5 Numerics for elliptic Feynman integrals

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The Standard Model involves several heavy particles: the Z and W bosons, the Higgs boson, and the top quark. Precision studies of these particles require, on the theoretical side, quantum corrections at the two-loop order and beyond. It is a well-known fact that, starting from two loops, Feynman integrals with massive particles can no longer be expressed in terms of multiple polylogarithms. This immediately raises the following question. What is the larger class of functions needed to express the relevant Feynman integrals? For single-scale two-loop Feynman integrals related to a single elliptic curve we now have the answer: they are expressed as iterated integrals of modular form [1]. This brings us to a second question: is there an efficient method to evaluate these functions numerically in the full kinematic range? In this contribution, we review how this can be done. This review is mainly based on Refs. [2, 3].

Efficient numerical evaluation methods rely on three ingredients: (i) an (iterated) integral representation, used to transform the arguments into the region of convergence, (ii) a (nested) sum representation, defined in the region of convergence, which can be truncated and gives a numerical approximation, and (iii) methods to accelerate the convergence of the truncated series. Let us illustrate this strategy for the numerical evaluation of the dilogarithm [4], defined by

$$\mathrm{Li}_2(x) = \int_0^x \frac{\mathrm{d}t_1}{t_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{1-t_2} = \sum_{n=1}^{\infty} \frac{x^n}{n^2}. \tag{5.1}$$

The power series expansion can be evaluated numerically, provided $|x| < 1$. Using the functional equations

$$\mathrm{Li}_2(x) = -\mathrm{Li}_2\left(\frac{1}{x}\right) - \frac{\pi^2}{6} - \frac{1}{2}\left(\ln(-x)\right)^2, \qquad \mathrm{Li}_2(x) = -\mathrm{Li}_2(1-x) + \frac{\pi^2}{6} - \ln(x)\ln(1-x),$$

any argument of the dilogarithm can be mapped into the region $|x| \leq 1$ and $-1 \leq \mathrm{Re}(x) \leq 1/2$. The numerical computation can be accelerated by using an expansion in $z = -\ln(1-x)$ and the Bernoulli numbers B_i:

$$\mathrm{Li}_2(x) = \sum_{i=0}^{\infty} B_i \frac{z^{i+1}}{(i+1)!}. \tag{5.2}$$

Multiple polylogarithms are defined for $z_k \neq 0$ by [5–7]

$$G(z_1, \ldots, z_k; y) = \int_0^y \frac{\mathrm{d}y_1}{y_1 - z_1} \int_0^{y_1} \frac{\mathrm{d}y_2}{y_2 - z_2} \cdots \int_0^{y_{k-1}} \frac{\mathrm{d}y_k}{y_k - z_k}. \tag{5.3}$$

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This represents multiple polylogarithms as iterated integrals. Alternatively, we may define multiple polylogarithms through a nested sum

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = \sum_{n_1>n_2>\cdots>n_k>0} \frac{{x_1}^{n_1}}{{n_1}^{m_1}} \cdots \frac{{x_k}^{n_k}}{{n_k}^{m_k}}. \tag{5.4}$$

With the shorthand notation

$$G_{m_1,\ldots,m_k}(z_1,\ldots,z_k;y) = G(\underbrace{0,\ldots,0}_{m_1-1},z_1,\ldots,z_{k-1},\underbrace{0,\ldots,0}_{m_k-1},z_k;y), \tag{5.5}$$

where all z_j for $j = 1,\ldots,k$ are assumed to be non-zero, the two notations are related by

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = (-1)^k G_{m_1,\ldots,m_k}\left(\frac{1}{x_1},\frac{1}{x_1x_2},\ldots,\frac{1}{x_1\cdots x_k};1\right). \tag{5.6}$$

The numerical evaluation of multiple polylogarithms follows the same strategy [8]. Using the integral representation, one transforms all arguments into a region, where the sum representation gives a converging power series expansion. In addition, the Hölder convolution is used to accelerate the convergence of the series expansion. The Hölder convolution reads (with $z_1 \neq 1$ and $z_k \neq 0$)

$$G\left(z_1,\ldots,z_k;1\right) = \sum_{j=0}^{k} (-1)^j\, G\left(1-z_j,1-z_{j-1},\ldots,1-z_1;\frac{1}{2}\right) G\left(z_{j+1},\ldots,z_k;\frac{1}{2}\right). \tag{5.7}$$

Multiple polylogarithms are a special case of iterated integrals. Let us briefly review Chen's definition of iterated integrals [9]: let M be an n-dimensional manifold and

$$\gamma \;:\; [0,1] \to M \tag{5.8}$$

a path with start point $x_\mathrm{i} = \gamma(0)$ and endpoint $x_\mathrm{f} = \gamma(1)$. Suppose further that ω_1, …, ω_k are differential 1-forms on M. Let us write

$$f_j\left(\lambda\right) \mathrm{d}\lambda = \gamma^* \omega_j \tag{5.9}$$

for the pull-backs to the interval $[0,1]$. For $\lambda \in [0,1]$ the k-fold iterated integral of ω_1, …, ω_k along the path γ is defined by

$$I_\gamma\left(\omega_1,\ldots,\omega_k;\lambda\right) = \int\limits_0^\lambda \mathrm{d}\lambda_1 f_1\left(\lambda_1\right) \int\limits_0^{\lambda_1} \mathrm{d}\lambda_2 f_2\left(\lambda_2\right) \cdots \int\limits_0^{\lambda_{k-1}} \mathrm{d}\lambda_k f_k\left(\lambda_k\right). \tag{5.10}$$

For multiple polylogarithms, we have $\omega_j = \mathrm{d}\ln(\lambda - z_j)$. A second special case is given by iterated integrals of modular forms [10]:

$$\omega_j = 2\pi\mathrm{i}\; f_j\left(\tau\right)\; \mathrm{d}\tau, \tag{5.11}$$

where $f_j(\tau)$ is a modular form. This type of iterated integral occurs in physics for the equal-mass sunrise integral [1, 11–14] and the the kite integral [15, 16]. A physical application is the two-loop electron self-energy in quantum electrodynamics, if the mass of the electron is not neglected [3,17]. This is a single-scale problem and we set $x = p^2/m^2$. In all these examples, the

complication is related to the equal-mass sunrise integral, which cannot be expressed in terms of multiple polylogarithms. This is related to the fact that the system of differential equations for this Feynman integral contains an irreducible second-order differential operator [18–20]

$$L = x\,(x-1)\,(x-9)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(3x^2 - 20x + 9\right)\frac{\mathrm{d}}{\mathrm{d}x} + x - 3. \tag{5.12}$$

Let ψ_1 and ψ_2 be two independent solutions of the homogeneous equation

$$L\,\psi = 0. \tag{5.13}$$

ψ_1 and ψ_2 can be taken as the periods of the elliptic curve

$$E \;:\; w^2 - z\,(z+4)\left[z^2 + 2\,(1+x)\,z + (1-x)^2\right] \;=\; 0. \tag{5.14}$$

One defines the modulus k and the complementary modulus k' by

$$k^2 = \frac{16\sqrt{x}}{(1+\sqrt{x})^3\,(3-\sqrt{x})}, \qquad k'^2 = 1 - k^2. \tag{5.15}$$

In a neighbourhood of $x = 0$, the periods may be taken as

$$\psi_{1,0} = \frac{4K\,(k)}{(1+\sqrt{x})^{\frac{3}{2}}\,(3-\sqrt{x})^{\frac{1}{2}}}, \qquad \psi_{2,0} = \frac{4\mathrm{i}K\,(k')}{(1+\sqrt{x})^{\frac{3}{2}}\,(3-\sqrt{x})^{\frac{1}{2}}}. \tag{5.16}$$

The complete elliptic integral $K(k)$ can be computed efficiently from the arithmetic-geometric mean

$$K\,(k) = \frac{\pi}{2\,\mathrm{agm}\,(k',1)}. \tag{5.17}$$

The periods ψ_1 and ψ_2 generate a lattice. Any other basis of the lattice again gives two independent solutions of the homogeneous differential equation (Eq. (5.13)). It is a standard convention to normalise one basis vector of the lattice to one: $(\psi_2, \psi_1) \to (\tau, 1)$ where $\tau = \psi_2/\psi_1$ and $\mathrm{Im}\tau > 0$. Let us now consider a change of basis:

$$\begin{pmatrix}\psi_2' \\ \psi_1'\end{pmatrix} = \gamma \begin{pmatrix}\psi_2 \\ \psi_1\end{pmatrix}, \qquad \gamma = \begin{pmatrix}a & b \\ c & d\end{pmatrix}. \tag{5.18}$$

The transformation should be invertible and preserve $\mathrm{Im}(\psi_2'/\psi_1') > 0$, therefore, $\gamma \in \mathrm{SL}_2\,(\mathbb{Z})$. In terms of τ and τ', this yields

$$\tau' = \frac{a\tau + b}{c\tau + d}. \tag{5.19}$$

This is a modular transformation and we write $\tau' = \gamma(\tau)$. Let us denote the complex upper half plane by $\mathbb{H}$. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$, if (i) f transforms under Möbius transformations as $f\,(\tau') = (c\tau + d)^k \cdot f(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, (ii) f is holomorphic on $\mathbb{H}$, and (iii) f is holomorphic at infinity. Furthermore, one defines modular forms for congruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ by requiring property (i) only

for $\gamma \in \Gamma$ (plus holomorphicity on $\mathbb{H}$ and at the cusps). Relevant to us will be the congruence subgroup $\Gamma_1(6)$, defined by

$$\Gamma_1(6) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod 6,\ c \equiv 0 \bmod 6 \right\}. \tag{5.20}$$

With $\psi_{1,0}$ and $\psi_{2,0}$ defined by Eq. (5.16), we set

$$\tau_0 = \frac{\psi_{2,0}}{\psi_{1,0}}, \qquad q_0 = \mathrm{e}^{2\mathrm{i}\pi\tau_0}. \tag{5.21}$$

We then change the variable from x to τ_0 (or q_0) [12]. The differential equation for the master integrals $\vec{I}$ relevant to the two-loop electron self-energy then reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau_0} \vec{I} = \varepsilon\ A(\tau_0)\ \vec{I}, \tag{5.22}$$

where $A(\tau_0)$ is an ε-independent matrix whose entries are modular forms for $\Gamma_1(6)$ [1, 14]. It follows immediately that all master integrals can be expressed in terms of iterated integrals of modular forms.

Let us now discuss how to evaluate numerically iterated integrals of modular forms in an efficient way. The essential point is that modular forms have a q-expansion. Using

$$2\pi\mathrm{i}\ \mathrm{d}\tau_0 = \frac{\mathrm{d}q_0}{q_0}\,, \tag{5.23}$$

we may integrate term-by-term and obtain the q_0-expansion of the master integrals. Truncating the q_0-series to the desired accuracy gives a polynomial in q_0. This needs to be done only once. The resulting polynomial can then be evaluated for different values of q_0 (or x) numerically. Note that the conversion from x to q_0 is also fast, since the complete elliptic integrals can be computed efficiently with the help of the arithmetic-geometric mean. Let us give an example. One finds for the ε^2-term of the sunrise integral [3]

$$I_6^{(2)} = 3\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) - 3\sqrt{3}\left[q_0 - \frac{5}{4}q_0^2 + q_0^3 - \frac{11}{16}q_0^4 + \frac{24}{25}q_0^5 - \frac{5}{4}q_0^6 + \frac{50}{49}q_0^7 - \frac{53}{64}q_0^8 + q_0^9\right] + \mathcal{O}\left(q_0^{10}\right). \tag{5.24}$$

We have $q_0 = 0$ for $x = 0$ and Eq. (5.24) gives a fast convergent series in a neighbourhood of $x = 0$. We are interested in evaluating the master integrals in the full kinematic range $x \in \mathbb{R}$. This raises the question: for which values $x \in \mathbb{R}$ do the q_0-series for the master integrals converge? Or phrased differently, for which values $x \in \mathbb{R}$ do we have $|q_0| < 1$? It turns out that we have $|q_0| < 1$ for $x \in \mathbb{R}\backslash\{1, 9, \infty\}$, corresponding to $p^2 \in \mathbb{R}\backslash\{m^2, 9m^2, \infty\}$ [2]. Thus, the q_0-series for the master integrals converge for all real values of x except three points. Let us stress that the q_0-series give the correct real and imaginary part of the master integrals, as specified by Feynman's iδ prescription. To cover the three remaining points, $x \in \{1, 9, \infty\}$, we recall that the periods ψ_1 and ψ_2 are not uniquely determined. By using four different choices for the pair of periods (ψ_1, ψ_2), we may define q_0, q_1, q_9 and q_∞ such that (i) the integration kernels are modular forms of $\Gamma_1(6)$ and (ii) $q_j = 0$ for $x = j$ [3]. This gives expansions around all singular points of the system of differential equations or—phrased differently—around all cusps of $\Gamma_1(6)$. In particular, there is always a choice such that $|q_j| \lessapprox 0.163$ for all real values

of x. Truncation of the q-series to order $\mathcal{O}(q^{30})$ gives for the finite part of the two-loop electron self-energy a relative precision better than 10^{-20} for all real values p^2/m^2.

Although we focused on the two-loop electron self-energy, we expect the methods discussed here to be applicable to any single-scale Feynman integral related to a single elliptic curve. This is a significant step beyond Feynman integrals evaluating to multiple polylogarithms and puts single-scale Feynman integrals related to a single elliptic curve on the same level of understanding as Feynman integrals evaluating to multiple polylogarithms. With the ongoing research on Feynman integrals beyond multiple polylogarithms [1, 2, 11–16, 20–50] we may expect more results—in particular, on multiscale Feynman integrals beyond multiple polylogarithms—to be coming soon.

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