9 Vertex functions in QCD—preparation for beyond two loops

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Abstract

We summarise the algorithm to determine the two-loop off-shell three-point vertex functions of QCD before outlining the steps required to extend the results to three and more loops.

9.1 Introduction

In our current generation of high-energy particle accelerators involving hadron collisions, a major source of background is radiation derived from the strong sector. As this is governed by quantum chromodynamics (QCD), to quantify the background effects one must carry out high loop order computations. There has been remarkable activity and progress in this direction since around the turn of the millennium. The primary focus has been with the evaluation of on-shell n-point gluonic and fermionic amplitudes to several loop orders, both analytically and numerically. Indeed, such results have been crucial in ensuring that the Higgs particle was observed at CERN's LHC. However, having information on the off-shell Green's functions, such as the three-point vertices of QCD, is also important for theory as well as experiment. For instance, various articles in this direction have appeared over the years. A non-exhaustive literature for this status of three- and four-point functions at various external momenta configurations is given by Refs. [1–11]. There are various theoretical reasons for having such off-shell Green's functions. One is that knowing, say, the two-loop off-shell vertex functions enables higher-loop n-point on-shell amplitudes to be modelled numerically. This could be an interim position in the absence of the technology to compute them fully explicitly. Such an approach is not uncommon. Equally, in solving QCD beyond the perturbative limit analytically to probe deep infrared properties using the Schwinger–Dyson formalism, approximations must be made in order to solve the infinite tower of Green's functions. Until recent years, the validity of such approximations could not be fully quantified. However, with explicit perturbative results, for instance, such error analyses have been made possible. For instance, one approximation in solving two- and three-point Schwinger–Dyson equations is to neglect the summed graphs deriving from the quartic gluon vertex. Work in this direction over a period of time [12–16] has confirmed that such a step does not affect final results by more than a few percent. Equally, the Schwinger–Dyson method has been applied to finding the behaviour of the vertex functions. While similar approximations have been made, such analyses must be consistent with explicit perturbative results where *no* approximation is made at a particular loop order to drop a subset of contributing graphs. As an aside, lattice gauge theory calculations of vertex functions equally have to match on to perturbative results. Therefore, in light of these different areas of activity, there is a clear need to compute QCD n-point, and specifically vertex functions, off-shell as well as on-shell. For the former, which is the focus of this article, we will review the status of the two-loop evaluation of the three-point vertices as well as outline the algorithm to extend this to higher-

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loop order. While the discussion will be technical in nature, we will pool together all the necessary ingredients for the goal to be obtained at three loops.

While it is not immediately obvious, it is the case that the route to achieve this will involve higher-level mathematics extracted, for instance, from an algebraic geometry approach. Indeed, this also lies at the heart of on-shell amplitude computations. This technology has revolutionalized the programme of loop calculations. An example of this can be seen in the results for two-loop off-shell vertex results of Ref. [17], where harmonic polylogarithms based on a specific type of polynomials, known as cyclotomic, [18], appeared. One corollary of such results is the possibility of effecting renormalization schemes other than the canonical $\overline{\rm MS}$ one, which is universally accepted as the default scheme. Although this is the scheme with which one can carry out very high loop order calculations, it is not a kinematic one and retains no data within the β function, for instance, of the information on the subtraction point. In Ref. [19], the momentum subtraction scheme, denoted MOM, was introduced and the R ratio studied [20]. Extending Ref. [19] to the next order in Ref. [21] produced the three-loop MOM renormalization group functions. This allowed for studies of physical quantities at a loop order where scheme effects were apparent [22]. One consequence is that choosing alternative renormalization schemes could lead to a different way of estimating theory errors in measurements. In other words, similar to an experiment estimating a measured quantity in different schemes, the average of the result could be a more sound way of assessing truncation errors as an alternative to using values at different scales. With fully off-shell vertex functions, for instance, this idea can be extended beyond the symmetric point subtraction of the MOM case to have a region bounding the central value.

This section is organised as follows. The method used to evaluate three-point off-shell vertex functions is discussed next, with reference to the triple-gluon vertex. This forms the basis for higher-loop computations, with the algorithm being outlined in Section 9.3. Concluding remarks are made in Section 9.4.

9.2 Current status

At the outset, it is worth reviewing aspects of the early QCD vertex evaluations. By this we mean that our focus will be on cases where there is *no* nullification of an external momentum. This is important, since, in the computation of the QCD β-function to very high loop order, the extraction of the $\overline{\rm MS}$ coupling constant renormalization constant can be facilitated by setting the momentum of one of the external fields of the vertex function to zero. This is a mathematical shortcut, since the ultraviolet divergence is not contaminated by any infrared ones. By contrast, this infrared-safe procedure does not produce the *correct* finite part of the vertex functions, so it is not an appropriate method for gaining insight into any aspect of the kinematic properties of the vertex functions themselves. To be more concrete in the discussion, we will focus on the triple-gluon vertex function of Fig. B.9.1, which represents

$$\left\langle A^a_\mu(p_1) A^b_\nu(p_2) A^c_\sigma(-p_1-p_2) \right\rangle = f^{abc} \Sigma^{\rm ggg}_{\mu\nu\sigma}(p_1,p_2) = f^{abc} \sum_{k=1}^{14} \mathcal{P}^{\rm ggg}_{(k)\mu\nu\sigma}(p_1,p_2) \Sigma^{\rm ggg}_{(k)}(p_1,p_2) \, , \quad (9.1)$$

where f^{abc} are the colour group structure constants. The momenta p_i satisfy energy–momentum conservation

$$\sum_{n=1}^{3} p_i \;=\; 0 \quad (9.2)$$

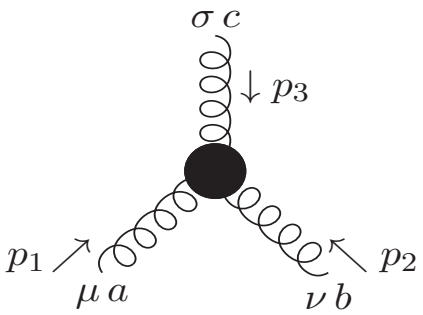

Fig. B.9.1: Triple-gluon vertex function

and the underlying Lorentz invariants that the three-point functions depend on are expressed in terms of two dimensionless variables, x and y, and one mass scale, μ, which are defined by

$$x = \frac{p_1^2}{p_3^2}\,, \qquad y = \frac{p_2^2}{p_3^2}\,, \qquad p_3^2 = -\mu^2\,, \tag{9.3}$$

and we assume that none of p_i^2 vanishes. In Eq. (9.1), we have decomposed the vertex into its 14 scalar amplitudes $\Sigma^{\rm ggg}_{(k)}(p_1,p_2)$ with respect to a basis of Lorentz tensors $\mathcal{P}^{\rm ggg}_{(k)\,\mu\nu\sigma}(p_1,p_2)$. With this structure in mind for the other two three-point vertices, the full one-loop vertex functions were studied in Ref. [19] in the early years following the discovery of asymptotic freedom.

Two important main early papers that stand out are Refs. [19, 23]. The former focused on the vertex functions at the fully symmetric subtraction point defined by $x = y = 1$ and introduced the MOM kinematic renormalization scheme known as MOM for momentum subtraction. Unlike the $\overline{\rm MS}$ scheme, the renormalization is carried out at this specific symmetric point and the finite part of the vertex functions is absorbed into the renormalization constants. Therefore, the β-functions contain kinematic data. The motivation of Ref. [19] was to study whether the convergence of the perturbative series could be improved in this new scheme. The other article [23] reported a systematic study of each fully off-shell three-point vertex with a view to writing each in terms of amplitudes dictated by external gluons being transverse. As such, it has served as the default vertex function convention, where Schwinger–Dyson techniques are used to approximate other Green's functions. Consequently, there have been a large number of one-loop studies of the three three-point vertices for different external momentum configurations, as noted earlier. In some cases, these studies have been at two loops, but for the most part one or more external gluon legs were on-shell and quarks have been massless, except in the case of Refs. [3, 7]. In the main, the evaluation has been by standard quantum field theory techniques via Feynman graphs. However, modern string-inspired methods have been used [11, 24] for off-shell one-loop vertex functions. The case where a gluon, for example, is on-shell must be treated separately from the configuration introduced in Eq. (9.3), owing to potential infrared singularities in taking the on-shell limit from the fully off-shell results.

Studies of the vertex functions for the special cases where one or more external lines are on-shell has direct applications to experimental set-ups. One of the reasons why these were computed was, in the main, that the calculational tools for the off-shell case were not developed until much later. Several main components were necessary for this, with the main breakthrough arriving in the form of the Laporta algorithm [25]. This is a procedure of relating scalar Feynman integrals of a particular n-point function at a specified loop order to core or master integrals of r-point functions with $r \leq n$ and the same loop order, the connection between integrals

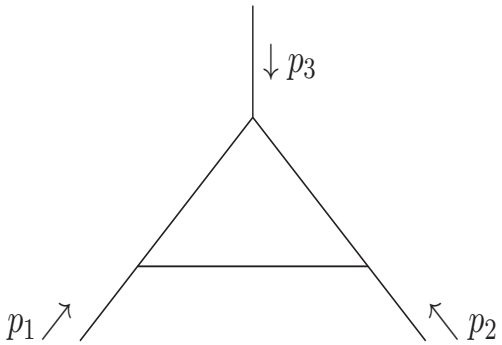

Fig. B.9.2: One-loop three-point master integral $I_1(x, y)$

being made via integration by parts. Then, starting with the most complicated integral, the relations derived from integration by parts could be solved algebraically. While such a large set of equations clearly contains a degree of redundancy, the whole process can be encoded for a computer to handle and several packages to do so are publicly available [26–32]. The second breakthrough necessary to complete this task was the determination of the master integrals. For three-point functions, these had to be constructed by specialised methods [33–36] to two loops, as integration by parts had been exhausted by the Laporta algorithm. To give a flavour of the resultant mathematical structure, the one-loop master integral of Fig. B.9.2 is, for instance, given by [33–35]

$$I_1(x,y) = -\frac{1}{\mu^2}\left[\Phi_1(x,y) + \Psi_1(x,y)\epsilon + \left[\frac{\zeta(2)}{2}\Phi_1(x,y) + \chi_1(x,y)\right]\epsilon^2 + O(\epsilon^3)\right] \tag{9.4}$$

in $d = 4 - 2\epsilon$ dimensions, where $\zeta(z)$ is the Riemann zeta function. Here, the functions are related to polylogarithms $\mathrm{Li}_n(z)$. For instance

$$\Phi_1(x,y) = \frac{1}{\lambda}\left[2\mathrm{Li}_2(-\rho x) + 2\mathrm{Li}_2(-\rho y) + \ln\left(\frac{y}{x}\right)\ln\left(\frac{(1+\rho y)}{(1+\rho x)}\right) + \ln(\rho x)\ln(\rho y) + \frac{\pi^2}{3}\right] , \tag{9.5}$$

with

$$\rho(x,y) = \frac{2}{[1-x-y+\lambda(x,y)]} , \qquad \lambda(x,y) = \left[1-2x-2y+x^2-2xy+y^2\right]^{\frac{1}{2}} , \tag{9.6}$$

with the other functions of Eq. (9.4) given in [33–35] too. While the $O(\epsilon)$ terms may not, at first sight, appear to be necessary, they are required for various reasons. One is that, at higher loops, these one-loop expressions are multiplied by the counterterms. Thus, when a pole in ϵ multiplies a term that is $O(\epsilon)$, then that will contribute to the finite part of the vertex function at the next loop order. Accordingly, one needs the master integrals to at least $O(\epsilon^2)$ at one loop for a three-loop evaluation. We have indicated this since it could be the case that, in the reduction using the Laporta algorithm, a spurious pole in ϵ arises, which we discuss later. This is not an uncommon occurrence but the latest Laporta algorithm packages now have tools to circumvent this possibility. These technical issues aside, the full off-shell three-point QCD vertex functions are available to two loops with more details provided in Ref. [17].

9.3 Three-loop strategy

One reason for detailing the formalism to carry out the two-loop computations is that it points the way for higher-loop corrections. On that basis, we outline the next parts of the jigsaw to construct the three-loop extension of Ref. [17]. First, we assume that the procedure of the general algorithm for the Green's functions is applied to obtain the three-loop scalar amplitudes, as illustrated in Eq. (9.1). From these, the large set of scalar Feynman integrals is assembled; these must be reduced to the master integrals. The Laporta algorithm can, in principle, be applied in the three-loop case, using one of the latest packages that have the built-in improvements, such as the refined algebraic reduction of the Kira package [32]. However, to speed the integration by parts procedure, it is not inconceivable that a faster algorithm could be developed. For instance, for many years, the Mincer package served the multiloop community well for three-loop massless two-point graphs in four dimensions [37,38]. It implemented the star–triangle rule to produce an efficient code to evaluate even the heaviest fully gluonic three-loop graphs. With the need for more precision experimentally, the four-loop Forcer package [39, 40] has superseded Mincer in the journey to hit the latest precision benchmark. Each has been encoded in the symbolic manipulation language Form [41, 42]. With increasing loop order, the evaluation time for a Green's function increases. However, the Forcer algorithm implements a *new* integration rule to handle an internal topology that has no three-loop antecedents and hence is a purely four-loop feature. We have mentioned this since Forcer, like Mincer, applies only to two-point functions. However, the same new rule should be applicable or adaptable to three-loop three-point functions, since such a configuration emerges when one slices the vertex off a two-point function, where that vertex contains one of the external legs. The remaining graph would retain the internal topology of the two-point four-loop case. Therefore, an adaptation of the new feature of Forcer could, in principle, be transferred to the three-point case to provide an efficient alternative to the application of the Laporta algorithm for massless three-point functions.

While such technology is already in effect in situ, the main obstacle to the full implementation of a three-loop evaluation is the determination of the required three-loop master integrals. In recent years, this field has advanced, with progress made in understanding the mathematical properties of high-order Feynman integrals. Examples of such articles include Refs. [43, 44], which provide novel procedures to compute Feynman graphs. The background to this is that there is a wide range of tools to evaluate a graph. One method is to introduce the Schwinger parameter representation of each propagator and convert the L-loop d-dimensional space–time integral into an integral over Schwinger parameters. The resulting integral has a large number of parameter integrations to be carried out and there is no guarantee that this can be achieved analytically. This is to be preferred over a numerical approach, as the latter, if a Monte Carlo approach is used, could require a sizeable amount of computer resources to obtain reasonable accuracy. In certain instances, an analytical evaluation is possible and, in essence, uses algebraic geometry to produce an integration strategy. Such higher mathematics is relevant, since the integrand contains polynomials of the parameters, which represent higher-dimensional geometries. Established mathematical theorems are then effected, which determine which parameter integration order is to be used, with the guiding principle being linear reducibility. By this, we mean that after each parameter integration the polynomial degree reduces but the key to achieve this is to have the polynomial factor off a smaller polynomial involving only factors linear in the next variable to be integrated. It is this linearity that is key, as it allows one to use the machinery of hyperlogarithms to carry out the integration over that Schwinger parameter.

What was not immediately evident is whether this procedure could be iterated without obstruction and that when it terminates the value of the integral is found. It has now been shown that, if an integral is linearly reducible [45,46], in this sense there is at least one choice of integration order that allows the integral to be determined. While this is, in essence, the general current position, it is known that, to three loops, the three-point vertex master integrals are all linearly reducible. Thus, in principle, the required master integrals can be determined.

The actual practicalities of this have yet to be carried out. However, several packages are available to assist with this task. For instance, converting a scalar Feynman integral into Schwinger parameter representation via the underlying graph polynomials is now a standard feature of integration packages, such as HYPERINT [47]. This package is appropriate for an analytical determination, since any evaluation can be written in various hyperlogarithm representations. It has features that allow one to find the order of integration over the parameter variables to ensure that there is no obstruction to the linear reducibility. In principle, one can expand to several orders in the ϵ expansion in $d = 4-2\epsilon$ dimensions. However, for terms beyond the leading few, the parameter integration can become tedious, especially for high loop orders. Therefore, a more appropriate strategy would be one where only the first term of the ϵ expansion of a master integral was required, which would then require the Laporta reduction to be constrained to producing a basis of masters that is finite. There is a caveat with this because one is using dimensional regularisation, which means that the reduction produces factors of rational polynomials in d. Such functions can include poles in $(d-4)$, which are termed spurious poles. This is in the sense that while they correspond to a divergence it is not necessarily one due to the divergence of an actual graph. There are now ways to circumvent this, which work hand in hand with another property of the beauty of computing in d dimensions. This was analysed in depth in Refs. [48, 49], where it was shown that d-dimensional integrals can be related to the corresponding topology in $(d+2)$ dimensions plus a sum of others that have the same core topology but with propagators missing. Such higher-dimensional integrals can be incorporated in the Laporta reduction process and have been implemented in version 2.11 of the REDUZE package [27]. The advantage is that, with the increase in dimensionality in the higher-dimensional integral, it is not as ultraviolet divergent as its lower-dimensional counterpart. Thereby, in principle, one reduces the evaluation of the more difficult master integrals to finite higher-dimensional ones, which should therefore be more accessible to the HYPERINT package.

In summarising the algorithm to extend the two-loop QCD off-shell vertex functions, it is worth noting that, for the triple-gluon vertex, there will be 2382 three-loop graphs to evaluate and 63 992 at four loops. For both the other three-point vertices, the numbers of graphs in each case are the same and are 688 and 17 311, respectively, at three and four loops. Thus, the evaluation of even just the three-loop vertex functions will require a substantial amount of work and computing time. This would especially be the case at four loops without access to appropriate computers to build the necessary databases of integral relations. In the interim, there is a potential alternative to gain some insight into or estimate of the three-loop contributions. In the period between the early work of Celmaster and Gonsalves [19] and its extension to the next order in Ref. [21], a method was developed [9] where the vertex functions were computed, at the fully symmetric point, numerically at two loops in QCD. The approach was to apply a large momentum expansion of the vertex functions to very high order. This produced a set of two-point integrals, which were evaluated using MINCER [37, 38]. Provided that enough terms were computed, the approximate value of the contributing graphs could

be accurately estimated numerically. The stability and accuracy of the expansion could be checked by choosing different external momenta to play the role of the large momentum. What was remarkable when the analytic two-loop expressions became available [21] was how accurate the large-momentum MINCER-based expansion values were. The only major difference was for a colour group Casimir coefficient in one three-loop MOM β function, which turned out to be of the order 0.01 [9]. The numerical coefficient was small and the expansion needed to a higher accuracy than was computationally available at the time [9]. With advances in symbolic manipulation, such as the provision of the FORCER program, which is significantly more efficient than MINCER, such an interim numerical evaluation of the vertex functions would at least give information on the magnitude of the next-order corrections. As a corollary, it would provide the four-loop MOM β functions numerically.

9.4 Discussion

To recap, we have reviewed recent results in the determination of the three-point vertex functions of QCD at two loops. We have for the most part concentrated on the off-shell case; it would not have been possible to achieve this without the earlier work on different external momentum configurations. While the two-loop off-shell results followed a long time after the one-loop case, the main reason for this was lack of the required computational technology. The last decade has seen a revolution in this direction with the Laporta algorithm [25], as well as a systematic way of computing master integrals from high-level mathematics. Consequently, the road to achieve the extension to three loops is, in principle, possible. One useful corollary of such a computation would be the extension of the renormalization group functions to four loops in kinematic schemes such as MOM. To go to higher orders beyond three, this depends on whether the linear reducibility of four-loop masters can be established. One case that we have not touched on is that of the four-point functions. The technology to compute the full off-shell one-loop amplitudes is already available. However, the current situation is that the relevant two-loop off-shell masters have not been computed. Moreover, it has not been established whether they are linearly reducible in order that the hyperlogarithm approach can be applied. This at present appears to be an open question for future work. Finally, including massive quarks in three- and four-point functions is another direction that needs consideration. However, this is not straightforward at two loops, since the three-point masters with one mass scale and off-shell momentum configuration are not yet known.

References

[1] P. Pascual and R. Tarrach, *Nucl. Phys.* **B174** (1980) 123 [Erratum: **B181** (1981) 546]. doi:10.1016/0550-3213(80)90193-5, doi:10.1016/0550-3213(81)90540-X

[2] A.I. Davydychev *et al.*, *Phys. Rev.* **D54** (1996) 4087 [Erratum: **D59** (1999) 109901]. arXiv:hep-ph/9605348, doi:10.1103/PhysRevD.54.4087, doi:10.1103/PhysRevD.59.109901

[3] A.I. Davydychev *et al.*, *JHEP* **08** (2001) 050. arXiv:hep-ph/0105072, doi:10.1088/1126-6708/2001/08/050

[4] M. Binger and S.J. Brodsky, *Phys. Rev.* **D74** (2006) 054016. arXiv:hep-ph/0602199, doi:10.1103/PhysRevD.74.054016

[5] A.I. Davydychev *et al.*, *Phys. Rev.* **D58** (1998) 036007. arXiv:hep-ph/9801380, doi:10.1103/PhysRevD.58.036007

[6] A.I. Davydychev and P. Osland, *Phys. Rev.* **D59** (1999) 014006. arXiv:hep-ph/9806522, doi:10.1103/PhysRevD.59.014006

[7] A.I. Davydychev *et al.*, *Phys. Rev.* **D63** (2001) 014022. arXiv:hep-ph/0008171, doi:10.1103/PhysRevD.63.014022

[8] K.G. Chetyrkin and A. Retey, Three loop three linear vertices and four loop similar to MOM beta functions in massless QCD. arXiv:hep-ph/0007088

[9] K.G. Chetyrkin and T. Seidensticker, *Phys. Lett.* **B495** (2000) 74. arXiv:hep-ph/0008094, doi:10.1016/S0370-2693(00)01217-X

[10] C. Kellermann and C.S. Fischer, *Phys. Rev.* **D78** (2008) 025015. arXiv:0801.2697, doi:10.1103/PhysRevD.78.025015

[11] N. Ahmadiniaz and C. Schubert, *Nucl. Phys.* **B869** (2013) 417. arXiv:1210.2331, doi:10.1016/j.nuclphysb.2012.12.019

[12] A. Blum *et al.*, *Phys. Rev.* **D89** (2014) 061703. arXiv:1401.0713, doi:10.1103/PhysRevD.89.061703

[13] A.L. Blum *et al.*, *EPJ Web Conf.* **137** (2017) 03001. arXiv:1611.04827, doi:10.1051/epjconf/201713703001

[14] M. Mitter *et al.*, *Phys. Rev.* **D91** (2015) 054035. arXiv:1411.7978, doi:10.1103/PhysRevD.91.054035

[15] R. Alkofer *et al.*, *Ann. Phys.* **324** (2009) 106. arXiv:0804.3042, doi:10.1016/j.aop.2008.07.001

[16] R. Alkofer and M. Mitter, *PoS* **ConfinementX** (2012) 061. arXiv:1302.0307, doi:10.22323/1.171.0061

[17] J.A. Gracey, *Phys. Rev.* **D90** (2014) 025014. arXiv:1406.0649, doi:10.1103/PhysRevD.90.025014

[18] J. Ablinger *et al.*, *J. Math. Phys.* **52** (2011) 102301. arXiv:1105.6063, doi:10.1063/1.3629472

[19] W. Celmaster and R.J. Gonsalves, *Phys. Rev.* **D20** (1979) 1420. doi:10.1103/PhysRevD.20.1420

[20] W. Celmaster and R.J. Gonsalves, *Phys. Rev.* **D21** (1980) 3112. doi:10.1103/PhysRevD.21.3112

[21] J.A. Gracey, *Phys. Rev.* **D84** (2011) 085011. arXiv:1108.4806, doi:10.1103/PhysRevD.84.085011

[22] J.A. Gracey, *Phys. Rev.* **D90** (2014) 094026. arXiv:1410.6715, doi:10.1103/PhysRevD.90.094026

[23] J.S. Ball and T.-W. Chiu, Phys. Rev. **D22** (1980) 2550, [Erratum: **D23**,3085(1981)]. doi:10.1103/PhysRevD.22.2550, doi:10.1103/PhysRevD.23.3085.2

[24] N. Ahmadiniaz, C. Schubert, PoS QCD-TNT-III (2013) 002. arXiv:1311.6829, doi:10.22323/1.193.0002

[25] S. Laporta, *Int. J. Mod. Phys.* **A15** (2000) 5087. arXiv:hep-ph/0102033, doi:10.1142/S0217751X00002157

[26] C. Studerus, *Comput. Phys. Commun.* **181** (2010) 1293. arXiv:0912.2546, doi:10.1016/j.cpc.2010.03.012

[27] A. von Manteuffel and C. Studerus, Reduze 2—distributed Feynman integral reduction. arXiv:1201.4330

[28] A.V. Smirnov, *JHEP* **10** (2008) 107. arXiv:0807.3243, doi:10.1088/1126-6708/2008/10/107

[29] R.N. Lee, Presenting LiteRed: a tool for the Loop InTEgrals REDuction. arXiv:1212.2685

[30] R.N. Lee, *J. Phys. Conf. Ser.* **523** (2014) 012059. arXiv:1310.1145, doi:10.1088/1742-6596/523/1/012059

[31] C. Anastasiou and A. Lazopoulos, *JHEP* **07** (2004) 046. arXiv:hep-ph/0404258, doi:10.1088/1126-6708/2004/07/046

[32] P. Maierhoefer *et al.*, *Comput. Phys. Commun.* **230** (2018) 99. arXiv:1705.05610, doi:10.1016/j.cpc.2018.04.012

[33] A.I. Davydychev, *J. Phys.* **A25** (1992) 5587. doi:10.1088/0305-4470/25/21/017

[34] N.I. Usyukina and A.I. Davydychev, *Phys. Atom. Nucl.* **56** (1993) 1553 [*Yad. Fiz.* **56N11** (1993) 172]. arXiv:hep-ph/9307327

[35] N.I. Usyukina and A.I. Davydychev, *Phys. Lett.* **B332** (1994) 159. arXiv:hep-ph/9402223, doi:10.1016/0370-2693(94)90874-5

[36] T.G. Birthwright *et al.*, *JHEP* **09** (2004) 042. arXiv:hep-ph/0407343, doi:10.1088/1126-6708/2004/09/042

[37] S.G. Gorishnii *et al.*, *Comput. Phys. Commun.* **55** (1989) 381. doi:10.1016/0010-4655(89)90134-3

[38] S.A. Larin *et al.*, The FORM version of MINCER, NIKHEF-H-91-18.

[39] T. Ueda *et al.*, *PoS* **LL2016** (2016) 070. arXiv:1607.07318, doi:10.22323/1.260.0070

[40] B. Ruijl *et al.*, Forcer, a FORM program for the parametric reduction of four-loop massless propagator diagrams. arXiv:1704.06650

[41] J.A.M. Vermaseren, New features of FORM, arXiv:math-ph/0010025

[42] M. Tentyukov and J.A.M. Vermaseren, *Comput. Phys. Commun.* **181** (2010) 1419. arXiv:hep-ph/0702279, doi:10.1016/j.cpc.2010.04.009

[43] F. Brown, On the decomposition of motivic multiple zeta values. arXiv:1102.1310

[44] O. Schnetz, *Phys. Rev.* **D97** (2018) 085018. arXiv:1606.08598, doi:10.1103/PhysRevD.97.085018

[45] F. Brown, *Commun. Math. Phys.* **287** (2009) 925. arXiv:0804.1660, doi:10.1007/s00220-009-0740-5

[46] F.C.S. Brown, On the periods of some Feynman integrals. arXiv:0910.0114

[47] E. Panzer, *Comput. Phys. Commun.* **188** (2015) 148. arXiv:1403.3385, doi:10.1016/j.cpc.2014.10.019

[48] O.V. Tarasov, *Phys. Rev.* **D54** (1996) 6479. arXiv:hep-th/9606018, doi:10.1103/PhysRevD.54.6479

[49] O.V. Tarasov, *Nucl. Phys.* **B502** (1997) 455. arXiv:hep-ph/9703319, doi:10.1016/S0550-3213(97)00376-3