}$ with $\mathcal{O}(\alpha)$ corrections for $\mathrm{e}^-\mathrm{e}^+ \to 4\mathrm{f}$. Such a computation is beyond current capabilities, but may be possible before a full $\mathrm{NNLO}^{\mathrm{ee4f}}$ calculation is available. A comparison of future NNLO calculations in the EFT and the conventional DPA may also enable these corrections to be constrained.

7.3 Summary and outlook

The prospects of reducing the theoretical uncertainty of the total W pair production cross-section near the threshold to the level of $\sim 0.01\%$ required to fully exploit the high statistics at a future circular $\mathrm{e}^-\mathrm{e}^+$ collider have been investigated within the EFT approach, building on results for the NLO and dominant NNLO corrections. The estimates in Section 7.2.1 suggest that $\mathcal{O}(\alpha^2)$ corrections beyond the leading Coulomb effects [15] are of the order

$$\Delta\sigma_{\mathrm{NNLO}} \approx 0.1\% \times \sigma_{\mathrm{Born}} \tag{7.33}$$

at the threshold and are therefore mandatory to reach FCC-ee precision. In Sections 7.2.2 and 7.2.3, the dominant, Coulomb-enhanced three-loop effects have been estimated to be of the order

$$\Delta\sigma_{\mathrm{N^3LO}} \approx \text{few} \times 0.01\% \times \sigma_{\mathrm{Born}}\,, \tag{7.34}$$

based on computations or estimates of representative examples of $\mathcal{O}(\alpha^3/v^2, \alpha^3/v)$ effects. These corrections are either part of the $\mathrm{NNLO}^{\mathrm{EFT}}$ result or can be computed once this result is available. The effect of the remaining $\mathcal{O}(\alpha^3)$ corrections without Coulomb enhancement is expected to be below the FCC-ee target accuracy. However, the accuracy of the $\mathrm{NNLO}^{\mathrm{EFT}}$ calculation is limited by non-resonant and kinematic corrections. An extrapolation of the difference of the $\mathrm{NLO}^{\mathrm{EFT}}$ and $\mathrm{NLO}^{\mathrm{ee4f}}$ calculations suggests the magnitude

$$\Delta\sigma^{(2)}_{4\mathrm{f}} \approx 0.02\% \times \sigma_{\mathrm{Born}}. \tag{7.35}$$

Related estimates, $\Delta\sigma_{\mathrm{N^3LO}} \approx 0.02\%$ and $\Delta\sigma^{(\text{non-res})}_{\mathrm{NNLO}} \approx 0.016\%$, have been obtained using scaling arguments and an extrapolation of the accuracy of the DPA [41]. Our results suggest that a theory-induced systematic error of the mass measurement from a threshold scan of

$$\Delta M_{\mathrm{W}} = (0.15 - 0.45)\ \mathrm{MeV} \tag{7.36}$$

should be achievable, where the lower value results from assuming that the non-resonant corrections are under control. In addition to the corrections considered here, it is also essential to reduce the uncertainty from ISR corrections and QCD corrections for hadronic final states to the required accuracy. It would also be desirable to bring the precision for differential cross-sections to a similar level to that of the total cross-section.

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References

[1] H. Baer *et al.*, The international linear collider technical design report, vol. 2: physics (2013), arXiv:1306.6352

[2] M. Baak *et al.*, Working group report: precision study of electroweak interactions, APS DPF Community Summer Study (Snowmass 2013), arXiv:1310.6708

[3] P. Azzurri *et al.,* Physics behind precision (2017). arXiv:1703.01626

[4] A. Abada *et al.*, *Eur. Phys. J.* **C79** (2019) 474. doi:10.1140/epjc/s10052-019-6904-3

[5] M. Dong and G. Li, CEPC Conceptual Design Report, vol. 2: physics & detector (2018), arXiv:1811.10545

[6] A. Aeppli *et al.*, *Nucl. Phys.* **B428** (1994) 126. arXiv:hep-ph/9312212, doi:10.1016/0550-3213(94)90195-3

[7] S. Jadach *et al.*, *Phys. Lett.* **B417** (1998) 326. arXiv:hep-ph/9705429, doi:10.1016/S0370-2693(97)01253-7

[8] S. Jadach *et al.*, *Comput. Phys. Commun.* **140** (2001) 432. arXiv:hep-ph/0103163, doi:10.1016/S0010-4655(01)00288-0

[9] W. Beenakker *et al.*, *Nucl. Phys.* **B548** (1999) 3. arXiv:hep-ph/9811481, doi:10.1016/S0550-3213(99)00110-8

[10] A. Denner *et al.*, *Phys. Lett.* **B475** (2000) 127. arXiv:hep-ph/9912261, doi:10.1016/S0370-2693(00)00059-9

[11] A. Denner *et al.*, *Nucl. Phys.* **B587** (2000) 67. arXiv:hep-ph/0006307, doi:10.1016/S0550-3213(00)00511-3

[12] A. Denner *et al.*, *Phys. Lett.* **B612** (2005) 223. arXiv:hep-ph/0502063, doi:10.1016/j.physletb.2005.03.007

[13] A. Denner *et al.*, *Nucl. Phys.* **B724** (2005) 247. arXiv:hep-ph/0505042, doi:10.1016/j.nuclphysb.2005.06.033

[14] M. Beneke *et al.*, *Nucl. Phys.* **B792** (2008) 89. arXiv:0707.0773, doi:10.1016/j.nuclphysb.2007.09.030

[15] S. Actis *et al.*, *Nucl. Phys.* **B807** (2009) 1. arXiv:0807.0102, doi:10.1016/j.nuclphysb.2008.08.006

[16] M. Beneke *et al.*, *Nucl. Phys.* **B842** (2011) 414. arXiv:1007.5414, doi:10.1016/j.nuclphysb.2010.09.009

[17] M. Beneke *et al.*, *Phys. Rev. Lett.* **93** (2004) 011602. arXiv:hep-ph/0312331, doi:10.1103/PhysRevLett.93.011602

[18] M. Beneke *et al.*, *Nucl. Phys.* **B686** (2004) 205. arXiv:hep-ph/0401002, doi:10.1016/j.nuclphysb.2004.03.016

[19] C.W. Bauer *et al.*, *Phys. Rev.* **D63** (2001) 114020. arXiv:hep-ph/0011336, doi:10.1103/PhysRevD.63.114020

[20] M. Beneke and Y. Kiyo, *Phys. Lett.* **B668** (2008) 143. arXiv:0804.4004, doi:10.1016/j.physletb.2008.08.031

[21] V S. Fadin *et al.*, *Phys. Rev.* **D49** (1994) 2247. doi:10.1103/PhysRevD.49.2247

[22] K. Melnikov and O.I. Yakovlev, *Phys. Lett.* **B324** (1994) 217. arXiv:hep-ph/9302311 doi:10.1016/0370-2693(94)90410-3

[23] A.V. Belitsky, *Phys. Lett.* B442 (1998) 307. arXiv:hep-ph/9808389 doi:10.1016/S0370-2693(98)01249-0

[24] T. Becher *et al.*, *J High Energy Phys.* 07 (2008) 030. arXiv:0710.0680, doi:10.1088/1126-6708/2008/07/030

[25] V.S. Fadin *et al.*, *Phys. Rev.* **D52** (1995) 1377. arXiv:hep-ph/9501214, doi:10.1103/PhysRevD.52.1377

[26] G. Sterman, *Nucl. Phys.* **B281** (1987) 310. doi:10.1016/0550-3213(87)90258-6

[27] S. Catani and L. Trentadue, *Nucl. Phys.* **B327** (1989) 323. doi:10.1016/0550-3213(89)90273-3

[28] P. Falgari, Ph.D. thesis, RWTH Aachen University, 2008. http://darwin.bth.rwth-aachen.de/opus3/volltexte/2009/2682/

[29] J. Blümlein *et al.*, *Nucl. Phys.* **B855** (2012) 508. arXiv:1107.4638, doi:10.1016/j.nuclphysb.2011.10.009

[30] M. Skrzypek, *Acta Phys. Pol.* **B23** (1992) 135. https://inspirehep.net/literature/335990

[31] M. Beneke *et al.*, *Phys. Lett.* **B454** (1999) 137. arXiv:hep-ph/9903260, doi:10.1016/S0370-2693(99)00343-3

[32] M. Beneke *et al.*, *JHEP* **02** (2018) 125. arXiv:1711.10429, doi:10.1007/JHEP02(2018)125

[33] P.A. Baikov *et al.*, *Phys. Rev. Lett.* **101** (2008) 012002. arXiv:0801.1821, doi:10.1103/PhysRevLett.101.012002

[34] D. Kara, *Nucl. Phys.* **B877** (2013) 683. arXiv:1307.7190, doi:10.1016/j.nuclphysb.2013.10.024

[35] S. Actis *et al.*, *Nucl. Phys.* **B811** (2009) 182. arXiv:0809.3667, doi:10.1016/j.nuclphysb.2008.11.024

[36] A. Freitas, *JHEP* **04** (2014) 070. arXiv:1401.2447, doi:10.1007/JHEP04(2014)070

[37] I. Dubovyk *et al.*, *Phys. Lett.* **B783** (2018) 86. arXiv:1804.10236, doi:10.1016/j.physletb.2018.06.037

[38] M. Beneke *et al.*, *Nucl. Phys.* **B855** (2012) 695. arXiv:1109.1536, doi:10.1016/j.nuclphysb.2011.10.021

[39] M. Beneke, *et al.*, *Phys. Rev. Lett.* **115** (2015) 192001. arXiv:1506.06864, doi:10.1103/PhysRevLett.115.192001

[40] J. Piclum and C. Schwinn, *JHEP* **03** (2018) 164. arXiv:1801.05788, doi:10.1007/JHEP03(2018)164

[41] S. Jadach and M. Skrzypek, *Eur. Phys. J.* **C79** (2019) 756. arXiv:1903.09895, doi:10.1140/epjc/s10052-019-7255-9