3 NNLO corrections in four dimensions

Contribution* by: R. Pittau [pittau@ugr.es]

3.1 Introduction

Currently, four-dimensional techniques applied to higher-order calculations are under active investigation [1–8]. The main motivation for this is the need to simplify perturbative calculations necessary to cope with the precision requirements of the future LHC and FCC experiments.

In this contribution, I review the four-dimensional regularisation or renormalization (FDR) approach [9] to the computation of NNLO corrections in four dimensions. In particular, I describe how fully inclusive NNLO final-state quark-pair corrections [10]

$$\sigma^{\rm NNLO} = \sigma_{\rm B} + \sigma_{\rm V} + \sigma_{\rm R} \qquad \text{with} \qquad \begin{cases} \sigma_{\rm B} = \int {\rm d}\Phi_n \sum_{\rm spin} |A_n^{(0)}|^2 \\ \sigma_{\rm V} = \int {\rm d}\Phi_n \sum_{\rm spin} \left\{ A_n^{(2)} (A_n^{(0)})^* + A_n^{(0)} (A_n^{(2)})^* \right\} \\ \sigma_{\rm R} = \int {\rm d}\Phi_{n+2} \sum_{\rm spin} \left\{ A_{n+2}^{(0)} (A_{n+2}^{(0)})^* \right\} \end{cases} \tag{3.1}$$

are computed in FDR by directly enforcing gauge invariance and unitarity in the definition of the regularised UV- and IR-divergent integrals. The IR-divergent parts of the amplitudes are depicted in Fig. C.3.1 and ${\rm d}\Phi_m := \delta\left(P - \sum_{i=1}^m p_i\right) \prod_{i=1}^m {\rm d}^4 p_i \delta_+(p_i^2)$.

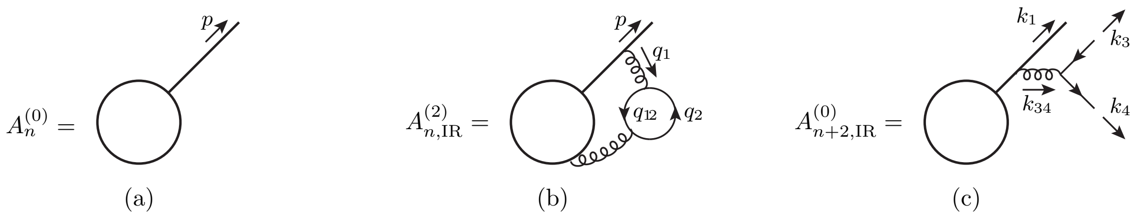

Fig. C.3.1: The lowest-order amplitude (a), the IR-divergent final-state virtual quark-pair correction (b), and the IR-divergent real component (c). The empty circle stands for the emission of $n-1$ particles. Additional IR finite corrections are created if the gluons with momenta q_1 and k_{34} are emitted by off-shell particles contained in the empty circle.

In Section 3.2, I recall the basics of FDR. The following sections deal with its use in the context of the calculation of $\sigma^{\rm NNLO}$ in Eq. (3.1).

3.2 FDR integration and loop integrals

The main idea of FDR can be sketched out with the help of a simple one-dimensional example [11]. More details can be found in the relevant literature [9, 10, 12–16]. Let us assume that one needs to define the UV divergent integral

$$I = \lim_{\Lambda \to \infty} \int_0^\Lambda {\rm d}x \frac{x}{x+M}, \tag{3.2}$$

*This contribution should be cited as:
R. Pittau, NNLO corrections in four dimensions, DOI: 10.23731/CYRM-2020-003.163, in: Theory for the FCC-ee, Eds. A. Blondel, J. Gluza, S. Jadach, P. Janot and T. Riemann,
CERN Yellow Reports: Monographs, CERN-2020-003, DOI: 10.23731/CYRM-2020-003, p. 163.

where M stands for a physical energy scale. FDR identifies the UV divergent pieces in terms of integrands that do not depend on M, the so-called FDR vacua, and separates them by rewriting

$$\frac{x}{x+M} = 1 - \frac{M}{x} + \frac{M^2}{x(x+M)}. \tag{3.3}$$

The first term in the right-hand side of Eq. (3.3) is the vacuum responsible for the linear $\mathcal{O}(\Lambda)$ UV divergence of I and $1/x$ generates its $\ln\Lambda$ behaviour. From the *definition* of FDR integration, both divergent contributions need to be subtracted from Eq. (3.2). The subtraction of the $\mathcal{O}(\Lambda)$ part is performed over the full integration domain $[0,\Lambda]$, while the logarithmic divergence is removed over the interval $[\mu_\mathrm{R},\Lambda]$ only. The arbitrary separation scale $\mu_\mathrm{R} \neq 0$ is needed to keep adimensional and finite the arguments of the logarithms appearing in the subtracted and finite parts. Thus

$$I_\mathrm{FDR} := I - \lim_{\Lambda\to\infty}\left(\int_0^\Lambda \mathrm{d}x - \int_{\mu_\mathrm{R}}^\Lambda \mathrm{d}x\frac{M}{x}\right) = M\ln\frac{M}{\mu_\mathrm{R}}. \tag{3.4}$$

The advantage of the definition in Eq. (3.4) is two-fold.

- The UV cut-off Λ is traded for μ_R, which is interpreted, straight away, as the renormalization scale.
- Other than logarithmic UV divergences never contribute.

The use of Eq. (3.4) is inconvenient in practical calculations, owing to the explicit appearance of μ_R in the integration interval. An equivalent definition is obtained by adding an auxiliary unphysical scale μ to x,

$$x \to \bar{x} := x + \mu, \tag{3.5}$$

and introducing an integral operator $\int_0^\infty[\mathrm{d}x]$, defined in such a way that it annihilates the FDR vacua before integration. Thus

$$I_\mathrm{FDR} = \int_0^\infty[\mathrm{d}x]\frac{\bar{x}}{\bar{x}+M} = \int_0^\infty[\mathrm{d}x]\left(1-\frac{M}{\bar{x}}+\frac{M^2}{\bar{x}(\bar{x}+M)}\right) := M^2\lim_{\mu\to 0}\int_0^\infty \mathrm{d}x\frac{1}{\bar{x}(\bar{x}+M)}\bigg|_{\mu=\mu_\mathrm{R}}, \tag{3.6}$$

where $\mu\to 0$ is an asymptotic limit. Note that, in order to keep the structure of the subtracted terms as in Eq. (3.3), the replacement $x\to\bar{x}$ must be performed in *both the numerator and the denominator* of the integrated function.

This strategy can be extended to more dimensions and to integrands that are rational functions of the integration variables, as is the case of multiloop integrals. For instance, typical two-loop integrals contributing to $\sigma_\mathrm{V}(\gamma^*\to\mathrm{jets})$ and $\sigma_\mathrm{V}(\mathrm{H}\to\mathrm{b}\bar{\mathrm{b}}+\mathrm{jets})$ are

$$K_1 := \int\left[\mathrm{d}^4q_1\right]\left[\mathrm{d}^4q_2\right]\frac{1}{\bar{q}_1^2\bar{D}_1\bar{D}_2\bar{q}_2^2\bar{q}_{12}^2}, \qquad K_2^{\rho\sigma\alpha\beta} := \int\left[\mathrm{d}^4q_1\right]\left[\mathrm{d}^4q_2\right]\frac{q_2^\rho q_2^\sigma q_1^\alpha q_1^\beta}{\bar{q}_1^4\bar{D}_1\bar{D}_2\bar{q}_2^2\bar{q}_{12}^2}, \tag{3.7}$$

where $q_{12} := q_1+q_2$, $\bar{D}_{1,2} = \bar{q}_1^2 + 2(q_1\cdot p_{1,2})$, $p_{1,2}^2 = 0$, and $\bar{q}_i^2 := q_i^2-\mu^2$ $(i=1,2,12)$, in the same spirit as Eq. (3.5).

FDR integration keeps shift invariance in any of the loop integration variables and the possibility of cancelling reconstructed denominators, e.g.,

$$\int\left[\mathrm{d}^4q_1\right]\left[\mathrm{d}^4q_2\right]\frac{\bar{q}_1^2}{\bar{q}_1^4\bar{D}_1\bar{D}_2\bar{q}_2^2\bar{q}_{12}^2} = K_1. \tag{3.8}$$

Since, instead,

$$\int \left[d^4q_1\right]\left[d^4q_2\right] \frac{q_1^2}{\bar{q}_1^4 \bar{D}_1 \bar{D}_2 \bar{q}_2^2 \bar{q}_{12}^2} \neq K_1\,,$$

this last property is maintained only if the replacement $q_i^2 \to \bar{q}_i^2$ is also made in the numerator of the loop integrals whenever q_i^2 is generated by Feynman rules. This is called *global prescription* (GP), often denoted $q_i^2 \overset{\text{GP}}{\to} \bar{q}_i^2$.

GP and shift invariance guarantee results that do not depend on the chosen gauge [12,14]. Nevertheless, unitarity should also be maintained. This requires that any given UV divergent subdiagram produce the same result when computed or manipulated separately or when embedded in the full diagram. Such a requirement is called *subintegration consistency* (SIC) [15]. Enforcing SIC in the presence of IR-divergent integrals, such as those in Eq. (3.7), needs extra care. In fact, the IR treatments of σ_V and σ_R should match each other. In the next sections, I describe how this is achieved in the computation of the observable in Eq. (3.1).

3.3 Keeping unitarity in the virtual component

Any integral contributing to σ_V has the form

$$I_\text{V} = \int \left[d^4q_1\right]\left[d^4q_2\right] \frac{N_\text{V}}{\bar{D}\bar{q}_2^2\bar{q}_{12}^2}, \tag{3.9}$$

where $\bar{D}$ collects all q_2-independent propagators and N_V is the numerator of the corresponding Feynman diagram. I_V can be subdivergent or globally divergent for large values of the integration momenta. For example, K_1 in Eq. (3.7) only diverges when $q_2 \to \infty$, while K_2 also diverges when $q_{1,2} \to \infty$. This means that FDR prescribes the subtraction of a *global vacuum* (GV) involving both integration variables in K_2, while the *subvacuum* (SV) developed when $q_2 \to \infty$ should be removed from both K_1 and K_2. In addition, IR infinities are generated by the on-shell conditions $p_{1,2}^2 = 0$. Even though IR divergences are automatically regulated when barring the loop denominators, a careful SIC preserving treatment is necessary in order not to spoil unitarity. Since the only possible UV subdivergence is produced by the quark loop in Fig. C.3.1(b), this is accomplished as follows [10].

- One does not apply GP to the contractions $g_{\rho\sigma}q_2^\rho q_2^\sigma$ when $g_{\rho\sigma}$ refers to indices external to the UV divergent subdiagram.
- One replaces everywhere $\bar{q}_1^2 \to q_1^2$ *after* GV subtraction.

The external indices entering the calculation of σ_V in Eq. (3.1) are denoted $\hat{\rho}$ and $\hat{\sigma}$ in Fig. C.3.2(a,b). Using this convention, one can rephrase the first rule as follows: $g_{\rho\sigma}q_2^\rho q_2^\sigma = q_2^2 \overset{\text{GP}}{\to} \bar{q}_2^2$, but $g_{\hat{\rho}\hat{\sigma}}q_2^\rho q_2^\sigma := \hat{q}_2^2 \overset{\text{GP}}{\to} q_2^2$, which gives, for instance,

$$\begin{aligned} g_{\rho\sigma}K_2^{\rho\sigma\alpha\beta} &\overset{\text{GP}}{\to} \bar{K}_2^{\alpha\beta} = \int \left[d^4q_1\right] \frac{q_1^\alpha q_1^\beta}{\bar{q}_1^4\bar{D}_1\bar{D}_2} \int \left[d^4q_2\right] \frac{1}{\bar{q}_{12}^2} = 0, \text{ but} \\ g_{\hat{\rho}\hat{\sigma}}K_2^{\rho\sigma\alpha\beta} &\overset{\text{GP}}{\to} \hat{K}_2^{\alpha\beta} = \int \left[d^4q_1\right]\left[d^4q_2\right] \frac{q_2^2 q_1^\alpha q_1^\beta}{\bar{q}_1^4\bar{D}_1\bar{D}_2\bar{q}_2^2\bar{q}_{12}^2} \neq 0, \end{aligned} \tag{3.10}$$

where $\bar{K}_2^{\alpha\beta}$ vanishes because the shift $q_2 \to q_2 - q_1$ makes it proportional to the subvacuum $1/\bar{q}_2^2$, which is annihilated by the $\int[d^4q_2]$ operator. It can be shown [10,15] that integrals such as $\hat{K}_2^{\alpha\beta}$ generate the unitarity-restoring logarithms missed by $\bar{K}_2^{\alpha\beta}$.

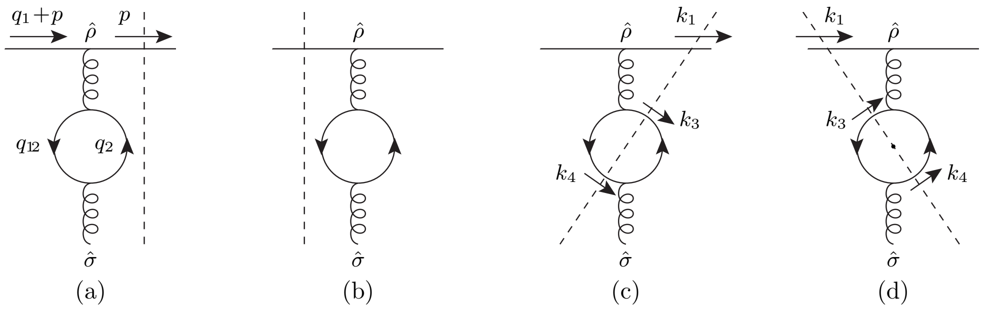

Fig. C.3.2: Virtual and real cuts contributing to the IR-divergent parts of $\sigma_{\rm V}$ (a,b) and $\sigma_{\rm R}$ (c,d).

As for the second rule, it states that a GV subtraction is needed first. In the case of $\hat K_2^{\alpha\beta}$, this is achieved by rewriting

$$\frac{1}{\bar D_1} = \frac{1}{\bar q_1^2} - \frac{2(q_1\cdot p_1)}{\bar D_1 \bar q_1^2}\,.$$

The first term gives a scaleless integral, annihilated by $\int [{\rm d}^4 q_1][{\rm d}^4 q_2]$, so that

$$\hat K_2^{\alpha\beta} = -2\int \left[{\rm d}^4 q_1\right]\left[{\rm d}^4 q_2\right] \frac{(q_1\cdot p_1) q_2^2 q_1^\alpha q_1^\beta}{\bar q_1^6 \bar D_1 \bar D_2 \bar q_2^2 \bar q_{12}^2}, \tag{3.11}$$

which is now only subdivergent when $q_2\to\infty$, as is K_1 in Eq. (3.7). After that, the replacement $\bar q_1^2 \to q_1^2$ produces

$$K_1 \to \tilde K_1 = \int {\rm d}^4 q_1 \left[{\rm d}^4 q_2\right] \frac{1}{q_1^2 D_1 D_2 \bar q_2^2 \bar q_{12}^2},\;\; \hat K_2^{\alpha\beta} \to \tilde K_2^{\alpha\beta} = -2\int {\rm d}^4 q_1 \left[{\rm d}^4 q_2\right] \frac{(q_1\cdot p_1) q_2^2 q_1^\alpha q_1^\beta}{q_1^6 D_1 D_2 \bar q_2^2 \bar q_{12}^2}. \tag{3.12}$$

All two-loop integrals $I_{\rm V}$ in Eq. (3.9) should be treated in this way. In the case of the $N_{\rm F}$ part of $\sigma_{\rm V}(\gamma^* \to \text{jets})$ and $\sigma_{\rm V}({\rm H}\to {\rm b\bar b} + \text{jets})$, this produces three master integrals, which can be computed as described in Appendix D of Ref. [10].

After loop integration, $\sigma_{\rm V}$ contains logarithms of μ^2 of both UV and IR origin. The former should be replaced by logarithms of $\mu_{\rm R}^2$, as dictated by Eq. (3.6), while the latter compensate the IR behaviour of $\sigma_{\rm R}$. To disentangle the two cases, it is convenient to renormalise $\sigma_{\rm V}$ first. This involves expressing the bare strong coupling constant $a^0 := \alpha_{\rm S}^0/4\pi$ and the bare bottom Yukawa coupling $y_{\rm b}^0$ in terms of $a := \alpha_{\rm S}^{\overline{\rm MS}}(s)/4\pi$ and $y_{\rm b}$ extracted from the the bottom pole mass $m_{\rm b}$. The relevant relations in terms of $L := \ln \mu^2/(p_1-p_2)^2$ and $L'' := \ln \mu^2/m_{\rm b}^2$ are [10]

$$a^0 = a\left(1 + a\delta_a^{(1)}\right), \qquad y_{\rm b}^0 = y_{\rm b}\left(1 + a\delta_y^{(1)} + a^2\left(\delta_y^{(2)} + \delta_a^{(1)}\delta_y^{(1)}\right)\right), \tag{3.13}$$

with

$$\delta_a^{(1)} = \frac{2}{3}N_{\rm F} L, \qquad \delta_y^{(1)} = -C_{\rm F}\left(3L''+5\right), \qquad \delta_y^{(2)} = C_{\rm F} N_{\rm F}\left(L''^2 + \frac{13}{3}L'' + \frac{2}{3}\pi^2 + \frac{151}{18}\right). \tag{3.14}$$

After renormalization, the remaining μ^2s are the IR ones.

3.4 Keeping unitarity in the real component

The integrands in σ_{R} of Eq. (3.1) are represented in Fig. C.3.2(c,d). They are of the form

$$J_{\mathrm{R}} = \frac{N_{\mathrm{R}}}{S s_{34}^{\alpha} s_{134}^{\beta}}, \qquad s_{i\cdots j} := (k_i + \cdots + k_j)^2, \qquad 0 \le \alpha, \quad \beta \le 2, \tag{3.15}$$

where N_{R} is the numerator of the amplitude squared and S collects the remaining propagators. Depending on the values of α and β, J_{R} becomes infrared divergent when integrated over Φ_{n+2}. These IR singularities must be regulated consistently with the SIC preserving treatment of σ_{V} described in Section 3.3.

The changes $q_2^2 \overset{\mathrm{GP}}{\to} \bar{q}_2^2$ and $q_{12}^2 \overset{\mathrm{GP}}{\to} \bar{q}_{12}^2$ in the virtual cuts of Fig. C.3.2(a,b) imply the Cutkosky relation

$$\frac{1}{(\bar{q}_2^2 + \mathrm{i}0^+)(\bar{q}_{12}^2 + \mathrm{i}0^+)} \leftrightarrow \left(\frac{2\pi}{\mathrm{i}}\right)^2 \delta_+(\bar{k}_3^2)\delta_+(\bar{k}_4^2), \tag{3.16}$$

with $\bar{k}_{3,4}^2 := k_{3,4}^2 - \mu^2$. Hence, one replaces in Eq. (3.1) $\Phi_{n+2} \to \tilde{\Phi}_{n+2}$, where the phase space $\tilde{\Phi}_{n+2}$ is such that $k_3^2 = k_4^2 = \mu^2$ and $k_i^2 = 0$ when $i \neq 3, 4$. In Ref. [10], it is proven that SV subtraction in σ_{V} does not alter Eq. (3.16). Analogously, the correspondence between cuts (a) and (d)

$$\frac{1}{(q_1 + p)^2 + \mathrm{i}0^+} \leftrightarrow \frac{2\pi}{\mathrm{i}} \delta_+(k_1^2) \tag{3.17}$$

is not altered by GV subtraction. Finally, k_3^2, k_4^2, and $(k_3 + k_4)^2 = s_{34}$ in N_{R} of Eq. (3.15) should be treated using the same prescriptions imposed on q_2^2, q_{12}^2, and q_1^2 in N_{V} of Eq. (3.9), respectively. This means replacing

$$k_{3,4}^2 \to \bar{k}_{3,4}^2 = 0, \qquad (k_3 \cdot k_4) = \frac{1}{2}\left(s_{34} - k_3^2 - k_4^2\right) \to \frac{1}{2}(s_{34} - \bar{k}_3^2 - \bar{k}_4^2) = \frac{1}{2} s_{34}, \tag{3.18}$$

where the last equalities are induced by the delta functions in Eq. (3.16). These changes should be made everywhere in N_{R}, except in contractions induced by the external indices $\hat{\rho}$ and $\hat{\sigma}$ in cuts (c,d). In this case

$$g_{\hat{\rho}\hat{\sigma}} k_{3,4}^{\rho} k_{3,4}^{\sigma} \to k_{3,4}^2 = \mu^2, \qquad g_{\hat{\rho}\hat{\sigma}} k_3^{\rho} k_4^{\sigma} \to (k_3 \cdot k_4) = \frac{s_{34} - 2\mu^2}{2}. \tag{3.19}$$

In the case of the N_{F} part of $\sigma_{\mathrm{R}}(\gamma^* \to \mathrm{jets})$ and $\sigma_{\mathrm{R}}(\mathrm{H} \to \mathrm{b}\bar{\mathrm{b}} + \mathrm{jets})$, integrating J_{R} over $\tilde{\Phi}_4$ and taking the asymptotic $\mu \to 0$ limit produces the phase space integrals reported in Appendix E of Ref. [10].

3.5 Results and conclusions

Using the approach outlined in Sections 3.3 and 3.4, one reproduces the known $\overline{\mathrm{MS}}$ results for the N_{F} components of $\sigma^{\mathrm{NNLO}}(\mathrm{H} \to \mathrm{b}\bar{\mathrm{b}} + \mathrm{jets})$ and $\sigma^{\mathrm{NNLO}}(\gamma^* \to \mathrm{jets})$ [10]

$$\begin{aligned} \sigma^{\mathrm{NNLO}}(\mathrm{H} \to \mathrm{b}\bar{\mathrm{b}} + \mathrm{jets}) &= \Gamma_{\mathrm{BORN}}(y_{\mathrm{b}}^{\overline{\mathrm{MS}}}(M_{\mathrm{H}}))\left\{1 + a^2 C_{\mathrm{F}} N_{\mathrm{F}}\left(8\zeta_3 + \frac{2}{3}\pi^2 - \frac{65}{2}\right)\right\}, \\ \sigma^{\mathrm{NNLO}}(\gamma^* \to \mathrm{jets}) &= \sigma_{\mathrm{BORN}}\left\{1 + a^2 C_{\mathrm{F}} N_{\mathrm{F}}\left(8\zeta_3 - 11\right)\right\}. \end{aligned} \tag{3.20}$$

This shows, for the first time, that a fully four-dimensional framework to compute NNLO quark-pair corrections can be constructed based on the requirement of preserving gauge invariance and unitarity. The basic principles leading to a consistent treatment of all the parts contributing to the NNLO results in Eq. (3.20) are also expected to remain valid when considering more complicated environments. A general four-dimensional NNLO procedure including initial-state IR singularities is currently under investigation.

References

[1] R.A. Fazio *et al.*, *Eur. Phys. J.* **C74** (2014) 3197. arXiv:1404.4783, doi:10.1140/epjc/s10052-014-3197-4

[2] O.A. Battistel *et al.*, *Mod. Phys. Lett.* **A13** (1998) 1597. doi:10.1142/S0217732398001686

[3] A.L. Cherchiglia *et al.*, *Int. J. Mod. Phys.* **A26** (2011) 2591. arXiv:1008.1377, doi:10.1142/S0217751X11053419

[4] R.J. Hernández-Pinto *et al.*, *JHEP* **02** (2016) 044. arXiv:1506.04617, doi:10.1007/JHEP02(2016)044

[5] G.F.R. Sborlini *et al.*, *JHEP* **08** (2016) 160. arXiv:1604.06699, doi:10.1007/JHEP08(2016)160

[6] G.F.R. Sborlini *et al.*, *JHEP* **10** (2016) 162. arXiv:1608.01584, doi:10.1007/JHEP10(2016)162

[7] F. Driencourt-Mangin *et al.*, *JHEP* **02** (2019) 143. arXiv:1901.09853, doi:10.1007/JHEP02(2019)143

[8] R. Runkel *et al.*, *Phys. Rev. Lett.* **122** (2019) 111603 arXiv:1902.02135, doi:10.1103/PhysRevLett.122.111603

[9] R. Pittau, JHEP **11** (2012) 151. arXiv:1208.5457, doi:10.1007/JHEP11(2012)151

[10] B. Page and R. Pittau, *Eur. Phys. J.* **C79** (2019) 361. arXiv:1810.00234 doi:10.1140/epjc/s10052-019-6865-6

[11] R. Pittau, *Nucl. Phys.* **B950** (2020), 114835. arXiv:1902.01767, doi:10.1016/j.nuclphysb.2019.114835

[12] A.M. Donati and R. Pittau, *JHEP* **04** (2013) 167. arXiv:1302.5668, doi:10.1007/JHEP04(2013)167

[13] R. Pittau, *Eur. Phys. J.* **C74** (2014) 2686. arXiv:1307.0705, doi:10.1140/epjc/s10052-013-2686-1

[14] A.M. Donati and R. Pittau, *Eur. Phys. J.* **C74** (2014) 2864. arXiv:1311.3551, doi:10.1140/epjc/s10052-014-2864-9

[15] B. Page and R. Pittau, *JHEP* **11** (2015) 183. arXiv:1506.09093, doi:10.1007/JHEP11(2015)183

[16] A. Blondel *et al.*, Standard Model theory for the FCC-ee Tera-Z stage, (CERN-2019-003). arXiv:1809.01830, doi:10.23731/CYRM-2019-003