Quantum field theory and the electroweak Standard Model

A.V. Bednyakov
Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia

Abstract
In these lecture notes, we discuss the basics of quantum field theory, key ideas underlying the construction of the electroweak Standard Model, and some phenomenological manifestations of the latter. In addition, the present status, issues, and prospects of the SM are briefly covered.

Keywords
Quantum field theory; Standard Model; Electroweak interactions; Lectures.

1 Introduction

The Standard Model (SM), see Refs. [1–3], turns out to be an incredibly successful theory, which survived many stringent experimental tests. Even after almost half a century, no significant deviations from the SM predictions have been found. Moreover, the discovery, see Refs. [4,5], of the Higgs boson at the LHC in 2012 was the final step in finalizing the SM. It is fair to say that it fully deserves the following fancy name:

The Absolutely Amazing Theory of Almost Everything. [6]

Let us mention a few excellent lectures (e.g., Refs. [7–11]) and textbooks (e.g., Refs. [12–14]) that can convince the reader that it is indeed the case. Since the history of the SM is rather long, it is obvious that it is not possible to discuss all the peculiarities of the SM in the set of three lectures. So the main task of the course is to review some basic facts and underlying principles of the model and emphasize key features of the latter.

Let us start with a brief overview of the SM particle content. The SM particles fall into two categories: fermions (half-integer spin) from bosons (integer spin). The former traditionally[1] associated with "matter", while the latter take the role of "force carriers" that mediate interactions between spin-1/2 particles.

In the SM, we have three *generations* involving two types of fermions - *quarks* and *leptons*. In total, there are

- 6 quarks of different flavour ($q = u, d, c, s, t, b$),
- 3 charged ($l = e, \mu, \tau$) and 3 neutral ($\nu_l = \nu_e, \nu_\mu, \nu_\tau$) leptons.

All of them participate in *weak* interactions. Both quarks q and charged leptons l take part in the electromagnetic interactions. In addition, quarks carry a *colour* charge and are influenced by the strong force. In the SM these interactions are due to the exchange of spin-1 (or vector) bosons:

- 8 gluons mediate the strong force between quarks;
- 4 electroweak (EW) bosons are responsible for the electromagnetic (photon - γ) and weak ($Z, W^\pm$) interactions.

There is also a spin-0 Higgs boson h. As it will be obvious from the subsequent discussion it plays a very important role in the construction of the SM. It turns out that only gluons and photons (γ) are assumed to be massless. All other *elementary* particles are massive due to the *Higgs* mechanism.

[1] The distinction is outdated: the fermions also mediate interactions between bosons.

In the SM the properties of the particle interactions can be read off the SM *Lagrangian* $\mathcal{L}_{SM}$. One can find its compact version on the famous CERN T-shirt. However, there is a lot of structure behind the short expression and it is *quantum field theory* or QFT (see, e.g., Refs. [14–18]) that allows us to derive the full Lagrangian and understand why the T-shirt Lagrangian is unique in a sense.

The form of $\mathcal{L}_{SM}$ is *restricted* by various kinds of (postulated) *symmetries*. Moreover, the SM is a *renormalizable* model. The latter fact allows us to use *perturbation theory* (PT) to make high-precision predictions for a vast number of observables and confront the model with experiment. All these peculiarities will be discussed during the lectures, which have the following structure.

We begin by introducing basics of quantum field theory in Section 2. Then we emphasize the role of symmetries in particles physics in Section 3. In Section 4 we use the *gauge principle* to construct the electroweak SM. The discussion of some experimental tests of the SM theoretical predictions can be found in Section 5. Final remarks and conclusions are provided in Section 6.

2 Basics of quantum field theory

2.1 Units, notation and all that

Before we begin our discussion of quantum fields, let us set up our notation. We work in natural units with the speed of light $c = 1$ and the (reduced) Planck constant $\hbar = 1$. As a consequence, all the quantities in particle physics are expressed in powers of electron-Volts (eV). To recover ordinary units, one uses the following convenient conversion factors:

[] $\hbar \simeq 6.58 \cdot 10^{-22}\ \text{MeV} \cdot s, \qquad \hbar c \simeq 1.97 \cdot 10^{-14}\ \text{GeV} \cdot \text{cm}$ []. (1)

In high-energy physics (HEP) we usually *require* that our theory should respect Lorentz *symmetry*. Due to this, a rotation or a boost in some direction, which can be parametrized by $\Lambda_{\mu\nu}$:

$$x_\mu \to x'_\mu = \Lambda_{\mu\nu} x_\nu, \tag{2}$$

does not change the value of scalar product

$$px \equiv p_\mu x_\mu = g_{\mu\nu} p_\mu x_\nu = p_0 x_0 - \mathbf{p}\cdot\mathbf{x}, \quad g_{\mu\nu} = \text{diag}(1,-1,-1,-1) \tag{3}$$

of any two four-vectors, e.g., space-time coordinates x_μ and energy-momenta p_μ

$$\begin{aligned} x_\mu &= \{x_0, \mathbf{x}\}, \text{ with time } t \equiv x_0, \\ p_\mu &= \{p_0, \mathbf{p}\}, \text{ with energy } E \equiv p_0. \end{aligned}$$

A well-known and very important example of a Lorentz invariant quantity is the particle *mass*. The latter corresponds to the "length" of the four-momentum vector $p^2 = E^2 - \mathbf{p}^2 = m^2$ and is the key property of a particle. Now let us switch to our main topic and discuss how fields are used to account for relativistic particles.

2.2 Quantum scalar field

A convenient way to deal with (quantum) fields is to consider the *Action* functional[2]. For the simplest (scalar) field, i.e., a function $\phi(x) \equiv \phi(t, \mathbf{x})$, the Action can be written as

$$\mathcal{A}[\phi(x)] = \int d^4x \underbrace{\mathcal{L}(\phi(x), \partial_\mu\phi)}_{\text{Lagrangian (density)}} = \int d^4x \underbrace{\left(\partial_\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi\right)}_{\phi^\dagger\cdot K\cdot\phi}. \tag{4}$$

[2]Contrary to ordinary functions that produce numbers from numbers, a *functional* takes a function and produces a number.

Given the Lagrangian $\mathcal{L}$, one can derive the *equations of motions* (EOM) via the *Action Principle*, which we describe now. The variation of the action due to tiny (infinitesimal) shifts in the field $\phi'(x) = \phi(x) + \delta\phi(x)$ can be cast into

$$\underbrace{\mathcal{A}[\phi'(x)] - \mathcal{A}[\phi(x)]}_{\delta\mathcal{A}[\phi(x)]=0} = \int d^4x \left[\underbrace{\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} \right)}_{(\partial_\mu^2 + m^2)\phi = 0} \delta\phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right)}_{\text{surface term}=0} \right]. \quad (5)$$

If we require that $\delta\mathcal{A}[\phi(x)] = 0$ for *any* variation $\delta\phi(x)$ of *some* $\phi(x)$, we will immediately deduce that this can be achieved only for *specific* $\phi(x)$ that satisfy EOM. These *particular* fields are usually called "on-mass-shell". In the case of the scalar field $\phi(x)$ we derive the Klein-Gordon (KG) equation, which is related in a straightforward way to the quadratic form K in Eq. (4):

$$-K\phi(x) = \left(\partial_\mu^2 + m^2\right)\phi = \left(\partial_t^2 - \nabla^2 + m^2\right)\phi(x) = 0. \quad (6)$$

After Fourier transformation (FT) Eq. (6) leads to the energy-momentum relation for the non-interacting particle,i.e.,

$$\phi(x) = \frac{1}{(2\pi)^4} \int d^4p \phi(p) e^{-ipx} \Rightarrow \left(p^2 - m^2\right)\phi(p) = \left(p_0^2 - \mathbf{p}^2 - m^2\right)\phi(p) = 0. \quad (7)$$

General solution of the homogeneous KG equation can be written as a sum (integral) over plane waves with $p_0^2 = \vec{p}^2 + m^2$

$$\phi(t, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[a(\mathbf{p}) e^{-i\omega_p t + i\mathbf{p}\mathbf{x}} + b(\mathbf{p}) e^{+i\omega_p t - i\mathbf{p}\mathbf{x}} \right], \quad (8)$$

where $\omega_p \equiv +\sqrt{\mathbf{p}^2 + m^2}$. For further convenience we explicitly write the terms corresponding to $p_0 = +\omega_p$ and $p_0 = -\omega_p$. The *negative-energy* solution with $p_0 < 0$ poses a serious problem if the field Eq. (8) is interpreted as *a wave-function* of *single* particle in the context of relativistic quantum mechanics (RQM). A single-particle interpretation fails to account for the appearance of negative-energy modes, and a new formalism is required to deal with such situations (see, e.g., Refs. [14]). Moreover, in RQM space coordinates play a role of dynamical variables and are represented by operators, while time is an evolution parameter. Obviously, a *consistent* relativistic theory should treat space and time on equal footing.

In QFT we interpret $\phi(\mathbf{x}, t)$ satisfying Eq. (6) as a *quantum* field, i.e., an *operator*[3]. The space coordinates $\mathbf{x}$ can be treated as a *label* for infinitely many dynamical variables, and we are free to choose a system of reference, in which we evolve these variables. As a consequence, a single field can account for an infinite number of particles, which correspond to field *excitations*.

Rewriting Eq. (8) in the compact QFT notation $[a(\mathbf{p}) \to a_{\mathbf{p}}^-, b(\mathbf{p}) \to b_{\mathbf{p}}^-]$

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[a_{\mathbf{p}}^- e^{-ipx} + b_{\mathbf{p}}^+ e^{+ipx} \right], \quad (9)$$

we interpret $a_{\mathbf{p}}^\pm$ and $b_{\mathbf{p}}^\pm$ as *operators* obeying

$$a_{\mathbf{p}}^- a_{\mathbf{p}'}^+ - a_{\mathbf{p}'}^+ a_{\mathbf{p}}^- \equiv \left[a_{\mathbf{p}}^-, a_{\mathbf{p}'}^+ \right] = \delta^3(\mathbf{p} - \mathbf{p}'), \quad \left[b_{\mathbf{p}}^-, b_{\mathbf{p}'}^+ \right] = \delta^3(\mathbf{p} - \mathbf{p}'). \quad (10)$$

[3]We use the Heisenberg picture, in which operators $\mathcal{O}_H(t)$ depend on time, while in the Schrödinger picture it is the states that evolve: $\langle\psi(t)|\mathcal{O}_S|\psi(t)\rangle = \langle\psi|\mathcal{O}_H(t)|\psi\rangle$ with $\mathcal{O}_S = \mathcal{O}_H(t=0)$, $|\psi\rangle = |\psi(t=0)\rangle$.

All other commutators are assumed to be zero, e.g., $\left[a^\pm_{\mathbf{p}}, a^\pm_{\mathbf{p}'}\right] = 0$. The operators satisfy $a^\pm_{\mathbf{p}} = (a^\mp_{\mathbf{p}})^\dagger$ and $b^\pm_{\mathbf{p}} = (b^\mp_{\mathbf{p}})^\dagger$, and for $a^\pm_{\mathbf{p}} \equiv b^\pm_{\mathbf{p}}$ the field is hermitian $\phi^\dagger(x) = \phi(x)$. The commutation relations, see Eq. (10), resemble the relations $[a^-, a^+] = 1$ for ladder operators $a^\pm$, which are usually considered to quantize harmonic oscillators. Following the analogy, we consider the *Fock* space that consists of a *vacuum* $|0\rangle$, which is *annihilated* by $a^-_{\mathbf{p}}$ (and $b^-_{\mathbf{p}}$) for every $\mathbf{p}$

$$\langle 0|0\rangle = 1, \quad a^-_{\mathbf{p}}|0\rangle = 0, \quad \langle 0|a^+_{\mathbf{p}} = (a^-_{\mathbf{p}}|0\rangle)^\dagger = 0,$$

and field excitations, which are *created* from the vacuum by acting with $a^+_{\mathbf{k}}$ (and/or $b^+_{\mathbf{k}}$) , e.g.,

$$|f_1\rangle = \int d\mathbf{k} \cdot f_1(\mathbf{k}) a^+_{\mathbf{k}}|0\rangle, \qquad \text{1-particle state;} \qquad (11)$$

$$|f_2\rangle = \int d\mathbf{k}_1 d\mathbf{k}_2 \cdot f_2(\mathbf{k}_1, \mathbf{k}_2) a^+_{\mathbf{k_1}} a^+_{\mathbf{k_2}}|0\rangle \qquad \text{2-particle state,} \qquad (12)$$

$$\ldots$$

Here various $f_i(\mathbf{k}, \ldots)$ are supposed to be square-integrable, so that, e.g., $\langle f_1|f_1\rangle = \int |f_1(\mathbf{k})|^2 d\mathbf{k} < \infty$. In spite of the fact that it is more appropriate to deal with such normalizable states, in QFT we usually consider (basis) states that have definite momentum $\mathbf{p}$, i.e., we assume that $f_1(\mathbf{k}) = \delta(\mathbf{k} - \mathbf{p})$.

The two set of operators $a^\pm$ and $b^\pm$ correspond to particles and antiparticles. It is worth emphasizing that in QFT all the particles of certain kind are excitations of a *single* field, and due to $a^+_{\mathbf{p}} a^+_{\mathbf{k}} = a^+_{\mathbf{k}} a^+_{\mathbf{p}}$, they are *indistinguishable* by construction.

Exploiting again the analogy with harmonic oscillators, we can introduce the Hamiltonian operator

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{part} + \hat{\mathcal{H}}_{antipart} = \int d\mathbf{p}\, \omega_p \left[n_{\mathbf{p}} + \bar{n}_{\mathbf{p}}\right] \qquad (13)$$

with $\bar{n}_{\mathbf{p}} \equiv b^+_{\mathbf{p}} b^-_{\mathbf{p}}$ and $n_{\mathbf{p}} \equiv a^+_{\mathbf{p}} a^-_{\mathbf{p}}$. The interpretation of the terms in Eq. (13) is straightforward: ($\bar{n}_{\mathbf{p}}$) $n_{\mathbf{p}}$ *counts* (anti-)particles with definite momentum $\mathbf{p}$ and there is a sum over the corresponding energies. In writing Eq. (13) we omit the infinite constant (zero-point energies) and from the very beginning assume that all the operators in $\hat{\mathcal{H}}$ are normal-ordered, i.e., all creation operators go before the annihilation ones. This corresponds to the assumption that the vacuum state has zero-energy $\hat{\mathcal{H}}|0\rangle = 0$.

It is easy to check that $[\hat{\mathcal{H}}, a^\pm_{\mathbf{p}}] = \pm\omega_{\mathbf{p}} a^\pm_{\mathbf{p}}$ and $[\hat{\mathcal{H}}, b^\pm_{\mathbf{p}}] = \pm\omega_{\mathbf{p}} b^\pm_{\mathbf{p}}$. As a consequence, single-particle states with definite momentum $\mathbf{p}$

$$|\mathbf{p}\rangle = a^+_{\mathbf{p}}|0\rangle, \quad \hat{\mathcal{H}}|\mathbf{p}\rangle = \omega_p|\mathbf{p}\rangle, \qquad |\bar{\mathbf{p}}\rangle = b^+_{\mathbf{p}}|0\rangle, \quad \hat{\mathcal{H}}|\bar{\mathbf{p}}\rangle = \omega_p|\bar{\mathbf{p}}\rangle \qquad (14)$$

are eigenvectors of the Hamiltonian with *positive* energies, and we avoid introduction of negative energies in our formalism from the very beginning. One can generalize Eq. (13) and "construct" the momentum $\hat{\mathbf{P}}$ and charge $\hat{Q}$ operators[4]:

$$\hat{\mathbf{P}} = \int d\mathbf{p}\, \mathbf{p} \left[n_{\mathbf{p}} + \bar{n}_{\mathbf{p}}\right], \quad \hat{\mathbf{P}}|0\rangle = 0|0\rangle, \qquad \hat{\mathbf{P}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle \qquad \hat{\mathbf{P}}|\mathbf{p}\rangle = \mathbf{p}|\bar{\mathbf{p}}\rangle, \qquad (15)$$

$$\hat{Q} = \int d\mathbf{p} \left[n_{\mathbf{p}} - \bar{n}_{\mathbf{p}}\right], \quad \hat{Q}|0\rangle = 0|0\rangle, \qquad \hat{Q}|\mathbf{p}\rangle = +|\mathbf{p}\rangle \qquad \hat{Q}|\bar{\mathbf{p}}\rangle = -|\bar{\mathbf{p}}\rangle. \qquad (16)$$

The charge operator $\hat{Q}$ distinguishes particles from anti-particles. One can show that the field $\phi^\dagger$ (ϕ) increases (decreases) the charge of a state

$$\left[\hat{Q}, \phi^\dagger(x)\right] = +\phi^\dagger(x), \quad \left[\hat{Q}, \phi(x)\right] = -\phi(x)$$

and consider the following amplitudes:

[4] It is worth pointing here that by construction both $\hat{Q}$ and $\hat{\mathbf{P}}$ do not depend on time and commute. In the next section, we look at this fact from a different perspective and connect it to various symmetries.

$$t_2 > t_1: \quad \langle 0|\underbrace{\phi(x_2)}_{a^-}\underbrace{\phi^\dagger(x_1)}_{a^+}|0\rangle \qquad t_1 > t_2: \quad \langle 0|\underbrace{\phi^\dagger(x_1)}_{b^-}\underbrace{\phi(x_2)}_{b^+}|0\rangle$$

Particle (charge $+1$) propagates from x_1 to x_2 — Antiparticle (charge -1) propagates from x_2 to x_1

We can take both possibilities into account in one function:

$$\underbrace{\langle 0|T[\phi(x_2)\phi^\dagger(x_1)]|0\rangle}_{-iD_c(x-y)} \equiv \theta(t_2-t_1)\langle 0|\phi(x_2)\phi^\dagger(x_1)|0\rangle + \theta(t_1-t_2)\langle 0|\phi^\dagger(x_1)\phi(x_2)|0\rangle, \qquad (17)$$

with T being the *time-ordering* operation ($\theta(t) = 1$ for $t \geq 0$ and zero otherwise).

Equation (17) give rise to the famous Feynman propagator, which has the following momentum representation:

$$D_c(x-y) = \frac{-1}{(2\pi)^4}\int d^4p \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}. \qquad (18)$$

The $i\epsilon$-prescription ($\epsilon \to 0$) picks up certain poles in the complex p_0 plane (see Fig. 1) and leads to the time-ordered expression Eq. (17). The propagator plays a key role in the construction of perturbation theory for interacting fields (see Section 2.5).

For the moment, let us mention a couple of facts about $D_c(x)$. It is a Green-function for the KG equation, i.e.,

$$\left(\partial_x^2 + m^2\right) D_c(x-y) = \delta(x-y). \qquad (19)$$

This gives us an alternative way to find the expression Eq. (18) by inverting the quadratic form introduced in Eq. (4). One can also see that $D_c(x-y)$ is a Lorentz and translational invariant function.

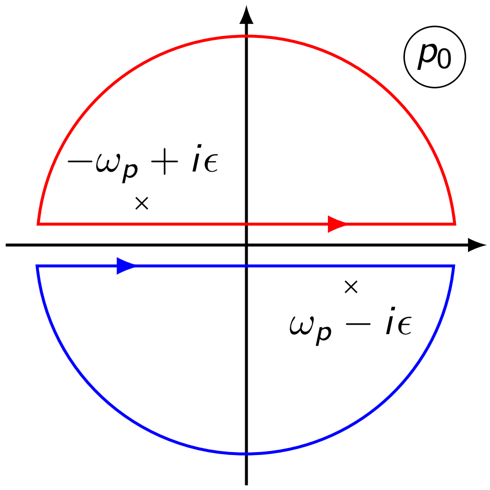


Fig. 1: Integration contours in p_0 plane.

The propagator of particles can be connected to the force between two classical static sources $J_i(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_i)$ located at $\mathbf{x}_i = (\mathbf{x}_1, \mathbf{x}_2)$. The sources disturb the vacuum $|0\rangle \to |\Omega\rangle$, since the Hamiltonian of the system is modified $\mathcal{H} \to \mathcal{H}_0 + J\cdot\phi$. Assuming for simplicity that $\phi = \phi^\dagger$, we can find the energy of the disturbed vacuum from

$$\langle\Omega|e^{-i\mathcal{H}T}|\Omega\rangle \equiv e^{-iE_0(J)T} \Rightarrow \text{in the limit } T\to\infty$$
$$= e^{\frac{i^2}{2!}\int dxdy J(x)\langle 0|T(\phi(x)\phi(y))|0\rangle J(y)} = e^{+\frac{i}{2}\int dxdy J(x)D_c(x-y)J(y)}$$

Evaluating the integral for $J(x) = J_1(x) + J_2(x)$ and neglecting "self-interactions", we get the contribution δE_0 to $E_0(J)$ due to interactions between two sources

$$\lim_{T\to\infty}\delta E_0 T = -\int dxdy J_1(x)D_c(x-y)J_2(y)$$
$$\delta E_0 = -\int \frac{d\mathbf{p}}{(2\pi)^3}\frac{e^{+i\mathbf{p}(\mathbf{x}_1-\mathbf{x}_2)}}{\mathbf{p}^2+m^2} = -\frac{1}{4\pi r}e^{-mr}, \qquad r = |\mathbf{x_1} - \mathbf{x_2}|$$

This is nothing else but the *Yukawa* potential. It is *attractive* and *falls off* exponentially over the distance scale $1/m$. Obviously, for $m = 0$ we get a Coulomb-like potential.

2.3 More degrees of freedom?

Up to now we were discussing simple scalar particles, which describe only one (in the case of hermitian field) degree of freedom (per space point). We can extend our formalism to account for fields involving several degrees of freedom by adding more (and more) *indices* to $\phi(x) \to \Phi^i_\alpha(x)$, and treating the latter as components of generalized $\Phi(x)$ in some field space. One conveniently splits the indices into two groups: *space-time* (α) and *internal* (i). Under space-time, e.g., Lorentz Eq. (2), transformations $x \to x'$, we also have $\Phi^i_\alpha(x) \to \Phi'^i_\alpha(x')$, where

$$\Phi'^i_\alpha(\Lambda x) = S_{\alpha\beta}(\Lambda)\Phi^i_\beta(x) \quad \text{(Lorentz transform)} \tag{20}$$

is a linear combination of "old" fields Φ^i_α having the same index i. Analogously, one considers rotations in the "internal" space $\Phi^i_\alpha(x) \to \Phi'^i_\alpha(x)$

$$\Phi'^i_\alpha(x) = \ U^{ij}\Phi^j_\alpha(x) \qquad \text{(Internal transform)} \tag{21}$$

that are characterized by some matrix U_{ij}, which acts only on internal indices. A quantum field in this case is represented as

$$\Phi^i_\alpha(x) = \frac{1}{(2\pi)^{3/2}} \sum_s \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[u^s_\alpha(\mathbf{p})\,(a^-_\mathbf{p})^i_s\, e^{-ipx} + v^{*s}_\alpha(\mathbf{p})(b^+_\mathbf{p})^i_s\, e^{+ipx}\right]. \tag{22}$$

Here the factors $e^{\pm ipx}$ with $p_0 = \omega_\mathbf{p}$ (plane waves) guarantee that every component of Φ^i_α satisfies the KG equation. The sum in Eq. (22) is over all polarization states, which are characterized by polarization "vectors" for particles $u^s_\alpha(\mathbf{p})$ annihilated by $(a^-_\mathbf{p})^i_s$, and anti-particles $v^{*s}_\alpha(\mathbf{p})$ created by $(b^+_\mathbf{p})^i_s$. The conjugated field $(\Phi^i_\alpha)^\dagger$ involves (conjugated) polarization vectors for (anti) particles that are (annihilated) created. Let us give a couple of examples:

- Quarks are *coloured fermions* Ψ^i_α and, e.g., $(a^-_\mathbf{p})^b_s$ annihilates the "blue" quark in a spin state s. The latter is characterized by a spinor $u^s_\alpha(\mathbf{p})$;
- There are *eight vector* gluons G^a_μ. So $(a^-_\mathbf{p})^a_s$ annihilates a gluon a in spin state s having polarization $u^s_\alpha(\mathbf{p}) \to \epsilon^s_\mu(\mathbf{p})$.

One can notice that the Lorentz transformations plays a key role in QFT. We can classify our fields as different *representations* of the corresponding *group*. Since in the SM (and, actually, in other *renormalizable* four-dimensional QFT models) we only deal with spin-0, spin-1/2, and spin-1 fields, let us elaborate on the formalism used to describe vector (spin-1) and fermion (spin-1/2) particles.

2.3.1 Massive vector fields

A charged vector field (e.g., a W-boson) can be written as

$$W_\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=1}^{3} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[\left(\epsilon^\lambda_\mu(\mathbf{p}) a^-_\lambda(\mathbf{p}) e^{-ipx} + \epsilon^{*\lambda}_\mu(\mathbf{p}) b^+_\lambda(\mathbf{p})\, e^{+ipx}\right)\right]. \tag{23}$$

A massive spin-1 particle has *3* independent polarization vectors, which satisfy

$$p_\mu \epsilon^\lambda_\mu(\mathbf{p}) = 0, \quad \epsilon^\lambda_\mu(\mathbf{p})\epsilon^{*\lambda'}_\mu(\mathbf{p}) = -\delta^{\lambda\lambda'}, \quad \sum_{\lambda=1}^{3} \epsilon^\lambda_\mu \epsilon^{*\lambda}_\nu = -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right) \quad [p_0 = \omega_\mathbf{p}].$$

The Feynman propagator can be found by considering time-ordered product of two fields

$$\langle 0|T(W_\mu(x)W^\dagger_\nu(y))|0\rangle = \frac{1}{(2\pi)^4} \int d^4p e^{-ip(x-y)} \left[\frac{-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)}{p^2 - m^2 + i\epsilon}\right] \quad [p_0 - \text{arbitrary}] \tag{24}$$

or by inverting the quadratic form of the (free) Lagrangian

$$\mathcal{L} = -\frac{1}{2} W^\dagger_{\mu\nu} W_{\mu\nu} + m^2 W^\dagger_\mu W_\mu, \quad W_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu.$$

One can show that one of the polarization vectors $\epsilon^L_\mu \simeq p_\mu/m + \mathcal{O}(m)$ and *diverges* in the limit $p_\mu \to \infty$ ($m \to 0$). This indicates that one should be careful when constructing models with massive vector fields. We will return to this issue later.

2.3.2 Massless vector fields

Massless (say photon) vectors are usually represented by

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=0}^{3} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[\epsilon^\lambda_\mu(\mathbf{p}) a^-_\lambda(\mathbf{p}) e^{-ipx} + \text{h.c.} \right]. \tag{25}$$

with

$$\epsilon^\lambda_\mu(\mathbf{p}) \epsilon^{*\lambda'}_\mu(\mathbf{p}) = g^{\lambda\lambda'}, \quad \epsilon^\lambda_\mu(\mathbf{p}) \epsilon^{*\lambda}_\nu(\mathbf{p}) = g_{\mu\nu}, \quad [a^-_\lambda(\mathbf{p}), a^+_{\lambda'}(\mathbf{p}')] = -g_{\lambda\lambda'} \delta_{\mathbf{p},\mathbf{p}'}.$$

The corresponding Feynman propagator can be given by

$$\langle 0 | T(A_\mu(x) A_\nu(y)) | 0 \rangle = \frac{1}{(2\pi)^4} \int d^4 p e^{-ip(x-y)} \left[\frac{-ig_{\mu\nu}}{p^2 + i\epsilon} \right]. \tag{26}$$

In spite of the fact that we sum over four polarizations in Eq. (25) only *two* of them are *physical*! This reflects the fact that the vector-field Lagrangian in the massless case $m = 0$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu}, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

is invariant under $A_\mu \to A_\mu + \partial_\mu \alpha(x)$ for arbitrary $\alpha(x)$ (*gauge* symmetry). Additional *conditions* (gauge-fixing) are needed to get rid of unphysical states.

2.3.3 Fermion fields

Spin-1/2 fermion fields (e.g., leptons) are represented by[5]

$$\psi^\alpha(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \sum_{s=1,2} \left[u^\alpha_s(\mathbf{p}) a^-_s(\mathbf{p}) e^{-ipx} + v^\alpha_s(\mathbf{p}) b^+_s(\mathbf{p}) e^{+ipx} \right],$$

where we explicitly write the *spinor* (Dirac) index α for u_s, v_s and the quantum operator ψ. The former satisfy the 4×4 matrix (Dirac) equations

$$(\hat{p} - m) u_s(\mathbf{p}) = 0, \quad (\hat{p} + m) v_s(\mathbf{p}) = 0, \quad \hat{p} \equiv \gamma_\mu p_\mu, \quad p_0 \equiv \omega_\mathbf{p} \tag{27}$$

and correspond to particles (u_s) or antiparticles (v_s). In Eq. (27) we use gamma-matrices

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu \equiv [\gamma_\mu, \gamma_\nu]_+ = 2 g_{\mu\nu} \mathbf{1} \quad \Rightarrow \gamma_0^2 = \mathbf{1}, \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -\mathbf{1}$$

to account for *two* spin states ($s = 1, 2$) of particles and antiparticles. Fermion fields transform under the Lorentz group $x' = \Lambda x$ as (*cf.* Eq. (20))

$$\psi'(x') = \mathcal{S}_\Lambda \psi(x), \qquad \psi'(x')^\dagger = \psi(x) \mathcal{S}^\dagger_\Lambda. \tag{28}$$

[5] There exists a charge-conjugation matrix $C = i\gamma_2$, which relates spinors for particles u and antiparticles v, e.g., $v = Cu^*$.

It turns out that the 4×4 matrix $\mathcal{S}_\Lambda^\dagger \neq \mathcal{S}_\Lambda^{-1}$ but $\mathcal{S}^{-1} = \gamma_0 \mathcal{S}^\dagger \gamma_0$. Due to this, it is convenient to introduce a *Dirac-conjugated* spinor $\bar{\psi}(x) \equiv \psi^\dagger \gamma_0$. The latter enters into

$$\begin{aligned} \bar{\psi}'(x')\psi'(x') &= \bar{\psi}(x)\psi(x), && \text{Lorentz } \textit{scalar}; \\ \bar{\psi}'(x')\gamma_\mu\psi'(x') &= \Lambda_{\mu\nu}\bar{\psi}(x)\gamma_\nu\psi(x), && \text{Lorentz } \textit{vector}. \end{aligned}$$

This allows us to convince ourselves that the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\hat{\partial} - m\right)\psi$$

is also a Lorentz scalar, i.e., respects Lorentz symmetry. Dirac-conjugated spinors are used to impose Lorentz-invariant normalization on u and v:

$$\bar{u}_s(\mathbf{p})u_r(\mathbf{p}) = 2m\delta_{rs}, \qquad \bar{v}_s(\mathbf{p})v_r(\mathbf{p}) = -2m\delta_{rs},$$

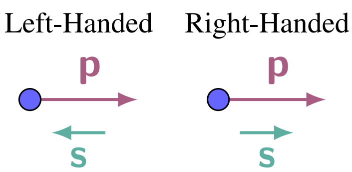
An important fact about quantum fermion fields is that, contrary to the case of scalar or vector (*boson*) fields, the creation/annihilation operators for fermions $a^\pm_{s,\mathbf{p}}$ and antifermions $b^\pm_{s,\mathbf{p}}$ *anticommute*:

$$\begin{aligned} \left[a^-_{r,\mathbf{p}}, a^+_{s,\mathbf{p}'}\right]_+ &= \left[b^-_{r,\mathbf{p}}, b^+_{s,\mathbf{p}'}\right]_+ = \delta_{sr}\delta(\mathbf{p} - \mathbf{p}') \\ \left[a^\pm_{r,\mathbf{p}}, a^\pm_{s,\mathbf{p}'}\right]_+ &= \left[b^\pm_{r,\mathbf{p}}, b^\pm_{s,\mathbf{p}'}\right]_+ = \left[a^\pm_{r,\mathbf{p}}, b^\pm_{s,\mathbf{p}'}\right]_+ = 0. \end{aligned}$$

Due to this, fermions obey the *Pauli principle*, e.g., $a^+_{r,\mathbf{p}}a^+_{r,\mathbf{p}} = 0$. Moreover, one can explicitly show that quantization of bosons (integer spin) with anticommutators or fermions (half-integer spin) with commutators leads to inconsistencies (violates the *Spin-Statistics* theorem).

Let us emphasize an important difference between the notions of *chirality* and *helicity*. Two independent solutions for *massive* fermions ($u_{1,2}$) can be chosen to correspond to two different *helicities* — projections of spin vector $\mathbf{s}$ onto direction of $\mathbf{p}$:

$$\mathcal{H} = \mathbf{s} \cdot \mathbf{n}, \quad \mathbf{n} = \mathbf{p}/|\mathbf{p}|. \tag{29}$$



In *free* motion it is *conserved* and serves as a good quantum number. However, it is not a Lorentz-invariant quantity. Indeed, we can flip the sign of particle momentum by moving with speed faster than $v = |\mathbf{p}|/p_0$. As a consequence, $\mathbf{n} \to -\mathbf{n}$ and $\mathcal{H} \to -\mathcal{H}$. However, *helicity* for a *massless* particle is the same for all inertial observers and coincides with *chirality*, which is a *Lorentz-invariant* concept.

By definition Left (ψ_L) and Right (ψ_R) *chiral* spinors are eigenvectors of

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \Rightarrow [\gamma_\mu, \gamma_5]_+ = 0, \quad \gamma_5^2 = 1, \quad \gamma_5^\dagger = \gamma_5, \tag{30}$$

where

$$\gamma_5\psi_L = -\psi_L, \qquad \gamma_5\psi_R = +\psi_R. \tag{31}$$

Any spinor ψ can be decomposed as

$$\psi = \psi_L + \psi_R, \qquad \psi_{L/R} = P_{L/R}\psi, \quad P_{L/R} = \frac{1 \mp \gamma_5}{2}. \tag{32}$$

To illustrate this fact, let us use the Dirac representation of 4×4 γ-matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_{L/R} = \frac{1}{2}\begin{pmatrix} 1 & \mp 1 \\ \mp 1 & 1 \end{pmatrix}$$

to re-write the spinors for particles u_λ and antiparticles v_λ of positive $\lambda = +1$ and negative $\lambda = -1$ helicities as a sum of left and right *chiral* parts ($\beta^+ = \sqrt{E+m}$):

$$u_\lambda(\mathbf{p}) = \frac{1}{2}\beta_+ \left(1 - \lambda\frac{p}{E+m}\right)\begin{pmatrix} \chi_\lambda \\ -\chi_\lambda \end{pmatrix} + \frac{1}{2}\beta^+ \left(1 + \lambda\frac{p}{E+m}\right)\begin{pmatrix} \chi_\lambda \\ \chi_\lambda \end{pmatrix}, \tag{33}$$

$$v_\lambda(\mathbf{p}) = \frac{1}{2}\beta^+ \left(1 - \lambda\frac{p}{E+m}\right)\begin{pmatrix} \chi_{-\lambda} \\ \chi_{-\lambda} \end{pmatrix} + \frac{1}{2}\beta^+ \left(1 + \lambda\frac{p}{E+m}\right)\begin{pmatrix} -\chi_{-\lambda} \\ \chi_{-\lambda} \end{pmatrix} \tag{34}$$

with $\mathbf{p} = p(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ and

$$\chi_1 = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \; \chi_{-1} = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}.$$

One can easily see that in the massless case[6]

$$P_L u_+ = 0, \quad P_R u_- = 0, \text{ for } \textit{particle}, \text{ the } \textit{spinor} \text{ chirality } \textit{coincides} \text{ with helicity},$$
$$P_L v_- = 0, \quad P_R v_+ = 0, \text{ for } \textit{antiparticle}, \text{ the } \textit{spinor} \text{ chirality is } \textit{opposite} \text{ to helicity}.$$

Moreover, we can rewrite the Dirac Lagrangian it terms of chiral components (Weyl spinors)

$$\mathcal{L} = i(\underbrace{\bar\psi_L\hat\partial\psi_L + \bar\psi_R\hat\partial\psi_R}_{\text{conserve chirality}}) - m(\underbrace{\bar\psi_L\psi_R + \bar\psi_R\psi_L}_{\text{break chirality}}), \tag{35}$$

and see that, indeed, it is the mass term that mixes two chiralities. Due to this, it violates *chiral* symmetry corresponding to the independent rotation of left and right components

$$\psi \to e^{i\gamma_5\alpha}\psi. \tag{36}$$

Consequently, if we drop the mass term, the symmetry of the Lagrangian is enhanced.

Up to now we were discussing the so-called Dirac mass term. For *neutral* fermions (e.g., neutrino) there is another possibility — a *Majorana* mass. Since charge-conjugation applied to fermion fields, $\psi \to \psi^c$, *flips* chirality, we can use ψ_L^c in place of ψ_R to write

$$\mathcal{L} = \frac{1}{2}(i\bar\psi_L\hat\partial\psi_L - m\bar\psi_L\psi_L^c). \tag{37}$$

As a consequence, to describe Majorana particles, we only need two components instead of four since antiparticles coincide with particles in this case. At the moment, the nature of neutrinos is unclear, and we refer to Ref. [20] for more elaborated discussion.

2.4 From free to interacting fields

Fields that describe non-interacting particles seems to be an abstraction. Nevertheless, all we have an intuition that in many cases we can neglect all (or some) interactions and treat real particles as free. Indeed, in HEP, a typical collision/scattering experiment deals with *"free"* initial and final states and

[6] One can also define *chirality of an antiparticle*, which is opposite to that of the corresponding *spinor* v, i.e., introduce v_R, such as $P_L v_R = v_R$, but $P_R v_R = 0$. In this case, for $m \to 0$ the chirality precisely corresponds to helicity and $v_R \to v_+$, etc.

considers *transitions* between these states due to *interactions*. To account for this in a quantum theory, one introduces the S-*matrix* with matrix elements

$$\mathcal{M} = \langle\beta|S|\alpha\rangle, \qquad \mathcal{M} = \delta_{\alpha\beta} + (2\pi)^4\delta^4(p_\alpha - p_\beta) i M_{\alpha\beta} \tag{38}$$

corresponding to the amplitudes for possible transitions between *in* $|\alpha\rangle$ and *out* $|\beta\rangle$ states:

$$|\alpha\rangle = \tilde{a}^+_{\mathbf{p}_1}...\tilde{a}^+_{\mathbf{p}_r}|0\rangle, \quad |\beta\rangle = \tilde{a}^+_{\mathbf{k}_1}...\tilde{a}^+_{\mathbf{k}_s}|0\rangle, \quad \tilde{a}^+_{\mathbf{p}} = (2\pi)^{3/2}\sqrt{2\omega_{\mathbf{p}}}a^+_{\mathbf{p}}, \tag{39}$$

where for convenience (see Eq. (50)) we re-scale our creation/annihilation operators. Given the matrix element $M_{\alpha\beta}$, one can calculate the differential probability (per unit volume per unit time) to evolve from $|\alpha\rangle$ to $|\beta\rangle$:

$$dw = \frac{n_1...n_r}{(2\omega_{p_1})...(2\omega_{p_r})}|M_{\alpha\beta}|^2 d\Phi_s. \tag{40}$$

Here n_i correspond to initial-state particle densities, and an element of phase space is given by

$$d\Phi_s = (2\pi)^4\delta^4\left(p_{in} - k_{out}\right)\frac{d\mathbf{k}_1}{(2\pi)^3(2\omega_{k_1})}\cdots\frac{d\mathbf{k}_i}{(2\pi)^3(2\omega_{k_i})} \tag{41}$$

with $p_{in} = \sum p_i$ and $k_{out} = \sum k_i$. Since we are usually interested in processes involving one (decay) or two particles (e.g., collision of two beams) in the initial state, it is more convenient to consider the differential decay width $d\Gamma$ in the rest frame of a particle with mass m, or cross-section $d\sigma$ of a process $2 \to s$:

$$d\Gamma = \Phi_\Gamma |M_{1\to s}|^2 d\Phi_s, \qquad \Phi_\Gamma = \frac{1}{2m}, \tag{42}$$

$$d\sigma = \Phi_\sigma |M|^2 d\Phi_s, \qquad \Phi_\sigma = \frac{1}{4\sqrt{(p_1p_2)^2 - p_1^2p_2^2}} \quad . \tag{43}$$

In Eq. (43) the factor Φ_σ is *Lorentz-invariant* and is expressed in terms of four-momenta of initial particles p_1 and p_2. The total width Γ and total cross-section σ can be obtained by integration over the momenta of final particles restricted by energy-momentum conservation due to the four-dimensional δ-function in Eq. (41).

In QFT, the S-matrix is written in terms of the time-ordered exponent

$$S = Te^{-i\int d^4x \mathcal{H}_I(x)} = Te^{i\int d^4x \mathcal{L}_I(x)}, \tag{44}$$

which involve the interaction Hamiltonian $\mathcal{H}_I$ (Lagrangian $\mathcal{L}_I$).

The interaction Lagrangian $\mathcal{L}_I = \mathcal{L}_{full} - \mathcal{L}_0$ is a sum of *Lorentz-invariant* terms having more than *two* fields and more ∂_μ than in the quadratic part $\mathcal{L}_0$, which, if considered alone, describes free particles. It is worth noting that in Eq. (44) we treat $\mathcal{L}_I$ ($\mathcal{H}_I$) as an operator built from *free*[7] quantum fields (i.e., certain combinations of $a^\pm$ and $b^\pm$).

The *time-ordering* operation, which was used to define particle propagators, is generalized in Eq. (44) to account for more than two fields originating from $\mathcal{L}_I$

$$T\Phi_1(x_1)...\Phi_n(x_n) = (-1)^k \Phi_{i_1}(x_{i_1})...\Phi_{i_n}(x_{i_n}), \qquad x^0_{i_1} > ... > x^0_{i_n}. \tag{45}$$

The factor $(-1)^k$ appears due to k possible permutations of *fermion* fields.

To conserve probability the (interaction) Lagrangian should be hermitian. Any scalar combination of quantum fields can, in principle, be included in $\mathcal{L}_I$, e.g.,

$$\mathcal{L}_I : \quad g\phi^3(x), \qquad \lambda\phi^4(x), \qquad y\bar{\psi}(x)\psi(x)\phi(x)$$

[7]More precisely, operators in the *interaction/Dirac* picture.

$$e\bar{\psi}(x)\gamma_\mu\psi(x)A_\mu(x), \qquad G\left[(\bar{\psi}_1\gamma_\mu\psi_2)\,(\bar{\psi}_3\gamma_\mu\psi_4)+\text{h.c.}\right] \quad .$$

The parameters (couplings) g, λ, e, y, and G are related to the "*strength*" of the interactions. An important property of any coupling in the QFT model is its (mass) *dimension*. The latter can be deduced from the fact that in the natural units the Action is dimensionless and $[\mathcal{L}] = 4$. One can notice that all the couplings (hidden) in the T-shirt Lagrangian are *dimensionless*. As it will be clear from subsequent discussion, this has crucial consequences for the self-consistency of the SM model.

2.5 Perturbation theory

In an interacting theory it is very hard, if not impossible, to calculate the S-matrix, see Eq. (44), exactly. Usually, we make an assumption that the couplings in $\mathcal{L}_I$ are small allowing us to treat the terms in $\mathcal{L}_I$ as *perturbations* to $\mathcal{L}_0$. As a consequence, we expand the T-exponent and restrict ourselves to a finite number of terms. In the simplest case of $\mathcal{L}_I = -\lambda\phi^4/4!$ we have at the nth order

$$\frac{i^n}{n!}\left[\frac{\lambda}{4!}\right]^n \langle 0|\tilde{a}^-_{\mathbf{k}_1}...\tilde{a}^-_{\mathbf{k}_s}\int dx_1...dx_n T\left[\phi(x_1)^4...\phi(x_n)^4\right]\tilde{a}^+_{\mathbf{p}_1}...\tilde{a}^+_{\mathbf{p}_r}|0\rangle. \tag{46}$$

To proceed, one utilizes the *Wick* theorem:

$$T\Phi_1...\Phi_n = \sum(-1)^\sigma\langle 0|T(\Phi_{i_1}\Phi_{i_2})|0\rangle...\langle 0|T(\Phi_{i_{m-1}}\Phi_{i_m})|0\rangle\ {:}\Phi_{i_{m+1}}...\Phi_{i_n}{:}, \tag{47}$$

where the sum goes over all possible ways to pair the fields. The Wick theorem Eq. (47) expresses *time-ordered* products of fields in terms of *normal-ordered* ones and propagators. As it was mentioned earlier the normal-ordered operation puts *all* annihilation operators originating from different Φs to the right. It also cares about fermions, e.g.,

$${:}a_1^- a_2^+ a_3^- a_4^- a_5^+ a_6^-{:} = (-1)^\sigma a_2^+ a_5^+ a_1^- a_3^- a_4^- a_6^-\,, \tag{48}$$

with σ corresponding to the number of fermion permutations (*cf.* Eq. (45)). In Fig. 2 a cartoon, which illustrates Eq. (47) for one of the contributions to $T[\mathcal{L}_I(x)\mathcal{L}_I(y)]$, is provided.

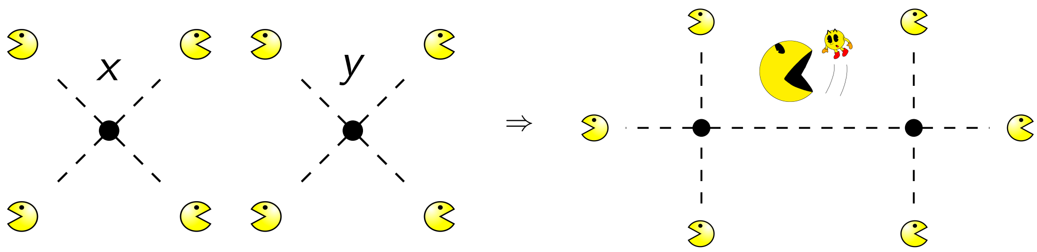


Fig. 2: The Wick theorem at work: one of the contributions.

After application of the Wick theorem we are left with the matrix elements of the form

$$\langle 0|\tilde{a}^-_{\mathbf{k}_1}...\tilde{a}^-_{\mathbf{k}_s}\ {:}\Phi_{i_{m+1}}...\Phi_{i_n}{:}\ \tilde{a}^+_{\mathbf{p}_1}...\tilde{a}^+_{\mathbf{p}_r}|0\rangle. \tag{49}$$

To get a *non-zero* result, all $a^-(a^+)$ in the normal product of fields from the Lagrangian have to be "killed" by (commuted with) $a^+(a^-)$ from the external states. For our *generalized* field, Eq. (22), we have

$$\left[\Phi^i_\alpha(x), (a^+_{\mathbf{p}})^i_s\right] = \underbrace{\frac{e^{-ipx}}{(2\pi)^{3/2}\sqrt{2\omega_p}}}_{\text{common to all fields}} u^s_\alpha(\mathbf{p}), \qquad \text{initial state polarization (particle);}$$

$$\left[(b^-_{\mathbf{p}})^i_s, \Phi^i_\alpha(x)\right] = \frac{e^{+ipx}}{(2\pi)^{3/2}\sqrt{2\omega_p}} v^{*s}_\alpha(\mathbf{p}), \qquad \text{final state polarization (antiparticle).} \qquad (50)$$

and one clearly sees that the factors in the denominators, Eq. (50), are avoided when the re-scaled $\tilde{a}^\pm$ (or $\tilde{b}^\pm$) operators, Eq. (39), are used.

All this machinery can be implemented in a set of *Feynman rules*, which are used to draw (and evaluate) *Feynman diagrams*. Every Feynman diagram involves *vertices*, *external* and *internal* lines. Internal lines connect two vertices and correspond to propagators. The expression for propagators can be derived from $\mathcal{L}_0$, e.g.,

$$\left.\begin{array}{c} \langle 0|T(\phi(x)\phi^\dagger(y)|0\rangle \\ \langle 0|T(\psi(x)\bar{\psi}(y)|0\rangle \\ \langle 0|T(W_\mu(x)W^\dagger_\nu(y)|0\rangle \end{array}\right\} = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip(x-y)}}{p^2-m^2+i\epsilon}\left\{\begin{array}{lll} 1 & & \phi; \\ \hat{p}+m & & \psi; \\ -g_{\mu\nu}+p_\mu p_\nu/m^2 & & W_\mu. \end{array}\right. \qquad (51)$$

One can notice that all the dependence on x_i of the integrand in Eq. (46) comes from either Eq. (50) or Eq. (51). As a consequence, it is possible to carry out the integration for *every* x_i

$$\int d^4x_i e^{-ix_i(p_1+...+p_n)} = (2\pi)^4\delta^4(p_1+...+p_n) \qquad (52)$$

and obtain a δ-function reflecting energy-momentum conservation at the corresponding vertex.

Depending on the direction of momenta, the external lines represent incoming or outgoing particles (see Table 1). Again, the corresponding factors (="polarization vectors") are derived from $\mathcal{L}_0$. Notice that we explicitly write the Lorentz indices for vector particles and suppress the Dirac indices for fermions. To keep track of the index contractions in the latter case, one uses *arrows* on the fermion lines.[8]

Table 1: Feynman rules for external states.

incoming scalar	1	*incoming* fermion	$u_s(\mathbf{p})$
outgoing scalar	1	*outgoing* fermion	$\bar{u}_s(\mathbf{p})$
incoming vector	$\epsilon^\lambda_\mu(\mathbf{p})$	*incoming* antifermion	$\bar{v}_s(\mathbf{p})$
outgoing vector	$\epsilon^{*\lambda}_\mu(\mathbf{p})$	*outgoing* antifermion	$v_s(\mathbf{p})$

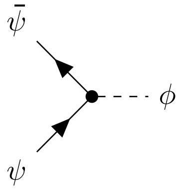
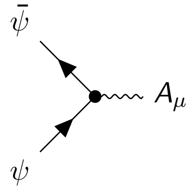
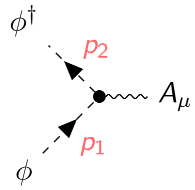
Let us turn to interaction vertices. The corresponding Feynman rules are derived from $\mathcal{A}_I = \int d^4\mathcal{L}_I$. It is convenient to do this by carrying out a Fourier transform to "convert" coordinate derivatives to momenta and considering variations of the action. In the case of $\mathcal{L}_I = -\lambda\phi^4/4!$ we have (all momenta are assumed to be incoming)

$$i\frac{\delta^4\mathcal{A}_I[\phi]}{\delta\phi(p_1)\delta\phi(p_2)\delta\phi(p_3)\delta\phi(p_4)}\bigg|_{\phi=0} \Rightarrow \underbrace{(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)}_{\text{conservation of energy-momentum}}\times[-i\lambda]. \qquad (53)$$

[8]There are subtleties when interactions involve Majorana fermions.

In a typical diagram all $(2\pi)^4\delta(...)$ factors (but *one*, which is accounted for in the definition, Eq. (38), of $M_{\alpha\beta}$) that reflect the energy-momentum conservation at each vertex, are removed by the momentum integration originating from propagators, Eq. (51). Due to this, we also omit these factors (see, Table 2 for examples).

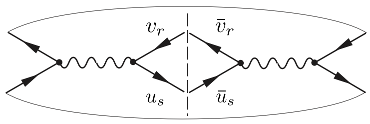
Table 2: Vertex Feynman rules. Derivatives in $\mathcal{L}_I$ correspond to particle momenta.

$\mathcal{L}_I = -y\bar{\psi}\psi\phi$	$\mathcal{L}_I = e\bar{\psi}\gamma_\mu\psi A_\mu$	$\mathcal{L}_I = ieA_\mu\left(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger\right)$
		
$-iy$	$ie\gamma_\mu$	$ie(p_1+p_2)_\mu$

Given Feynman rules, one can draw all possible diagrams that contribute to a process and evaluate the amplitude. We do not provide the precise prescription here (see Refs. [14–18] for details) but just mention the fact that one should keep in mind various *symmetry* factors and relative *signs* that can appear in real calculations.

In order to get probabilities, we have to *square* matrix elements, e.g.,

$$|M|^2 = MM^\dagger \Rightarrow \tag{54}$$



Sometimes we do not care about polarization states of initial or final particles. As a consequence, we have to *sum* the probabilities corresponding to different *final* polarizations, and *average* over the *initial* ones. That is where *spin-sum* formulas,e.g.,

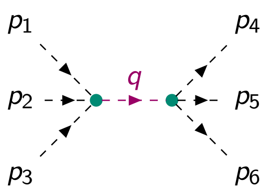
$$\sum_s u_s(\mathbf{p_1})\bar{u}_s(\mathbf{p_1}) = \hat{p}_1 + m, \qquad \sum_s v_s(\mathbf{p_2})\bar{v}_s(\mathbf{p_2}) = \hat{p}_2 - m \tag{55}$$

become useful

$$MM^\dagger \to \sum_{s,r}(\bar{u}_s A v_r)(\bar{v}_r A^\dagger u_s) = \mathrm{Tr}\left[(\hat{p}_1 + m)A(\hat{p}_2 - m)A^\dagger\right]. \tag{56}$$

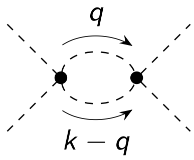
In this case we avoid explicit manipulations with spinors and utilize well-known and efficient machinery for gamma-matrix traces.

Let us continue by mentioning that only in *tree* graphs, such as



$$\Rightarrow \quad (2\pi)^4\delta^4\left(\sum_{i=1}^3 p_i - \sum_{i=4}^6 p_i\right)[-i\lambda]^2\,\frac{i}{q^2-m^2+i\epsilon},$$

all the integrations (due to propagators) are “killed” by vertex δ-functions. However, nothing forbids us from forming *loops*. In this case, we have *integrals* over unconstrained momenta, e.g., in the ϕ^4-theory



$$: \qquad I_2(k) \equiv \int \frac{d^4q}{[q^2+i\epsilon][(k-q)^2+i\epsilon]} \sim \int^\infty \frac{|q|^3 d|q|}{|q|^4} \sim \ln\infty,$$

which can lead to *divergent* (meaningless?) results. This is a manifestation of the ultraviolet (or *UV*) divergences due to *large* momenta ("small distances").

A natural question arises: Do we have to abandon QFT? Since we still use it, there are reasons *not* to do this. Indeed, we actually do not know physics up to infinitely small scales and our extrapolation can not be adequate in this case. To make sense of the integrals, we can *regularize* them, e.g., introduce a *"cut-off"* $|q| < \Lambda$,

$$I_2^\Lambda(k) = i\pi^2\left[\ln\frac{\Lambda^2}{k^2}+1\right] + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right) = i\pi^2\left[\ln\frac{\Lambda^2}{\mu^2} - \ln\frac{k^2}{\mu^2}+1\right] + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right) \tag{57}$$

or use another convenient possibility — *dimensional* regularization, when $d = 4$ space-time is formally continued to $d = 4 - 2\varepsilon$ dimensions:

$$I_2^{4-2\varepsilon}(k) = \mu^{2\varepsilon}\int\frac{d^{4-2\varepsilon}q}{q^2(k-q)^2} = i\pi^2\left(\frac{1}{\varepsilon} - \ln\frac{k^2}{\mu^2}+2\right) + \mathcal{O}(\varepsilon). \tag{58}$$

Both the regularized integrals are now convergent[9] and share the same logarithmic dependence on external momentum k. One can also notice a new (renormalization) scale μ, which appears in regularized integrals, and a (one-to-one) correspondence between a *logarithmically* divergent contribution $\log\Lambda^2/\mu^2$ in Eq. (57) and the pole term $1/\varepsilon$ in Eq. (58). However, the constant terms are *different*. How do we make sense of this ambiguity?

The crucial observation here is that the divergent pieces, which blow up when we try to remove the regulators ($\Lambda \to \infty$ or $\varepsilon \to 0$), are *local*, i.e., depend polynomially on external kinematic variables. This fact allows us to *cancel* them by the so-called counterterm (CT) vertices. The latter can be interpreted as new terms in $\mathcal{L}_I$. Moreover, in a *renormalizable* QFT model additional (divergent) contributions have the same form as the initial Lagrangian and thus can be "absorbed' into redefinition of fields and parameters.

One can revert the reasoning and assume that the initial Lagrangian is written in terms of the so-called *bare* (unobservable) quantities. The predictions of the model are finite since the explicit dependence of Feynman integrals on the cut-off Λ (or ε) is actually compensated by the implicit dependence of bare fields and parameters. In some sense the latter quantities represent our ignorance of dynamics at tiny scales: physical fields and parameters are always "dressed" by clouds of virtual particles.

It is obvious that working with *bare* quantities is not very convenient. One usually makes the dependence on Λ (or ε) explicit by introducing divergent Z-factors for *bare* fields (ϕ_B), masses (m_B^2), and couplings (λ_B), e.g.,

$$\mathcal{L}_{full} = \frac{1}{2}(\partial\phi_B)^2 - \frac{m_B^2}{2}\phi_B^2 + \frac{\lambda_B\phi_B^4}{4!} = \frac{Z_2}{2}(\partial\phi)^2 - \frac{Z_m m^2}{2}Z_2\phi^2 + \frac{Z_\lambda\lambda}{4!}(Z_2\phi^2)^2 \tag{59}$$

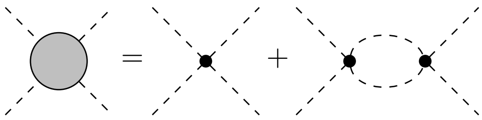
$$= \frac{(\partial\phi)^2}{2} - \frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4!} + \underbrace{\frac{(Z_2-1)}{2}(\partial\phi)^2 - \frac{(Z_mZ_2-1)m^2}{2}\phi^2 + (Z_4Z_2^2-1)\frac{\lambda\phi^4}{4!}}_{\text{counterterms}}. \tag{60}$$

Here ϕ, m and λ denote *renormalized* (finite) quantities. Since we can always subtract something finite from infinity, there is a certain freedom in this procedure. The different constant terms in Eq. (57) and

[9]We do not discuss the issue of possible infrared (IR) divergences here.

Eq. (58) are just a manifestation of this fact. So we have to impose additional *conditions* on Z, i.e., define a *renormalization* scheme. For example, in the minimal (MS) schemes we subtract only the divergent terms, e.g., only poles in ε, while in the so-called momentum-subtraction (MOM) schemes we require certain amplitudes (more generally *Green functions*) to have specific values at some fixed kinematics.

As an illustration, let us consider a scattering amplitude $2 \to 2$ in the ϕ^4 model calculated in perturbation theory:



$$+ \text{permutations} + \text{more loops} \tag{61}$$

$$= \lambda_B(\Lambda) - \frac{\lambda_B(\Lambda)^2}{2(16\pi^2)} \left(\ln \frac{\Lambda^2}{\mu^2} - \ln \frac{k^2}{\mu^2} + \ldots \right) + \ldots \tag{62}$$

$$= \left[\lambda(\mu) + \frac{3}{2} \frac{\lambda^2(\mu)}{16\pi^2} \ln \frac{\Lambda^2}{\mu^2} \right] - \frac{\lambda(\mu)^2}{2(16\pi^2)} \left(\ln \frac{\Lambda^2}{\mu^2} - \ln \frac{k^2}{\mu^2} + \ldots \right) + \ldots \tag{63}$$

$$= \lambda(\mu) + \frac{\lambda(\mu)^2}{2(16\pi^2)} \left(\ln \frac{k^2}{\mu^2} + \ldots \right) + \ldots . \tag{64}$$

In Eq. (61) the tree-level and one-loop diagrams contributing to the matrix element are presented. The corresponding expression in terms of the bare coupling $\lambda_B(\Lambda)$ that implicitly depends on the regularization parameter Λ is given in Eq. (62). We introduce a renormalized[10] coupling $\lambda(\mu)$ in Eq. (63) to make the dependence explicit:

$$\lambda_B(\Lambda) = \lambda(\mu) Z_\lambda = \lambda(\mu) \left(1 + \frac{3}{2} \frac{\lambda(\mu)}{16\pi^2} \ln \frac{\Lambda^2}{\mu^2} + \ldots \right). \tag{65}$$

The final result, Eq. (64), is finite (when $\Lambda \to \infty$) and can be confronted with experiment. It seems to depend on the auxiliary scale μ. The crucial point here is that *observables* (if all orders of PT are taken into account) actually do *not* depend on μ. Changing μ corresponds to a certain reshuffling of the PT series: some terms from loop corrections are absorbed into the re-scaled (*running*) couplings. This allows one to improve the "convergence"[11] of the finite-order result by a convenient choice of μ.

The scale-dependence of the *running* couplings is governed by renormalization-group equations (RGE). In the considered case we have

$$\lambda(\mu_0) \to \lambda(\mu), \quad \frac{d}{d\ln\mu}\lambda = \beta_\lambda(\lambda), \quad \beta_\lambda = \frac{3}{2} \frac{\lambda^2}{16\pi^2} + \ldots \quad . \tag{66}$$

The *beta-function* β_λ can be calculated order-by-order in PT. However, the (initial) value $\lambda(\mu_0)$ needed to solve Eq. (66) is *not predicted* and has to be extracted from experiment ("measured").

It is worth pointing out here that two different numerical values of the *renormalized* self-coupling, λ_1 and λ_2, do not necessarily correspond to different physics. Indeed, if they are fitted from measurements at different scales, e.g., μ_0 and μ, and are related by means of RGE, they represent the *same* physics (see Fig. 3).

2.6 Renormalizable or non-renormalizable?

Let us stress again that the model is called *renormalizable* if *all* the UV divergences that appear in loop integrals can be canceled by local counterterms due to renormalization of bare parameters and couplings of the *initial* Lagrangian $\mathcal{L}_{full}$. But what happens if there is a UV divergent amplitude but the structure

[10] We use minimal subtractions here and the factor of three comes from the fact that all three one-loop graphs (s, t and u) give rise to the same *divergent* term.

[11] Actually, the PT series are *asymptotic* (divergent) and we speak about the behavior of a limited number of first terms here.

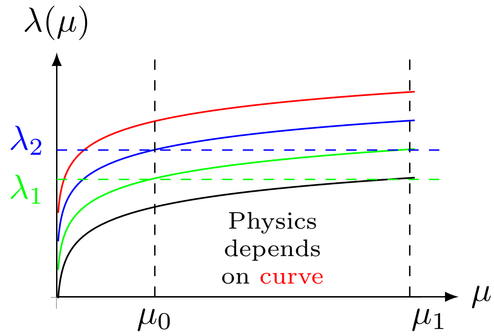


Fig. 3: Solutions of RGE for different boundary conditions.

of the required subtraction does not have a counter-part in $\mathcal{L}_{full}$, i.e., we do not have a coupling to absorb the infinity? Obviously, we can modify $\mathcal{L}_{full}$ and *add* the required term (and the coupling).

An example of such a situation can be found in the model with a scalar ϕ (e.g., Higgs) coupled to a fermion ψ (e.g., top quark) via the Yukawa interaction characterized by the coupling y

$$\mathcal{L}_I \ni \delta\mathcal{L}_Y = -y \cdot \bar{\psi}\psi\phi. \tag{67}$$

Let us assume for the moment that we set the self-coupling to zero $\lambda = 0$ and want to calculate the Higgs-scattering amplitude due to virtual top quarks (see, Fig. 4). We immediately realize that the contribution is divergent and without $\delta\mathcal{L}_4 = -\lambda\phi^4/4!$ we are not able to make sense of our model. Due to this, we are forced to consider the ϕ^4 term in a consistent theory.

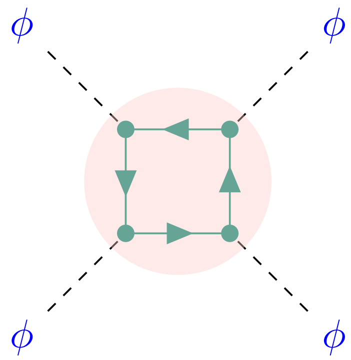


Fig. 4: One-loop correction to higgs self-interaction.

Since we modified $\mathcal{L}_{full}$, we have to re-calculate all the amplitudes form scratch. In principle, new terms in $\mathcal{L}_I$ will generate new diagrams, which can require new interactions to be added to $\mathcal{L}_I$. Will this process terminate? In the case of *renormalizable* models the answer is positive. We just need to make sure that $\mathcal{L}_I$ include *all* possible terms with *dimensionless* couplings[12], or, *equivalently*, local dimension-4 *operators* built from quantum fields and their derivatives.

On the contrary, if we have an avalanche of new terms with increasing dimensions, this is a signal of a *non-renormalizable* model. It looks like that we have to abandon such models since we need to measure an infinite number of couplings to predict something in this situation! However, it should be stressed that non-renormalizable models, contrary to renormalizable ones, involve couplings G_i with *negative* mass dimension $[G_i] < 0$! Due to this, not all of them are important at *low* energies, which satisfy

$$G_i E^{-[G_i]} \ll 1. \tag{68}$$

[12]Remember the T-shirt Lagrangian?

This explains the success of the *Fermi model* involving the dimension-6 four-fermion operator

$$-\mathcal{L}_I = G\bar{\Psi}_p\gamma_\rho\Psi_n \cdot \bar{\Psi}_e\gamma_\rho\Psi_\nu + \text{h.c.} \tag{69}$$

in the description of the β-decay $n \to p + e^- + \bar{\nu}_e$. The model turns out to be the harbinger of the modern electroweak theory. Although being non-renormalizable and not self-consistent, it provides us with the important information about the *electroweak* scale. The latter turns out to be related to the *measured* value of the Fermi constant G and corresponds to the scale, at which some new dynamics should appear to cure the inconsistencies.

Let us summarize what we have learned so far. In QFT we describe particles and their interactions by considering an *Action/Lagrangian*. We assume that general Lagrangian $\mathcal{L}$ is

– Lorentz (Poincare) invariant (a sum of Lorentz scalars),
– Local (involve finite number of partial derivatives),
– Real (hermitian) (respects unitarity=conservation of probability)

We split $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ into

- the free part $\mathcal{L}_0$ that determines *Feynman rules* for propagators and polarization vectors,
- the interaction Lagrangian $\mathcal{L}_I$ that gives rise to *Feynman rules* for interaction vertices.

Given Feynman rules we evaluate amplitudes and probabilities in *perturbation theory* (PT). Depending on the *dimension* of the couplings in $\mathcal{L}_I$ we distinguish *renormalizable* (self-consistent) and *non-renormalizable* (not self-consistent yet useful) models. To make sense of predictions of *renormalizable* models we utilize *regularization/renormalization*. The parameters of such models depend on scale and *RG* can be used to relate predictions at different scales. *Non-renormalizable* models, on the contrary, are treated only as low-energy *effective* approximations and give us a hint for a "breakdown" or *"new physics"* scale [19].

In principle, we provide (almost) all the necessary information that allows one, given some $\mathcal{L}$, to carry out *calculations* and confront the model with experiments. We put some important, yet very general, restrictions on $\mathcal{L}$. We can try to construct new models by trial and error, but it always nice to have some guiding principle. It is fair to say that modern physics is built around *symmetries*. Anticipating their role in the construction of the SM, let us consider this topic in more detail.

3 An ode to symmetry

The beauty of symmetries and their usefulness in ordinary life are beyond doubt. For example, an architect can design only half of the building (and use mirror symmetry to get the rest), or we can save a lot of time if employ symmetry arguments for cutting paper snowflakes.

Symmetries are intimately connected with *transformations*, which leave something *invariant*. The transformations can be *discrete*, such as (switching back to QFT)

$$\begin{aligned}
\text{Parity} &: \phi'(\mathbf{x},t) = P\phi(\mathbf{x},t) = \phi(-\mathbf{x},t),\\
\text{Time-reversal} &: \phi'(\mathbf{x},t) = T\phi(\mathbf{x},t) = \phi(\mathbf{x},-t),\\
\text{Charge-conjugation} &: \phi'(\mathbf{x},t) = C\phi(\mathbf{x},t) = \phi^\dagger(\mathbf{x},t),
\end{aligned}$$

or depend on *continuous* parameters. One distinguishes *space-time* from *internal* transformations (*cf.* with the distinction between two sets of indices that we attached to our *generalized* field, Eq. (22)). Lorentz boosts, rotations, and translations are typical examples of the former, while phase transformations belong to the latter (see Fig. 5). At the moment, let us consider *global* symmetries with parameters independent of space-time coordinates and postpone the discussion of x-dependent or *local* (*gauge*) transformations to Section 3.2.

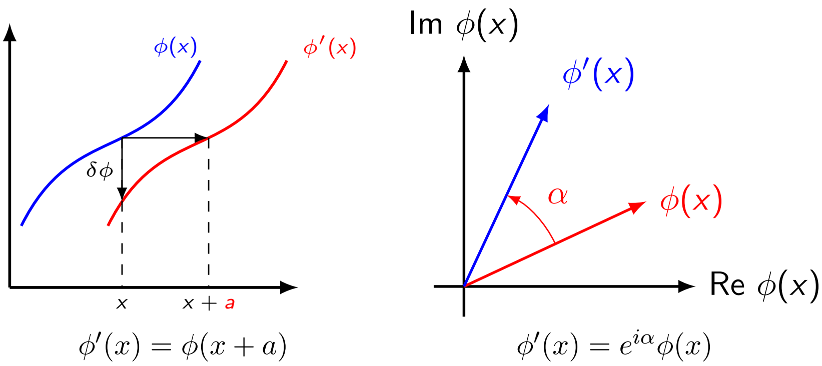


Fig. 5: Translations (left) and phase transformations (right).

3.1 Global symmetries

A convenient way to deal with symmetries in (quantum) field theories is to consider again the Action functional $\mathcal{A}[\phi]$. We can *define* a *symmetry* as *particular* infinitesimal variations $\delta\phi(x)$ that for *any* $\phi(x)$ leave $\mathcal{A}[\phi]$ invariant up to a surface term (*cf.* the Action Principle)

$$\mathcal{A}[\phi'(x)] - \mathcal{A}[\phi(x)] = \int d^4x\, \partial_\mu \mathcal{K}_\mu, \quad \phi'(x) \equiv \phi(x) + \delta\phi(x).$$

If we compare this with the general expression

$$\mathcal{A}[\phi'(x)] - \mathcal{A}[\phi(x)] = \int d^4x \left[\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right) \right].$$

and require in addition that ϕ *satisfy* EOM[13], we get a *local conservation law*

$$\partial_\mu J_\mu = 0, \quad J_\mu \equiv \mathcal{K}_\mu - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi. \tag{70}$$

The integration of Eq. (70) over *space* leads to *conserved* charge:

$$\frac{d}{dt} Q = 0, \qquad Q = \int d\mathbf{x} J_0. \tag{71}$$

If $\delta\phi = \rho_i \delta_i \phi$ depends on parameters ρ_i, we have a conservation law for every ρ_i. This is the essence of the *first Noether theorem* [21].

By means of the Noether theorem we can get almost at no cost the expressions for energy-momentum $P_\mu = (\mathcal{H}, \mathbf{P})$ and charge Q, which we used in Section 2.2. For example, P_μ is nothing else but the conserved "charges", which correspond to space-time translations. Indeed, the Noether current in this case is just the energy-momentum tensor $T_{\mu\nu}$

$$\phi'(x+a) = \phi(x), \qquad \text{expand in } a \Rightarrow \delta\phi(x) = -a_\nu \partial_\nu \phi(x), \tag{72}$$

$$\delta\mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \partial_\nu \left(-a_\nu \mathcal{L} \right) \Rightarrow J_\mu = -a_\mu \mathcal{L} + a_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi = a_\nu T_{\mu\nu}. \tag{73}$$

According to Eq. (71), for every a_μ we have $P_\nu = \int d\mathbf{x} T_{0\nu}$, i.e., conserved total energy-momentum. In the same way, we can apply the Noether theorem to phase transformations of our *complex* field and get

$$\phi'(x) = e^{i\alpha}\phi(x), \quad \delta\phi(x) = \; i\alpha\phi(x), \quad J_\mu = i(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger), \quad Q = \int d\mathbf{x} J_0. \tag{74}$$

[13] This requirement is crucial.

The corresponding quantum operators, i.e., $\hat{\mathcal{H}}$ in Eq. (13) or $\hat{Q}$ in Eq. (16), are obtained (modulo ordering issues) from these (classical) expressions by plugging in quantum field $\hat{\phi}$ from Eq. (9).

After quantization the operators corresponding to the *conserved* quantities

– can be used to define a convenient *basis* of states, e.g., we characterize our particle states by eigenvalues of P_μ, and Q:

$$|\mathbf{p}\rangle \equiv |\mathbf{p}, +1\rangle,\ |\bar{\mathbf{p}}\rangle \equiv |\mathbf{p}, -1\rangle \Rightarrow \hat{Q}|\mathbf{p}, q\rangle = q|\mathbf{p}, q\rangle, \hat{\mathbf{P}}|\mathbf{p}, q\rangle = \mathbf{p}|\mathbf{p}, q\rangle. \tag{75}$$

– act as *generators* of symmetries, e.g., for space-time translations we have a *unitary* operator $U(a)$

$$U(a) = \exp\left(i\hat{P}_\mu a_\mu\right), \tag{76}$$

which guarantees that *transition* probabilities between states do not change upon translations. In addition, classical relations between initial and transformed fields become *constraints* on quantum fields, e.g.,

$$\phi'(x+a) = \phi(x) \Rightarrow \hat{\phi}(x+a) = U(a)\hat{\phi}(x)U^\dagger(a). \tag{77}$$

It is worth mentioning that some symmetries can mix fields, e.g.,

$$\phi'_i(x') = S_{ij}(a)\phi_j(x) \Rightarrow \phi_i(x') = S_{ij}(a)U(a)\phi_j(x)U^\dagger(a), \quad x' = x'(x,a). \tag{78}$$

Typical examples are fields with non-zero spin, e.g., vectors and fermions that we discussed in Section 2.

3.2 Local (gauge) symmetries

In this section we revise local symmetries, which play essential role in the construction of *interacting* models. Let us consider the free Dirac Lagrangian

$$\mathcal{L}_0 = \bar{\psi}\left(i\hat{\partial} - m\right)\psi \tag{79}$$

and make the *global* $U(1)$-symmetry of $\mathcal{L}_0$

$$\psi \to \psi' = e^{ie\omega}\psi \tag{80}$$

local, i.e., $\omega \to \omega(x)$. In this case, the Lagrangian ceases to be invariant[14]:

$$\delta\mathcal{L}_0 = \partial_\mu\omega \cdot J_\mu, \qquad J_\mu = -e\bar{\psi}\gamma_\mu\psi. \tag{81}$$

To compensate this term, we *introduce the interaction* of the current J_μ with the photon field A_μ:

$$\mathcal{L}_0 \to \mathcal{L} = \mathcal{L}_0 + A_\mu J_\mu = \bar{\psi}\left[i(\hat{\partial} + ie\hat{A}) - m\right]\psi, \qquad A_\mu \to A'_\mu = A_\mu - \partial_\mu\omega. \tag{82}$$

The photon A_μ is an example of *gauge* field. To get the full QED Lagrangian, we should also add a kinetic term for the photon:

$$\mathcal{L}_{QED} = \bar{\psi}\left(i\hat{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}^2 \tag{83}$$

$$D_\mu = \partial_\mu + ieA_\mu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{84}$$

[14] Note that one can use this fact to get an expression for the Noether current J_μ.

Here we introduce a *covariant* derivative D_μ and a *field-strength* tensor $F_{\mu\nu}$. One can check that Eq. (83) is invariant under

$$\psi \to \psi' = e^{ie\omega(x)}\psi$$
$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\omega$$
$$D_\mu\psi \to D'_\mu\psi' = e^{ie\omega(x)}D_\mu\psi.$$

As a consequence, *gauge principle* forces us to add interactions. But there is price to pay. The *second* Noether theorem [21] states that theories possessing local or *gauge* symmetries are *redundant*, i.e., some degrees of freedom are not physical. This makes quantization non-trivial. To deal with this problem in QED, one usually adds a *gauge-fixing term* to the free vector-field Lagrangian:

$$\mathcal{L}_0(A) = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}\left(\partial_\mu A_\mu\right)^2 \equiv -\frac{1}{2}A_\mu K_{\mu\nu} A_\nu. \tag{85}$$

This term allows one to obtain the photon propagator by inverting $K_{\mu\nu}$:

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{-i\left[g_{\mu\nu} - (1-\xi)p_\mu p_\nu/p^2\right]}{p^2 + i\epsilon}e^{-ip(x-y)} \tag{86}$$

The propagator now involves an auxiliary parameter ξ, and Eq. (26) corresponds to $\xi = 1$ (Feynman gauge). The parameter controls the propagation of *unphysical* longitudinal polarization $\epsilon_\mu^L \propto p_\mu$. The polarization turns out to be harmless in QED since the corresponding terms *drop out* of physical quantities, e.g., due to current conservation

$$e_\mu^L J_\mu \propto p_\mu J_\mu = 0 \quad \text{[we have no source for unphysical } \gamma\text{]}. \tag{87}$$

One can see that the propagator has good UV behaviour and falls down as $1/p^2$ for large p. The gauge symmetry of QED is $U(1)$. It is *Abelian* since the order of two transformations is irrelevant (see Fig. 6).

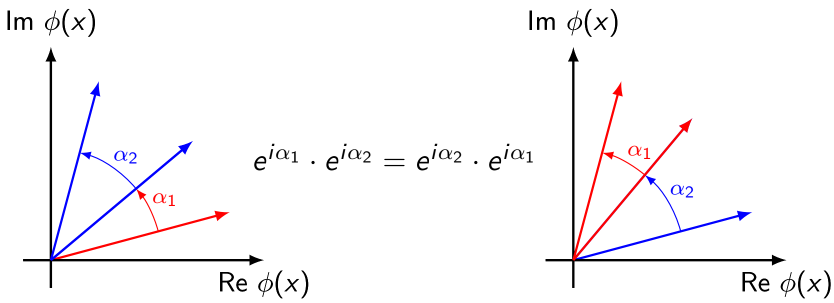


Fig. 6: $U(1)$ transformations commute with each other.

We can generalize $U(1)$ to the *Non-Abelian* case, which is relevant for the EW and QCD interactions. Let us consider the $SU(n)$ group, i.e., unitary $n \times n$ matrices U_{ij} depending on $n^2 - 1$ parameters ω^a and having $\det U = 1$:

$$\psi_i \to \psi'_i = U_{ij}(\omega)\psi_j, \quad U(\omega) = e^{igt^a\omega^a}. \tag{88}$$

In general, different transformations do not commute in the non-Abelian case. This fact is reflected in commutation relations for the group *generators* t^a, which obey the $su(n)$-algebra:

$$[t^a, t^b] = if^{abc}t^c, \qquad f^{abc} - \text{ structure constants }. \tag{89}$$

For *constant* ω^a the transformation, Eq. (88), is a symmetry of the Lagrangian

$$\mathcal{L}_0 = \bar{\psi}_i \left(i\hat{\partial} - m\right) \psi_i, \qquad i = 1, ..., n \tag{90}$$

describing n free fermions in the *fundamental* representation of $SU(n)$.

In order to make the symmetry local, we introduce a (matrix) *covariant derivative* depending on $n^2 - 1$ gauge fields W_μ^a:

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} - igt^a_{ij} W^a_\mu. \tag{91}$$

The transformation properties of W_μ^a should guarantee that for space-time dependent $\omega^a(x)$ the covariant derivative of ψ transforms in the same way as the field itself:

$$D'_\mu \psi' = U(\omega)(D_\mu \psi), \quad U(\omega) = e^{igt^a\omega^a}. \tag{92}$$

One can find that

$$W^a_\mu \to W'^a_\mu = W^a_\mu + \partial_\mu \omega^a + g f^{abc} W^b_\mu \omega^c \tag{93}$$
$$= W^a_\mu + (D_\mu)^{ab}\omega_b, \qquad (D_\mu)^{ab} \equiv \partial_\mu \delta^{ab} - ig(-if^{abc})W^c_\mu, \tag{94}$$

where we introduce the covariant derivative, Eq. (91), D^{ab}_μ with generators $(t^c)^{ab} = -if^{cab}$ in the *adjoint* representation. The field-strength tensor for each component of W^a_μ is given by the commutator

$$[D_\mu, D_\nu] = -igt^a \mathcal{F}^a_{\mu\nu}, \quad \mathcal{F}^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\mu W^a_\nu + g f^{abc} W^a_\mu W^b_\nu. \tag{95}$$

Contrary to the $U(1)$ case, $\mathcal{F}^a_{\mu\nu}$ contains an additional term quadratic in W^a_μ. Due to this, the gauge symmetry predicts not only interactions between fermions ψ (or fields in the fundamental representation of the gauge group) and W^a_μ but also *self-interactions* of the latter (the gauge fields are *"charged"* under the group).

Combining all the ingredients, we can write down the following Lagrangian for an $SU(n)$ gauge (Yang-Mills) theory :

$$\mathcal{L} = \bar{\psi}\left(i\hat{D} - m\right)\psi - \frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^a_{\mu\nu} = \mathcal{L}_0 + \mathcal{L}_I, \tag{96}$$

$$\mathcal{L}_0 = \bar{\psi}\left(i\hat{\partial} - m\right) - \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu, \tag{97}$$

$$\mathcal{L}_I = g\bar{\psi}^i_\alpha \gamma^\mu_{\alpha\beta} t^a_{ij} \psi^j_\beta W^a_\mu - \frac{g}{2} f^{abc} W^b_\mu W^c_\nu F^a_{\mu\nu} - \frac{g^2}{4} f^{abc} f^{ade} W^a_\mu W^b_\nu W^d_\mu W^e_\nu. \tag{98}$$

For illustration purposes we explicitly specify all the indices in the first term of interaction Lagrangian $\mathcal{L}_I$: the Greek ones correspond to Dirac (α, β) and Lorentz (μ) indices, while the Latin ones belong to different representations of $SU(n)$: i, j – fundamental, a – adjoint. One can also see that the strength of all interactions in $\mathcal{L}_I$ is governed by the single dimensionless coupling g.

To quantize a Yang-Mills theory, we generalize the QED gauge-fixing term and write, e.g.,

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}\left(F^a\right)^2, \qquad F^a = \partial_\mu W^a_\mu \tag{99}$$

with F^a being a gauge-fixing function. This again introduces unphysical states in the W^a_μ propagator. However, contrary to the case of QED, the *fermionic* current $J^a_\mu = g\bar{\psi}t^a\gamma_\mu\psi$ is not conserved and can produce longitudinal W^a_μ. Nevertheless, the *structure* of vector-boson self-interactions guarantees that at *tree* level amplitudes, in which *one* of W^a_μ has an unphysical polarization, *vanish* (see, e.g., Fig. 7).

Unfortunately, this is not sufficient to get rid of unphysical states completely. For example, a virtual gauge boson can produce *a pair* of unphysical polarizations. At tree level we, in principle, can avoid

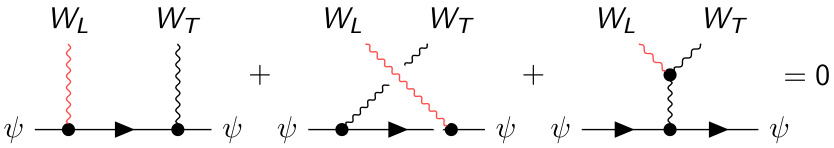


Fig. 7: Gauge symmetry at work: tree-level amplitudes with unphysical polarization (L) vanish.

them by restricting ourselves to physical external states. However, it is hard to control their appearance in loops. To deal with the problem in a *covariant* way, one introduces the so-called *Fadeev-Popov ghosts* $\bar{c}_a$ and c_a. They are *anticommuting* "scalars" and precisely cancel the annoying contribution[15]. The Lagrangian for the fictitious particles is related to the gauge-fixing function $F_a(W_\mu) = \partial_\mu W_\mu^a$ via

$$\begin{aligned}\mathcal{L}_{ghosts} &= -\bar{c}^a \frac{\partial F_a(W^\omega)}{\partial \omega_b} c^b = -\bar{c}^a \partial_\mu D_\mu^{ab} c^b \\ &= -\bar{c}^a \partial^2 c^a - g f^{abc} (\partial_\mu \bar{c}^a) c^b A_\mu^b.\end{aligned} \tag{100}$$

The ghosts are charged under $SU(n)$ and interact with gauge fields in the same way as the unphysical modes. However, there is an additional minus sign for the loops involving anticommuting ghosts (see, e.g., Fig. 8) that leads to the above-mentioned cancellations.

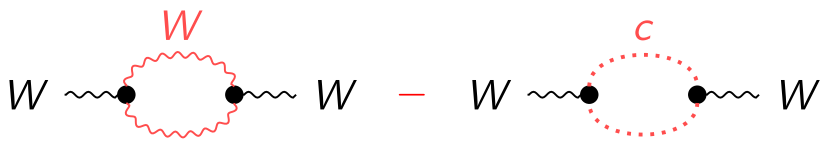


Fig. 8: Ghosts cancel contributions due to virtual unphysical states.

4 Gauge theory of electroweak interactions

4.1 From Fermi theory to the electroweak model

In 1957 R. Marshak and G. Sudarshan, R. Feynman and M. Gell-Mann modified the original Fermi theory of beta-decay to incorporate 100 % violation of parity discovered by C.S. Wu in 1956 :

$$-\mathcal{L}_{\text{Fermi}} = \frac{G_F}{2\sqrt{2}} (J_\mu^+ J_\mu^- + \text{h.c.}). \tag{101}$$

Here the current

$$J_\rho^- = (V-A)_\rho^{\text{nucleons}} + \overline{\Psi}_e \gamma_\rho (1-\gamma_5)\, \Psi_{\nu_e} + \overline{\Psi}_\mu \gamma_\rho (1-\gamma_5)\, \Psi_{\nu_\mu} + ... \tag{102}$$

is the difference between Vector (V) and Axial (A) parts. It is worth mentioning that under parity

$$\begin{aligned} V^0 &\overset{P}{\to} \quad V^0, \qquad & \mathbf{V} &\overset{P}{\to} -\mathbf{V}, \\ A^0 &\overset{P}{\to} -A^0, \qquad & \mathbf{A} &\overset{P}{\to} \quad \mathbf{A}. \end{aligned}$$

[15] In a sense, ghosts also fix the unitary issue in non-Abelian theories: *optical* theorem applied to Feynman diagrams relates imaginary parts of loop integrals to the *squared* matrix elements, which can be obtained by "cutting" loop propagators (see, e.g., Ref. [14] for details).

As a consequence, parity P is conserved for *pure* vector $V_\mu V_\mu$ and axial $A_\mu A_\mu$ interactions, while it is the mixed $A_\mu V_\mu$ terms play a role in parity violation. One can also convince oneself that the charge-conjugation symmetry C is also not respected in this case (see, e.g., a cartoon in Fig. 9). Nevertheless, Eq. (101) conserves combined CP-parity, and it is better not to use the Wu experiment to set the notion of left and right in a phone call with aliens made of antimatter [22].

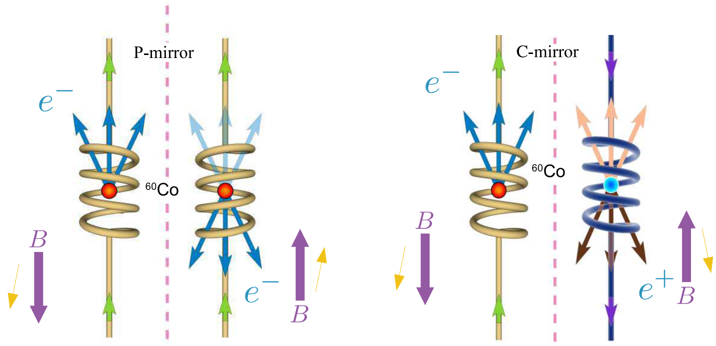


Fig. 9: A cartoon of the Wu experiment and its "distorted" images in P- and C-mirrors. One can see a correlation between the direction of the magnetic field (axial vector) and the direction of motion (polar vector) of the emitted electrons (positrons). The yellow arrows indicate the spin of the (anti) nuclei. The direction of the latter is correlated with that of emitted fermions. Adopted from Wikipedia.

The *current-current* interactions given in Eq. (101) can describe not only the proton beta-decay but also the muon decay $\mu \to e\nu_\mu\bar{\nu}_e$ or the process of $\nu_e e$ - scattering. Since the *Fermi* constant $G_F \simeq 10^{-5}$ GeV^{-1}, from simple *dimensional* grounds we have

$$\sigma(\nu_e e \to \nu_e e) \propto G_F^2 s, \qquad s = (p_e + p_\nu)^2. \tag{103}$$

With such a dependence on energy we eventually *violate unitarity*. This is another manifestation of the fact that non-renormalizable interactions are not self-consistent.

However, a modern view on the Fermi model treats it as an *effective* field theory [19] with certain *limits of applicability*. It perfectly describes low-energy experiments and one can fit the value of G_F very precisely (see Ref. [23]). The *magnitude* of G_F tells us something about a *more fundamental* theory (the SM in our case): around $G_F^{-1/2} \sim 10^2 - 10^3$ GeV there should be some "new physics" (NP) to cure the above-mentioned shortcomings. Indeed, by analogy with (renormalizable) QED we can introduce *mediators* of the weak interactions – electrically charged *vector* fields $W_\mu^\pm$:

$$\mathcal{L}_{\text{Fermi}} = -\frac{G_F}{2\sqrt{2}}(J_\mu^+ J_\mu^- + \text{h.c.}) \to \mathcal{L}_I = \frac{g}{2\sqrt{2}}(W_\mu^+ J_\mu^- + \text{h.c.}) \tag{104}$$

with a *dimensionless* coupling g. Since we know that weak interactions are *short-range*, the W-bosons should be *massive*. Given $\mathcal{L}_I$ we can calculate the tree-level scattering amplitude due to the exchange of $W^\pm$ between two fermionic currents:

$$T = i(2\pi)^4 \frac{g^2}{8} J_\alpha^+ \left[\frac{g_{\alpha\beta} - p_\alpha p_\beta / M_W^2}{p^2 - M_W^2}\right] J_\beta^- . \tag{105}$$

In the limit $|p| \ll M_W$, Eq. (105) reproduces the prediction of the effective theory (Fermi model) if we identify ("match")

$$\text{(effective theory)} \quad \frac{G_F}{\sqrt{2}} = \frac{g^2}{8{M_W}^2} \quad \text{(more fundamental theory)}. \tag{106}$$

At this point, it is good idea to compare the *chirality* structure of the W-coupling to fermions with that of the photon γ. In QED, the γ-fermion-fermion vertex conserves *chirality* and treats ψ_L and ψ_R on equal footing:

$$\mathcal{L}_I \ni -eA_\mu \cdot \bar{\psi}\gamma_\mu\psi = -eA_\mu\left[\bar{\psi}_L\gamma_\mu\psi_L + \bar{\psi}_R\gamma_\mu\psi_R + \cancel{\bar{\psi}_L\gamma_\mu\psi_R} + \cancel{\bar{\psi}_R\gamma_\mu\psi_L}\right].$$

As a consequence, in the high-energy limit ($m \to 0$) we have two helicity combinations, both for electrons and positrons, that give a non-zero amplitude. The weak vertex with W also conserves chirality but, due to postulated parity violation, involves only ψ_L

$$\mathcal{L}_I \ni -\frac{g}{2\sqrt{2}}W_\mu^+ \cdot \bar{\psi}_e\gamma_\mu(1-\gamma_5)\psi_{\nu_e} + \text{h.c.} = -\frac{g}{\sqrt{2}}W_\mu^+\left[\bar{\psi}_{eL}\gamma_\mu\psi_{\nu L}\right] + \text{h.c.},$$

and, thus, only one *helicity* combination take part in the ultra-relativistic processes involving W-bosons (see Fig. 10).

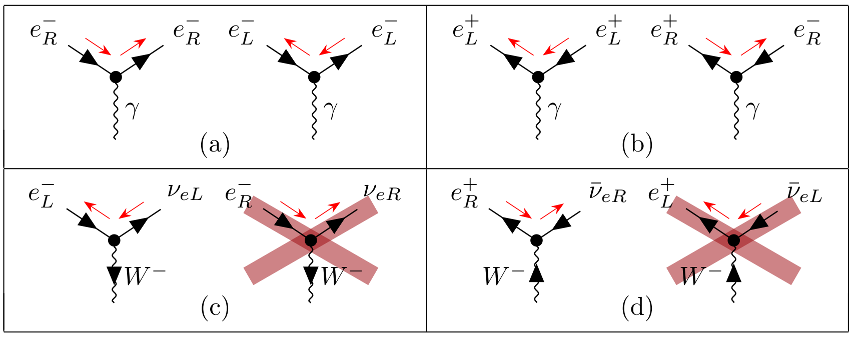


Fig. 10: Non-zero *helicity* combinations for electrons (a) and positrons (b) coupled to the photon in *massless* QED. In the case of W boson, only left-handed electrons (c) and right-handed positrons (d) contribute, if masses of the fermions are neglected. The red arrows represent helicities.

One interesting phenomenological consequence of the peculiar nature of the weak vertices is that it can be used to probe the (anti)quark content of the proton in Deep Inelastic Scattering (DIS) of (anti)neutrino. Indeed, let us consider a high-energy ($30\,\text{GeV} \lesssim E_\nu \lesssim 350\,\text{GeV}$) *muon* antineutrino produced in an accelerator-based beam. It can give an antimuon in the *charged-current* scattering either over the u quark, or over the $\bar{d}$ antiquark. Moreover, in the considered limit, the u quark should be *left-handed*, while $\bar{d}$ should be *right-handed*. The outgoing antimuon is also *right-handed* and to conserve helicity, the antineutrino cross-sections have the following form (we neglect the momentum transfer in the W-propagator, or, equivalently, use effective, Eq. (101), theory):

$$\frac{d\sigma_{\bar{\nu}q}}{d\Omega^*} = \frac{G_F s}{4\pi^2}\left(\frac{1+\cos\theta^*}{2}\right)^2, \qquad \sigma^{\bar{\nu}q} = \frac{G_F s}{3\pi}, \tag{107}$$

$$\frac{d\sigma_{\bar{\nu}\bar{q}}}{d\Omega^*} = \frac{G_F s}{4\pi^2}, \qquad \sigma^{\bar{\nu}\bar{q}} = \frac{G_F s}{\pi}. \tag{108}$$

Here θ^* is the scattering angle in the center-of-mass frame. Analogously, for the *left-handed* neutrino (see, Fig. 11), only incoming *left-handed* d or *right-handed* $\bar{u}$ can give a non-zero cross-section in the ultra-relativistic limit, so

$$\sigma^{\nu\bar{q}} = \frac{G_F s}{3\pi}, \qquad \sigma^{\nu q} = \frac{G_F s}{\pi}. \tag{109}$$

In the parton model the neutrino DIS over proton and neutron can be described by

$$\sigma_{\nu p} = \frac{G_F^2 s}{\pi}\left[f_d + \frac{1}{3}f_{\bar{u}}\right], \quad \sigma_{\nu n} = \frac{G_F^2 s}{\pi}\left[f_u + \frac{1}{3}f_{\bar{d}}\right], \tag{110}$$

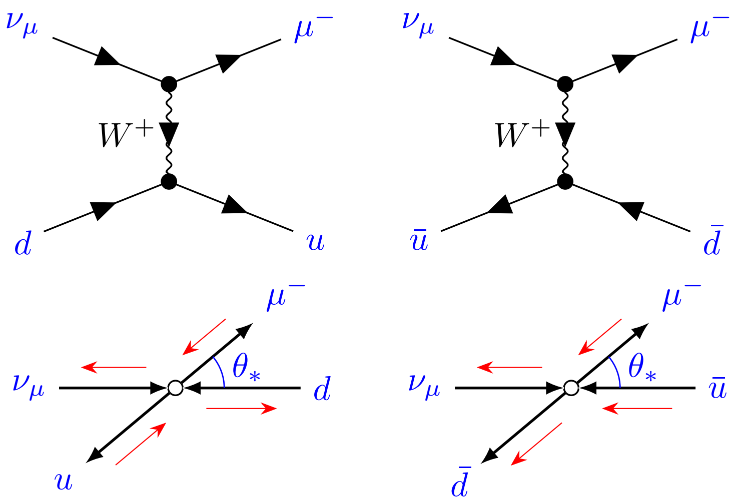


Fig. 11: Neutrino scattering on quarks and antiquarks. Red arrows represent helicity of the particles.

where $f_q = \int\limits_0^1 xq(x)dx$ corresponds to the fraction of *proton* momentum carried by the quark q, and we assumed that $f_u(\text{proton}) = f_d(\text{neutron})$, etc. For an isoscalar target that have equal number of protons and neutrons, there is an equal probability to scatter either on p or n, so averaging over these possibilities gives

$$\sigma_{\nu N} = \frac{1}{2}\left[\sigma_{\nu p} + \sigma_{\nu n}\right] = \frac{G_F^2 s}{2\pi}\left[f_q + \frac{1}{3}f_{\bar{q}}\right], f_q = f_d + f_u,$$
$$\sigma_{\bar{\nu} N} = \frac{G_F^2 s}{2\pi}\left[f_{\bar{q}} + \frac{1}{3}f_q\right]. \quad (111)$$

One consequence of Eq. (111) is that the experimentally measured ratio

$$\frac{\sigma_{\nu N}}{\sigma_{\bar{\nu} N}} = \frac{3f_q + f_{\bar{q}}}{f_q + 3f_{\bar{q}}} = 1.984 \pm 0.012 \quad (112)$$

probes the *antiquark* $\bar{q}$ content of the proton, and indicates that antiquarks carry a non-zero fraction of the proton momentum $f_{\bar{q}} \simeq 0.08$.

4.2 The electroweak gauge bosons in the Standard Model

One can see that by construction $W^\pm$ is electrically charged, and interact with fermions and photons. Due to this, we can consider the W-pair production process ($e^+e^- \to W^+W^-$) at a lepton-antilepton collider (e.g., Large Electron-Positron (LEP) collider at CERN). We pretend to know nothing about the Z boson, so only two diagrams contribute in our theory (see first two graphs in Fig. 16). It turns out that in this case the predicted cross-section for the *longitudinal* W-bosons *increases* with center-of-mass energy s and, again, eventually violates unitarity.

In addition, the W-boson propagator, Eq. (24), behaves rather badly in the UV region (due to the $p_\mu p_\nu/m^2$ term in the numerator) and in loops can lead to severe UV divergencies. To deal with these issues in the SM, we associate a *gauge* symmetry with $W^\pm$, much like we do with photon. It turns out that to introduce EW interactions we need to utilize the

$$SU(2)_L \otimes U(1)_Y \quad (113)$$

gauge group that has four generators or, equivalently, four gauge bosons. Three of them, W_μ, belong to *weak-isospin* $SU(2)_L$, while the photon-like B_μ mediates *weak-hypercharge* $U(1)_Y$ interactions. The

SM fermions are charged under the group described in Eq. (113). To account for the $(V-A)$ pattern only *left* fermions interact with W_μ and form $SU(2)_L$ doublets:

$$L = \begin{pmatrix} \nu_l \\ l^- \end{pmatrix}_L, Q = \begin{pmatrix} q_u \\ q_d \end{pmatrix}_L, \qquad q_u = u, c, t;\ q_d = d, s, b;\ l = e, \mu, \tau. \tag{114}$$

Since the generators of $SU(2)$ are just the Pauli matrices, we immediately write the following expression for the corresponding covariant derivative

$$D^L_\mu = \begin{pmatrix} \partial_\mu - \frac{i}{2}\left(gW^3_\mu + g'Y^f_L B_\mu\right) & -i\frac{g}{\sqrt{2}}W^+_\mu \\ -i\frac{g}{\sqrt{2}}W^-_\mu & \partial_\mu + \frac{i}{2}\left(gW^3_\mu - g'Y^f_L B_\mu\right) \end{pmatrix}. \tag{115}$$

The *right* fermions[16] are $SU(2)_L$ singlets and do not couple to W_μ:

$$D^R_\mu = \partial_\mu - ig'\frac{Y^f_R}{2}B_\mu. \tag{116}$$

The covariant derivatives involve two gauge couplings g, g' corresponding to $SU(2)_L$ and $U(1)_Y$, respectively. Different $Y^f_{L/R}$ denote weak hypercharges of the fermions and up to now the values are not fixed. Let us put some constraints on $Y^f_{L/R}$. The first restriction comes from the $SU(2)_L$ symmetry, i.e., $Y^u_L = Y^d_L \equiv Y^Q_L$, and $Y^\nu_L = Y^e_L \equiv Y^L_l$.

One can see that the EW interaction Lagrangian

$$\mathcal{L}_W = \mathcal{L}_{NC} + \mathcal{L}_{CC}, \tag{117}$$

in addition to the *charged-current* interactions of the form

$$\mathcal{L}^l_{CC} = \frac{g}{\sqrt{2}}\bar{\nu}^e_L\gamma_\mu W^+_\mu e_L + \text{h.c.} = \frac{g}{2\sqrt{2}}\bar{\nu}_e\gamma_\mu W^+_\mu\left(1-\gamma_5\right)e + \text{h.c.} \tag{118}$$

also involves *neutral-current* interactions

$$\mathcal{L}^l_{NC} = \bar{\nu}^e_L\gamma_\mu\left(\frac{1}{2}gW^3_\mu + \frac{Y^l_L}{2}g'B_\mu\right)\nu^e_L + \bar{e}_L\gamma_\mu\left(-\frac{1}{2}gW^3_\mu + \frac{Y^l_L}{2}g'B_\mu\right)e_L + g'\bar{e}_R\gamma_\mu\frac{Y^e_R}{2}B_\mu e_R. \tag{119}$$

It is obvious that we have to account for QED in the SM and should predict a photon field that couples to fermions with the correct values of the electric charges. Since both W^3_μ and B_μ are *electrically neutral*, they can mix

$$\begin{aligned} W^3_\mu &= Z_\mu\cos\theta_W + A_\mu\sin\theta_W \\ B_\mu &= -Z_\mu\sin\theta_W + A_\mu\cos\theta_W. \end{aligned} \tag{120}$$

Here we introduce the *Weinberg* angle θ_W. One can try to fix $\sin\theta_W$ and various $Y^f_{L/R}$ from the requirement that, e.g., A_μ has the same interactions as the photon in QED. Indeed, given fermion *electric* charges Q_f (see Table 3) in the units of the elementary charge e, one can derive the following relations:

$$\begin{aligned} g\sin\theta_W &= e(Q_\nu - Q_e) = e(Q_u - Q_d), \\ g'Y^l_L\cos\theta_W &= e(Q_\nu + Q_e) = -e, \\ g'Y^Q_L\cos\theta_W &= e(Q_u + Q_d) = \frac{1}{3}e, \end{aligned}$$

[16] In what follows we do not consider right-handed neutrino and refer again to Ref. [20].

$$g'Y_R^f \cos\theta_W = 2eQ_f, \qquad f = e,\, u,\, d. \tag{121}$$

As a consequence, $e = g\sin\theta_W$ and, e.g., $e = 3g'Y_L^Q\cos\theta_W$, so that

$$Y_L^l = -3Y_L^Q, \quad Y_R^e = -6Y_L^Q, \quad Y_R^u = 4Y_L^Q, \quad Y_R^d = -2Y_L^Q \tag{122}$$

are fixed in terms of one (arbitrary chosen) Y_L^Q. It is convenient to normalize the $U(1)_Y$ coupling g' so that $e = g'\cos\theta_W$, so $Y_L^Q = 1/3$. As a consequence, the photon field couples to the electric charge Q_f of a fermion f. The latter is related to the weak hypercharge and the third component of weak isospin T_3^f via the Gell-Mann–Nishijima formula:

$$\mathcal{L}_{NC} \ni \bar{f}\left[\left(gT_3^f\sin\theta_W + g'\frac{Y_f^L}{2}\cos\theta_W\right)P_L + \left(g'\frac{Y_f^R}{2}\cos\theta_W\right)P_R\right]\gamma_\mu f A_\mu \tag{123}$$

$$= e\bar{f}\left(T_3 + \frac{Y}{2}\right)\gamma_\mu f A_\mu = eQ_f\bar{f}\gamma_\mu f A_\mu, \tag{124}$$

where in Eq. (124) we assume that T_3 and Y are operators, which give T_3^f and Y_L^f, when acting on left components, and $T_3^f = 0$ and $Y_R^f = 2Q_f$ for right fermions.

The relations, Eq. (122), allow one to rewrite the neutral-current Lagrangian as

$$\mathcal{L}_{NC} = eJ_\mu^A A^\mu + g_Z J_\mu^Z Z_\mu, \qquad g_Z = \frac{g}{\cos\theta_W}, \tag{125}$$

where the photon A_μ and a new Z-boson couple to the currents of the form

$$J_\mu^A = \sum_f Q_f\bar{f}\gamma_\mu f, \qquad J_\mu^Z = \sum_f \bar{f}\left(c_L^f P_L + c_R^f P_R\right)f = \frac{1}{4}\sum_f \bar{f}\gamma_\mu\left(v_f - a_f\gamma_5\right)f \tag{126}$$

$$c_L^f = T_3^f - Q_f\sin^2\theta_W, \quad c_R^f = -Q_f\sin^2\theta_W, \quad v_f = 2T_3^f - 4Q_f\sin^2\theta_W, \quad a_f = 2T_3^f. \tag{127}$$

Here $T_3^f = \pm\frac{1}{2}$ for left up-type/down-type fermions. For example, in the case of u-quarks, $Q_u = 2/3$, $T_3^u = 1/2$, so

$$v_u = 1 - \frac{8}{3}\sin^2\theta_W, \qquad a_u = 1. \tag{128}$$

In Table 3, we summarize the SM fermion charges and the Z-boson couplings for $\sin^2\theta_W \simeq 0.23$.

Table 3: The values of the electric charge Q_f, the weak isospin T_3^f (for left particles), and the hypercharge for left Y_L^f and right Y_R^f SM fermions f. The Z-bosons coupling parameters c_L^f and c_R^f from Eq. (127) are also provided for $\sin^2\theta_W \simeq 0.23$.

fermion	Q_f	T_3^f	Y_L^f	Y_R^f	c_L^f	c_R^f
ν_l	0	$+\frac{1}{2}$	-1	0	$+\frac{1}{2}$	0
l^-	-1	$-\frac{1}{2}$	-1	-2	-0.27	$+0.23$
u	$+\frac{2}{3}$	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{4}{3}$	$+0.35$	-0.15
d	$-\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{2}{3}$	-0.42	$+0.08$

For completeness, let us give the expression for the charged-current interactions in the EW model

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}}\left(J_\mu^+ W^{+\mu} + J_\mu^- W^{-\mu}\right), \quad J_\mu^+ = \frac{1}{2}\sum_f \bar{f}_u\gamma_\mu\left(1 - \gamma_5\right)f_d, \tag{129}$$

where $f_u(f_d)$ is the up-type (down-type) component of an $SU(2)_L$ doublet f. The corresponding interaction vertices are given in Fig. 12. It is worth emphasizing that in the SM the couplings between fermions and gauge bosons exhibit *universality*.

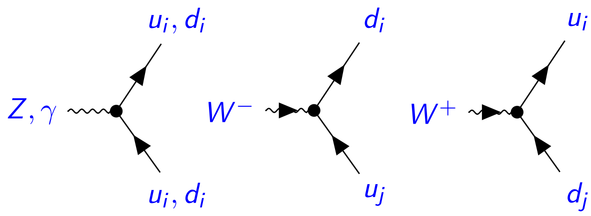


Fig. 12: Gauge-boson–quark vertices. Leptons interact with the EW bosons in the same way.

It turns out that it was *a prediction* of the electroweak SM that there should be an additional neutral gauge boson Z_μ. Contrary to the photon, the Z-boson also interacts with neutrinos. This crucial property was used in the experiment called *Gargamelle* at CERN, which presented the discovery in 1973 (Fig. 13). About ten years later both W and Z were directly produced at Super Proton Synchrotron (SPS) at CERN. Finally, in the early 90s a comprehensive analysis of the $e^+e^- \to f\bar{f}$ process, which was carried out at LEP, CERN, and at the Standford Linear Collider (SLC), SLAC, confirmed the SM predictions for the Z couplings to fermions, see Eg. (127).

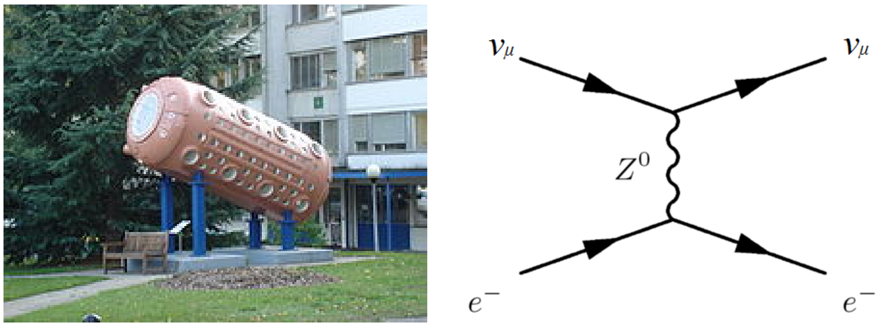


Fig. 13: The chamber of Gargamelle at CERN (left), ν_μ scattering due to Z-boson (right). From Wikipedia.

It is also worth mentioning the fact that the (hyper)-charge assignment, Eq. (122), satisfies very non-trivial constraints related to cancellation of *gauge anomalies*. Anomalies correspond to situations when a symmetry of the classical Lagrangian is violated at the quantum level. A well-known example is *Axial or Chiral or Adler–Bell–Jackiw(ABJ)* anomaly when the classical conservation law for the axial current J_μ^A is modified due to quantum effects:

$$J_\mu^A = \bar{\Psi}\gamma_\mu\gamma_5\Psi, \qquad \partial_\mu J_\mu^A = 2im\Psi\gamma_5\Psi + \underbrace{\frac{\alpha}{2\pi}F_{\mu\nu}\tilde{F}_{\mu\nu}}_{\text{anomaly}}, \qquad \tilde{F}_{\mu\nu} = 1/2\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{130}$$

The $F\tilde{F}$-term appears due to loop diagrams presented in Fig. 14.

There is nothing wrong when the anomalous current J_μ^A corresponds to a global symmetry and does not enter into $\mathcal{L}$. It just implies that a classically forbidden processes may actually occur in the quantum theory. For example, it is the anomaly in the *global* axial *flavour* symmetry that is responsible for the decay $\pi \to \gamma\gamma$. On the contrary, if an axial current couples to a gauge field, anomalies break gauge invariance, thus rendering the corresponding QFT inconsistent. In the SM left and right fermions (eigenvectors of γ_5) have different $SU(2)_L \times U(1)_Y$ quantum numbers, leaving space for potential

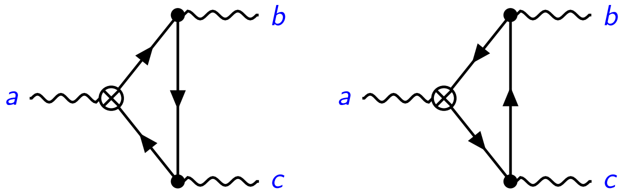


Fig. 14: Diagrams contributing to the anomaly of an axial current (crossed vertex).

anomalies. However, since we have to take into account all fermions which couple to a gauge field, there is a possibility that contributions from different species cancel each other due to a special assignment of fermion charges. Indeed, in the case of chiral[17] theories, anomalies are proportional to ($\gamma_5 = P_R - P_L$)

$$\text{Anom} \propto \text{Tr}[t^a, \{t^b, t^c\}]_L - \text{Tr}[t^a, \{t^b, t^c\}]_R, \tag{131}$$

where t^a are generators of the considered symmetries and the traces are over left (L) or right (R) fields. In the SM the requirement that all anomalies should be zero imposes the following conditions on fermion hypercharges:

$$0 = 2Y_L^Q - Y_R^u - Y_R^d, \qquad U(1)_Y - SU(3)_c - SU(3)_c, \tag{132a}$$
$$0 = N_c Y_L^Q + Y_L^l, \qquad U(1)_Y - SU(2)_L - SU(2)_L, \tag{132b}$$
$$0 = N_c \left[2(Y_L^Q)^3 - (Y_R^u)^3 - (Y_R^d)^3\right] + \left[2(Y_L^l)^3 - (Y_R^e)^3\right], \qquad U(1)_Y - U(1)_Y - U(1)_Y, \tag{132c}$$
$$0 = N_c \left[2Y_L^Q - Y_R^u - Y_R^d\right] + \left[2Y_L^l - Y_R^e\right], \qquad U(1)_Y - grav. - grav., \tag{132d}$$

where, in addition to the EW gauge group, we also consider strong interactions of quarks that have $N_c = 3$ colours. While the first three conditions come from the SM interactions, the last one, Eq. (132d) is due to the coupling to gravity. Other anomalies are trivially zero. One can see that the hypercharges introduced in Eq. (122) do satisfy the equations. It is interesting to note that contributions due to colour quarks miraculously cancel those of leptons and the cancellation works within a single generation. This put a rather strong restriction on possible new fermions that can couple to the SM gauge bosons: new particles should appear in a complete generation (quarks + leptons) in order not to spoil anomaly cancellation within the SM. Moreover, the anomaly cancellation condition can select viable models that go beyond the SM (BSM).

Due to the non-Abelian nature of the $SU(2)_L$ group, the gauge fields W_i have triple and quartic self-interactions (see Eq. (98)). Since W_3 is a linear combination of the Z-boson and photon, the same is true for Z and γ. In Fig. 15, self-interaction vertices for the EW gauge bosons are depicted. The triple vertex WWZ predicted by the SM allows one to cure the bad behavior of the $e^+e^- \to W^+W^-$

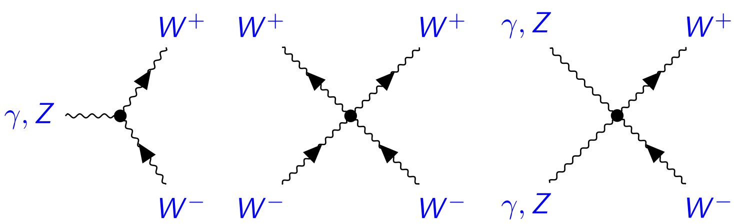


Fig. 15: Gauge-boson self-interaction vertices.

[17] that distinguish left and right fermions

cross-section, which we discussed in the beginning of the Section. Moreover, the coupling was tested experimentally at LEP2 (Fig. 16) and agreement with the SM predictions was found. Subsequent studies at hadron colliders (Tevatron and LHC) aimed at both quartic and triple gauge couplings (QGC and TGC, respectively) also show consistency with the SM and put limits on possible deviations (so-called anomalous TGC and QGC).

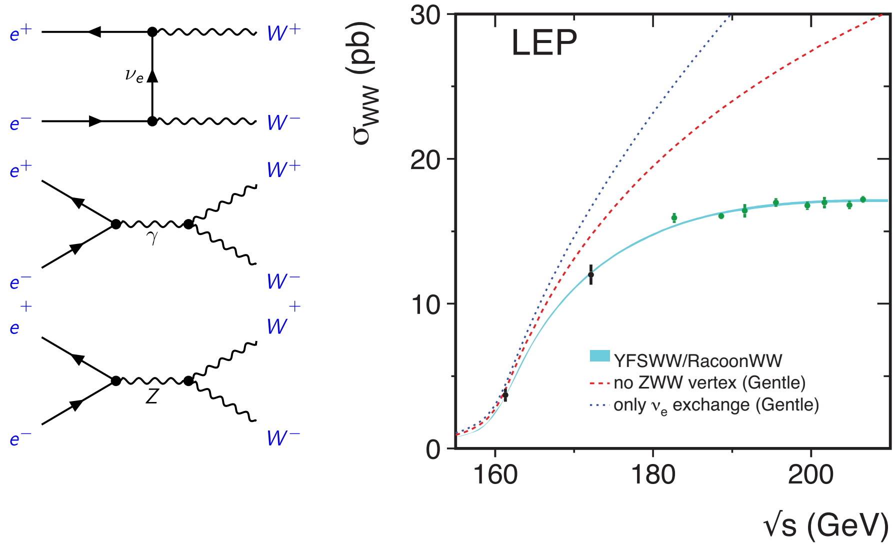


Fig. 16: $e^+e^- \to W^+W^-$.

Since we do not observe Z-bosons flying around like photons, Z_μ should have a non-zero mass M_Z and similar to $W^\pm$ give rise to Fermi-like interactions between *neutral* currents J^μ_Z at low energies. The relative strength of the *charged* and *neutral* current-current interactions $(J^Z_\mu J^\mu_Z)/(J^{+\mu} J^+_\mu)$ can be measured by the parameter ρ:

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2\theta_W}. \tag{133}$$

Up to now, we do not specify any relations between M_Z and M_W. Due to this, the value of ρ can, in principle, be arbitrary. However, it is a prediction of the full SM that $\rho \simeq 1$ (see below).

The fact that both W and Z should be massive poses a serious problem for theoretical description of the EW interactions. The naive introduction of the corresponding mass terms breaks the *gauge* symmetry, see Eq. (113). For example, $m_W^2 W^+_\mu W^-_\mu$ is forbidden due to $W_\mu \to W_\mu + \partial_\mu \omega + \ldots$. One can also mention an issue with unitarity, which arises in the scattering of longitudinal EW bosons due to gauge self-interactions in Fig. 15.

In addition, the symmetry also forbids *explicit* mass terms for fermions, since e.g., $m_\mu(\bar\mu_L \mu_R + \text{h.c.})$, which accounts for muon mass, mixes left and right fields that transform differently under the electroweak group, see Eq. (113). In the next section, we discuss how these problems can be solved by coupling the SM fermions and gauge bosons to the scalar (Higgs) sector (see also Ref. [24]).

4.3 Spontaneous symmetry breaking and hidden symmetry

We need to *generate* masses for $W^\pm_\mu$ and Z_μ (but not for A_μ) without *explicit* breaking of the gauge symmetry. Let us consider for simplicity *scalar* electrodynamics:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial_\mu \phi - V(\phi^\dagger\phi) - \frac{1}{4}F_{\mu\nu}^2 + ie\left(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger\right) A_\mu + e^2 A_\mu A_\mu \phi^\dagger \phi \equiv \mathcal{L}_1, \tag{134}$$

which is invariant under $U(1)$

$$\phi \to e^{ie\omega(x)}\phi, \quad A_\mu \to A_\mu + \partial_\mu \omega. \tag{135}$$

In Eq. (134) a *complex* scalar ϕ interacts with the photon A_μ. We can use *polar* coordinates to rewrite the Lagrangian in terms of new variables

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{e^2\rho^2}{2}\left(A_\mu - \frac{1}{e}\partial_\mu\theta\right)\left(A_\mu - \frac{1}{e}\partial_\mu\theta\right) - V(\rho^2/2) - \frac{1}{4}F_{\mu\nu}^2, \tag{136}$$

$$= \frac{1}{2}(\partial_\mu \rho)^2 + \frac{e^2\rho^2}{2}B_\mu B_\mu - V(\rho^2/2) - \frac{1}{4}F_{\mu\nu}^2(B), \tag{137}$$

where ρ is gauge invariant, while the $U(1)$ transformation (135) gives rise to a *shift* in θ:

$$\phi = \frac{1}{\sqrt{2}}\rho(x)e^{i\theta(x)}, \quad \rho \to \rho, \quad \theta \to \theta + e\omega. \tag{138}$$

One can also notice that $B_\mu \equiv A_\mu - \frac{1}{e}\partial_\mu\theta$ is also invariant! Moreover, since $F_{\mu\nu}(A) = F_{\mu\nu}(B)$, we can completely get rid of θ. As a consequence, the gauge symmetry becomes "hidden" when the system is described by the variables $B_\mu(x)$ and $\rho(x)$.

If in Eq. (134) we replace our *dynamical* field $\rho(x)$ by a constant $\rho \to v = \text{const}$, we get the mass term for B_μ. This can be achieved by considering the potential $V(\phi)$ of the form (written in terms of initial variables)

$$V = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2. \tag{139}$$

One can distinguish two different situations (see Fig. 17):

- $\mu^2 > 0$ — a *single* minimum with $\phi = 0$;
- $\mu^2 < 0$ — a valley of *degenerate* minima with $\phi \neq 0$.

In both cases we solve EOM for the homogeneous (in space and time) field. When $\mu^2 > 0$ the *solution* is unique and symmetric, i.e., it does not transform under $U(1)$. In the second case, in which we are interested here, the potential has non-trivial minima

$$\left.\frac{\partial V}{\partial \phi^\dagger}\right|_{\phi=\phi_0} = 0 \Rightarrow \phi_0^\dagger\phi_0 = -\frac{\mu^2}{2\lambda} = \frac{v^2}{2} > 0 \Rightarrow \phi_0 = \frac{v}{\sqrt{2}}e^{i\beta}, \tag{140}$$

which are *related* by *global* $U(1)$ transformations, Eq. (135), that change $\beta \to \beta + e\omega$. So, in spite of the fact that we do not break the symmetry *explicitly*, it is *spontaneously broken* (SSB) due to a particular choice of our solution (β).

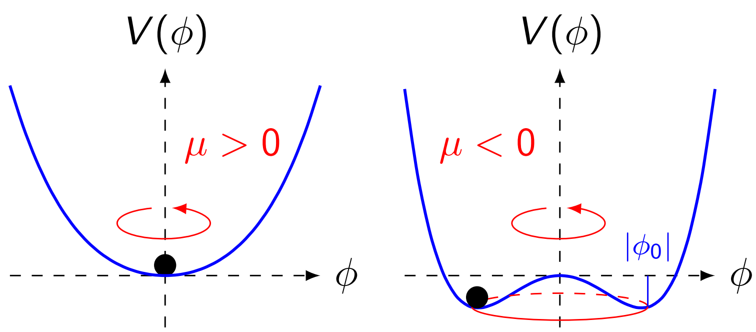


Fig. 17: A symmetric vacuum (left) and degenerate vacua (right).

In QFT we interpret ϕ_0 as a characteristic of our *vacuum* state, i.e., as a *vacuum expectation value* (VEV) or *condensate* of the quantum field:

$$\phi_0 = \langle 0|\phi(x)|0\rangle \overset{\beta=0}{=} \frac{v}{\sqrt{2}}. \tag{141}$$

Since we want to introduce particles as *excitations* above the vacuum, we have to shift the field:

$$\phi(x) = \frac{v+h(x)}{\sqrt{2}} e^{i\zeta(x)/v}, \qquad \langle 0|h(x)|0\rangle = 0, \quad \langle 0|\zeta(x)|0\rangle = 0. \tag{142}$$

As a consequence, Eq. (137) can be rewritten as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 + \frac{e^2 v^2}{2}\left(1+\frac{h}{v}\right)^2 B_\mu B_\mu - V(h) - \frac{1}{4}F^2_{\mu\nu}(B) \equiv \mathcal{L}_2, \tag{143}$$

$$V(h) = -\frac{|\mu|^2}{2}(v+h)^2 + \frac{\lambda}{4}(v+h)^4 = \frac{2\lambda v^2}{2}h^2 + \lambda v h^3 + \frac{\lambda}{4}h^4 - \frac{\lambda}{4}v^4. \tag{144}$$

The Lagrangian, Eq. (144), describes a massive vector field B_μ with $m_B^2 = e^2v^2$ and a massive scalar h with $m_h^2 = 2\lambda v^2$. We do not break the symmetry explicitly. It is again *hidden* in the relations between couplings and masses. This is the essence of the *Brout-Englert-Higgs-Hagen-Guralnik-Kibble* mechanism [25–27].

The Lagrangians $\mathcal{L}_1$, Eq. (134), and $\mathcal{L}_2$, Eq. (144), describe the same physics but written in terms of different quantities (variables). Eq. (134) involves a *complex* scalar ϕ with 2 (real) degrees of freedom (DOFs) and a *massless* gauge field (A_μ) also having 2 DOFs. It is manifestly gauge invariant but not suitable for perturbative expansion (ϕ has imaginary mass).

On the contrary, in $\mathcal{L}_2$ the gauge symmetry is hidden[18] and it is written in terms of *physical* DOFs, i.e., a *real* scalar h (1 DOF) and a *massive* vector B_μ (3 DOFs). In a sense, one *scalar* DOF (ζ) is "eaten" by the gauge field to become massive. It is important to note that the postulated *gauge* symmetry allows us to avoid the consequences of the *Goldstone* theorem, which states that if the vacuum breaks a *global* continuous symmetry there is a *massless* boson (Nambu-Goldstone) in the spectrum[19]. This boson is associated with 'oscillations" along the valley, i.e., in the *broken* direction (see Fig. 17). However, due to the local character of symmetry, χ is not physical anymore, its disappearance (or appearance, see below) reflects the *redundancy*, which was mentioned above.

In Section 2.6, we indicated that the massive-vector propagator has rather bad UV behavior and is not very convenient for doing calculations in PT. It looks like we gain nothing from the gauge principle. But it is not true. We can write the model Lagrangian in the *Cartesian* coordinates $\phi = \frac{1}{\sqrt{2}}(v+\eta+i\chi)$:

$$\mathcal{L}_3 = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{e^2v^2}{2}A_\mu A_\mu + \frac{1}{2}\partial_\mu\chi\,\partial_\mu\chi - \underline{evA_\mu\partial_\mu\chi} + \frac{1}{2}\partial_\mu\eta\,\partial_\mu\eta - \frac{2v^2\lambda}{2}\eta^2 + \frac{v^4\lambda}{4} \tag{145}$$

$$+ eA_\mu\chi\partial_\mu\eta - eA_\mu\eta\partial_\mu\chi - v\lambda\eta(\eta^2+\chi^2) - \frac{\lambda}{4}(\eta^2+\chi^2)^2 + \frac{e^2}{2}A_\mu A_\mu(2v\eta+\eta^2+\chi^2). \tag{146}$$

The "free" part, Eq. (145), of $\mathcal{L}_3$ seems to describe 5 real DOFs: a massive scalar η, a *massless* (would-be *Nambu-Goldstone*) boson χ and a massive A_μ. However, there is a mixing between the *longitudinal* component of A_μ and χ that spoils this naive counting (unphysical χ is "partially eaten" by A_μ) .

In spite of this subtlety, $\mathcal{L}_3$ is more convenient for calculations in PT. To quantize the model, one can utilize the gauge-fixing freedom and add the following expression to $\mathcal{L}_3$

$$\delta\mathcal{L}_{g.f.} = -\frac{1}{2\xi}(\partial_\mu A_\mu + ev\xi\chi)^2 = -\frac{1}{2\xi}(\partial_\mu A_\mu)^2 - \underline{ev\chi\partial_\mu A_\mu} - \frac{e^2v^2\xi}{2}\chi^2. \tag{147}$$

[18] One can also say that $\mathcal{L}_2$ corresponds to the *unitary* gauge, i.e., no unphysical "states" in the particle spectrum.

[19] Any non-derivative interactions violate the shift symmetry $\zeta \to \zeta + ev\omega$ for $\omega = \text{const}$

It removes the mixing from Eq. (145) and introduces a mass for χ, $m_\chi^2 = (e^2v^2)\xi$. In addition, the vector-boson propagator in this case looks like

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{-i\left[g_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2-\xi m_A^2}\right]}{p^2-m_A^2+i\epsilon}e^{-ip(x-y)}, \quad m_A = ev. \tag{148}$$

One can see that for $\xi\to\infty$ we reproduce Eq. (24), while for finite ξ the propagator behaves like $1/p^2$ as $p\to\infty$, thus making it convenient for PT calculations.

It should be mentioned that contrary to $\mathcal{L}_2$ the full Lagrangian corresponding to $\mathcal{L}_3$ involves also unphysical *ghosts*, which do not decouple in the considered case. Nevertheless, it is a relatively small price to pay for the ability to perform high-order calculations required to obtain high-precision predictions.

Let us switch back to the SM. We have three gauge bosons that should become massive. According to our reasoning, three symmetries should be broken by the SM vacuum to feed hungry $W_\mu^\pm$ and Z_μ with (would-be) Goldstone bosons

$$SU(2)_L\times U(1)_Y\to U(1)_{em}. \tag{149}$$

The photon should remain massless and correspond to the unbroken electromagnetic $U(1)_{em}$. This can be achieved by considering an $SU(2)_L$ doublet of scalar fields:

$$\Phi = \frac{1}{\sqrt{2}}\exp\left(i\frac{\zeta_j(x)\sigma^j}{2v}\right)\begin{pmatrix}0\\ v+h(x)\end{pmatrix}, \qquad \Phi_0\equiv\langle 0|\Phi|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix}, \tag{150}$$

where we decompose $\Phi(x)$ in terms of three (would-be) Goldstone bosons ζ_j and a Higgs h. The Pauli matrices σ_j represent broken generators of $SU(2)_L$. Let Φ also be charged under $U(1)_Y$:

$$\Phi\to\exp\left(ig\frac{\sigma^i}{2}\omega_a + ig'\frac{Y_H}{2}\omega'\right)\Phi. \tag{151}$$

We do not want to break $U(1)_{em}$ spontaneously so the vacuum characterized by the VEV Φ_0 should be invariant under $U(1)_{em}$, i.e., has no electric charge Q

$$e^{ieQ\theta}\Phi_0 = \Phi_0 \to Q\Phi_0 = 0. \tag{152}$$

The operator Q is a linear combination of diagonal generators of $SU(2)_L\times U(1)_Y$, $T_3 = \sigma_3/2$ and $Y/2$:

$$Q\Phi_0 = \left(T_3+\frac{Y}{2}\right)\Phi_0 = \frac{1}{2}\begin{pmatrix}1+Y_H & 0\\ 0 & -1+Y_H\end{pmatrix}\begin{pmatrix}0\\ \frac{v}{\sqrt{2}}\end{pmatrix}\stackrel{?}{=}0. \tag{153}$$

As a consequence, to keep $U(1)_{em}$ unbroken, we should set $Y_H = 1$. Since Φ transforms under the EW group, we introduce gauge interactions for the Higgs doublet to make sure that the scalar sector respects the corresponding local symmetry:

$$\mathcal{L}_\Phi = (D_\mu\Phi)^\dagger(D_\mu\Phi) - V(\Phi), \quad \text{with} \quad V(\Phi) = m_\Phi^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2. \tag{154}$$

For $m_\Phi^2 < 0$ the symmetry is spontaneously broken. In the *unitary* gauge (Goldstone bosons are gauged away: in Eq. (150) we put $\zeta_j = 0$) the first term in Eq. (154) can be cast into

$$\begin{aligned}|D_\mu\Phi|^2 &= \frac{1}{2}(\partial_\mu h)^2 + \frac{g^2}{8}(v+h)^2|W_\mu^1 + iW_\mu^2|^2 + \frac{1}{8}(v+h)^2(gW_\mu^3 - g'Y_HB_\mu)^2 \\ &= \frac{1}{2}(\partial_\mu h)^2 + \frac{g^2}{4}(v+h)^2W^+W^- \quad \left[\sqrt{2}W^\pm = W_\mu^1 \mp iW_\mu^2\right]\end{aligned} \tag{155}$$

$$+\frac{1}{8}(v+h)^2\left[Z_\mu(g\cos\theta_W+g'\sin\theta_W)+A_\mu(g\sin\theta_W-g'\cos\theta_W)\right]^2 \tag{156}$$

$$=\frac{1}{2}(\partial_\mu h)^2+M_W^2\left(1+\frac{h}{v}\right)^2 W^+W^-+\frac{M_Z^2}{2}\left(1+\frac{h}{v}\right)^2 Z_\mu Z_\mu, \tag{157}$$

where we *require* the photon to be massless after SSB, i.e.,

$$g\sin\theta_W-g'\cos\theta_W=0 \quad\Rightarrow\quad \sin\theta_W=\frac{g'}{\sqrt{g^2+g'^2}},\cos\theta_W=\frac{g}{\sqrt{g^2+g'^2}} \tag{158}$$

and, consequently,

$$g\cos\theta_W+g'\sin\theta_W=\sqrt{g^2+g'^2}, \qquad e=g\sin\theta_W=g'\cos\theta_W=\frac{gg'}{\sqrt{g^2+g'^2}}. \tag{159}$$

The masses of the Z and W-bosons are proportional to the EW gauge couplings

$$M_W^2=\frac{g^2v^2}{4}, \quad M_Z^2=\frac{(g^2+g'^2)v^2}{4}. \tag{160}$$

One can see that the Higgs-gauge boson vertices (Fig. 18) are related to the masses M_W and M_Z.

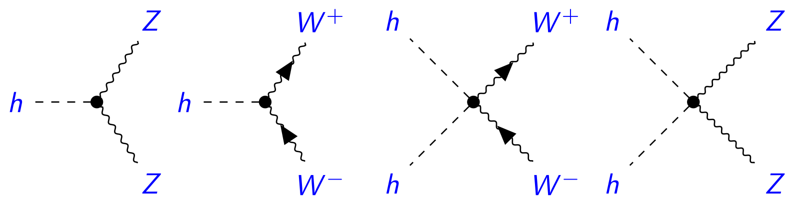


Fig. 18: Gauge-boson–Higgs interactions.

Earlier we emphasized that the introduction of the Z-boson cures the problem with unitarity in the process $e^-e^+\to W^-W^+$ for longitudinal W-bosons. Another important consequence of the SM gauge symmetry and the *existence* of the Higgs boson is the *unitarization* of massive vector-boson scattering. By means of simple power counting, one can easily convince oneself that the amplitude for (longitudinal) W-boson scattering originating from the quartic vertex in Fig. 15 scales with energy as E^4/M_W^4. However, in the SM, thanks to gauge symmetry, QGC and TGC are related. This results in E^2/M_W^2 behavior when Z/γ exchange is taken into account. Moreover, since the gauge bosons couple also to Higgs, we need to include the corresponding contribution to the total amplitude. It turns out that it is this contribution that cancels the E^2 terms and saves unitarity in the WW-scattering, as shown in Fig. 19. Obviously, this pattern is a consequence of the EW symmetry breaking in the SM and can be modified by the presence of new physics. Due to this, experimental studies of vector boson scattering (VBS) play a role in proving overall consistency of the SM.

Having in mind Eq. (106), one can derive the relation

$$G_F=\frac{1}{\sqrt{2}v^2}\Rightarrow v\simeq 246\text{ GeV}, \tag{161}$$

which gives a numerical estimate of v. One can also see that due to Eq. (160) we have (at the tree level)

$$\rho=\frac{M_W^2}{M_Z^2\cos^2\theta_W}=1. \tag{162}$$

Let us emphasize that it is a consequence of the fact that the SM Higgs is a weak *doublet* with *unit* hypercharge. Due to this, $\rho\simeq 1$ imposes important constraints on possible extensions of the SM Higgs

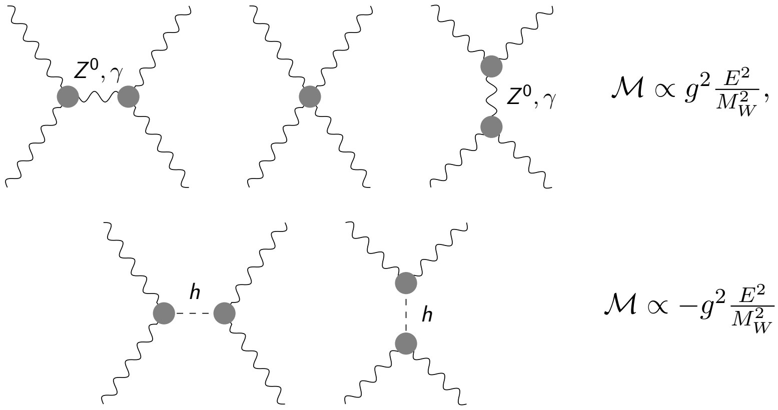


Fig. 19: WW-scattering and unitarity.

sector. For example, we can generalize Eq. (162) to account for n scalar $(2I_i + 1)$-plets $(i = 1, ..., n)$ that transform under $SU(2)_L$ and have hypercharges Y_i. In case they acquire VEVs v_i, which break the EW group, we have

$$\rho = \frac{\sum_i (I_i(I_i+1) - Y_i^2) v_i^2}{\sum_i 2 Y_i^2 v_i^2}. \tag{163}$$

Consequently, any non-doublet (with total weak isospin $I_i \neq 1/2$) VEV leads to a deviation from $\rho = 1$.

4.4 Fermion-Higgs interactions and masses of quarks and leptons

Since we fixed all the gauge quantum numbers of the SM fields, it is possible to construct the following *gauge-invariant* Lagrangian:

$$\mathcal{L}_Y = -y_e(\underset{+1}{\bar{L}}\ \underset{+1}{\Phi})\ \underset{-2}{e_R} - y_d(\underset{-\frac{1}{3}}{\bar{Q}}\ \underset{+1}{\Phi})\ \underset{-\frac{2}{3}}{d_R} - y_u(\underset{-\frac{1}{3}}{\bar{Q}}\ \underset{-1}{\Phi^c})\ \underset{\frac{4}{3}}{u_R} + \text{h.c.}, \tag{164}$$

which involves *dimensionless* Yukawa couplings y_f. It describes interactions between the Higgs field Φ, left fermion doublets, Eq. (114), and right singlets. In Eq. (164) we also indicate weak hypercharges of the corresponding fields. One can see that combinations of two doublets, $(\bar{Q}\Phi)$ etc., are invariant under $SU_L(2)$ but have a non-zero charge under $U(1)_Y$. The latter is compensated by hypercharges of right fermions. In addition, $U(1)_Y$ symmetry forces us to use a charge-conjugated Higgs doublet $\Phi^c = i\sigma_2\Phi^*$ with $Y = -1$ to account for Yukawa interactions involving u_R.

In the spontaneously broken phase with non-zero Higgs VEV, the Lagrangian $\mathcal{L}_Y$ can be written in the following simple form:

$$-\mathcal{L}_Y = \sum_f \frac{y_f v}{\sqrt{2}} \left(1 + \frac{h}{v}\right) \bar{f} f = \sum_f m_f \left(1 + \frac{h}{v}\right) \bar{f} f, \quad f = u,\ d,\ e, \tag{165}$$

where unitary gauge is utilized. One can see that SSB generates fermion masses m_f and, similarly to Eq. (157), *relates* them to the corresponding couplings of the Higgs boson h (see Fig 20a).

It is worth noting that Eq. (164) is not the most general renormalizable Lagrangian involving the SM scalars and fermions. One can introduce *flavour* indices and non-diagonal *complex* Yukawa matrices y_f^{ij} to account for a possible mixing between the SM fermions, i.e.,

$$\mathcal{L}_{\text{Yukawa}} = -y_l^{ij}(\bar{L}_i\Phi) l_{jR} - y_d^{ij}(\bar{Q}_i\Phi) d_{jR} - y_u^{ij}(\bar{Q}_i\Phi^c) u_{jR} + \text{h.c.}. \tag{166}$$

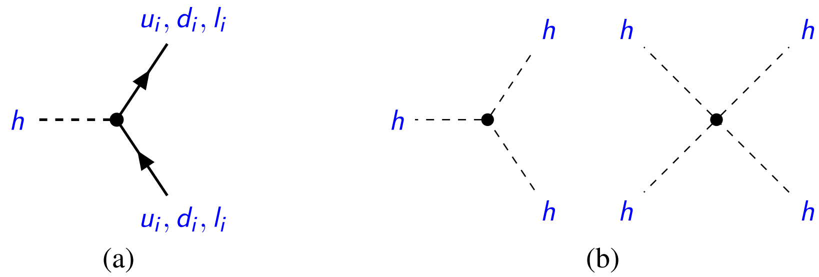


Fig. 20: Higgs–fermion couplings (a) and self-interactions of the Higgs boson (b).

Substituting $\Phi \to \Phi_0$ we derive the expression for fermion mass matrices $m_f^{ij} = y_f^{ij} \frac{v}{\sqrt{2}}$, which can be diagonalized by suitable unitary rotations of left and right fields. In the SM the Yukawa matrices, Eq. (166), are also diagonalized by the *same* transformations. This leads again (in the unitary gauge) to Eq. (165) but with the fields corresponding to the *mass* eigenstates. The latter *do not* coincide with *weak* states, which enter into $\mathcal{L}_W$, Eq. (117). However, one can rewrite $\mathcal{L}_W$ in terms of mass eigenstates. Due to large *flavour symmetry* of weak interactions, this introduces a single mixing matrix (the Cabibbo–Kobayashi–Maskawa matrix, or CKM), which manifests itself in the charged-current interactions $\mathcal{L}_{CC}$. A remarkable fact is that three generations are *required* to have $\mathcal{CP}$ violation in the quark sector. Moreover, a single CKM with only four physical parameters (angles and one phase) proves to be very successful in accounting for plethora of phenomena involving transitions between different flavours. We will not discuss further details but refer to the dedicated lectures on flavor physics [28].

5 The Standard Model: theory vs experiment

5.1 The Standard Model input parameters

Let us summarize and write down the full SM Lagrangian as

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{Gauge}}(g_s, g, g') + \mathcal{L}_{\text{Yukawa}}(y_u, y_d, y_l) + \mathcal{L}_{\text{Higgs}}(\lambda, m_\Phi^2) + \mathcal{L}_{\text{Gauge-fixing}} + \mathcal{L}_{\text{Ghosts}}. \tag{167}$$

The Yukawa part $\mathcal{L}_{\text{Yukawa}}$ is given in Eq. (166), while $\mathcal{L}_{\text{Higgs}} = -V(\Phi)$ is the Higgs potential from Eq. (154). After SSB the corresponding terms give rise to the Higgs couplings to the SM fermions (Fig. 20a) and Higgs self-interactions (Fig. 20b). The former are diagonal in the *mass* basis. The kinetic term for the Higgs field is included in

$$\mathcal{L}_{\text{Gauge}} = -\frac{1}{4}\underbrace{G^a_{\mu\nu}G^a_{\mu\nu}}_{SU(3)_c} - \frac{1}{4}\underbrace{W^i_{\mu\nu}W^i_{\mu\nu}}_{SU(2)_L} - \frac{1}{4}\underbrace{B_{\mu\nu}B_{\mu\nu}}_{U(1)_Y} + (D_\mu\Phi)^\dagger(D_\mu\Phi) \tag{168}$$

$$+\underbrace{\bar{L}_i\, i\hat{D}\, L_i + \bar{Q}_i\, i\hat{D}\, Q_i}_{SU(2)_L \text{ doublets}} + \underbrace{\bar{l}_{Ri}\, i\hat{D}\, l_{R_i} + \bar{u}_{Ri}\, i\hat{D} u_{R_i} + \bar{d}_{Ri}\, i\hat{D}\, d_{R_i}}_{SU(2)_L \text{ singlets}}, \tag{169}$$

where for completeness we also add the colour group $SU(3)_c$ responsible for the strong force. The first three terms in Eq. (168) introduce gauge bosons for the SM gauge groups and in the non-Abelian case account for self-interactions of the latter (Fig. 15). The fourth term in Eq. (168) written in the form shown in Eq. (157) accounts for gauge interactions of the Higgs field (Fig. 18). Finally, Eq. (169) gives rise to interactions between gauge bosons and the SM fermions (see, e.g., Fig. 12).

The SM Lagrangian, Eq. (167), depends on 18 physical[20] parameters — 17 dimensionless couplings (gauge, Yukawa, and scalar self-interactions) and only 1 mass parameter m_Φ^2 (see Table 4). It is

[20]We do not count unphysical gauge-fixing parameters entering into $\mathcal{L}_{\text{Gauge-fixing}}$ and $\mathcal{L}_{\text{Ghosts}}$.

worth emphasizing here that there is certain freedom in the definition of *input* parameters. In principle, one can write down the SM predictions for a set of 18 observables (e.g., physical particle masses or cross-sections at fixed kinematics) that can be measured in experiments. With the account of loop corrections the predictions become non-trivial functions of *all* the Lagrangian parameters. By means of PT it is possible to invert these relations and express these primary parameters in terms of the chosen measured quantities. This allows us to *predict* other *observables in terms of* a finite set of measured *observables*[21].

However, it is not always practical to strictly follow this procedure. For example, due to confinement we are not able to directly probe the strong coupling g_s and usually treat it as a scale-dependent parameter $(4\pi)\alpha_s = g_s^2$ defined in the modified minimal-subtraction ($\overline{\text{MS}}$) scheme. It is customary to use the value of $\alpha_s^{(5)}(M_Z) = 0.1181 \pm 0.011$ at the Z-mass scale as an input for theoretical predictions. A convenient choice of other input parameters is presented in Table 4. It is mostly dictated by the fact that the parameters from the "practical" set are measured with better precision than the others.

Let us discuss some of the so-called Z pole observables that, after being measured with high precision at LEP and SLC, serve as an important input for the determination of the SM parameters.

Table 4: Parameters of the SM.

18=	1	1	1	1	1	9	4
primary:	g_s	g	g'	λ	m_Φ^2	y_f	y_{ij}
practical:	α_s	M_Z^2	α	M_H^2	G_F	m_f	V_{CKM}

5.2 Z pole observables

The electroweak model provides precise predictions for the properties of the Z boson, which can be tested in the process $e^+e^- \to Z \to f\bar{f}$ (Fig. 21). The latter dominate the cross-section $e^+e^- \to f\bar{f}$, if the center-of-mass energy is tuned to be $\sqrt{s} \simeq m_Z$. As we known, the heaviest SM fermion, that can be produced at such energies, is the b-quark with mass $m_b \simeq 4\,\text{GeV}$. Due to this, we will neglect all m_f in the following considerations.

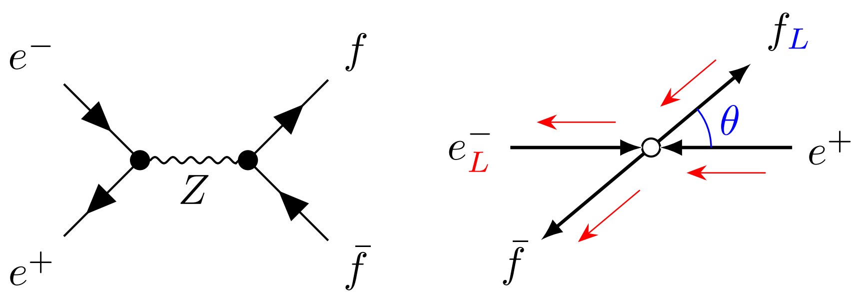


Fig. 21: The e^+e^- annihilation at $s \simeq m_Z^2$ (left), and one of the helicity combinations (right), which corresponds to the amplitude M_{LL} given in Eq.(170). Fermions are assumed to be massless.

Let us start by calculating the *tree-level* matrix elements for the processes involving fermions with certain helicity (= chirality). Since the Zff vertex conserve chirality, we will label the amplitudes by the *helicities* of the incoming e^- and outgoing f. For example, the *squared* amplitude

$$|M_{LL}|^2 = g_Z^4 |P(z)|^2 [c_L^e]^2 [c_L^f]^2 \,(1+\cos\theta)^2 \tag{170}$$

corresponds to the process $e_L^- e_R^+ \to f_L \bar{f}_R$, in which left-handed electron and right-handed positron annihilate to produce left-handed f and right-handed $\bar{f}$. In Eq. (170) the (Breit-Wigner) factor $|P(z)|^2 =$

[21] One can even avoid the introduction of *renormalizable* parameters and use *bare* quantities at the intermediate step.

$1/[(s-m_Z)^2+m_Z^2\Gamma_Z^2]$ originates from the Z-boson propagator, and the dependence on the scattering angle θ in the center-of-mass frame can again be understood from simple arguments based on helicity conservation (see Fig. 21). In the same way, one can obtain

$$|M_{RR}|^2 = g_Z^4|P(z)|^2[c_R^e]^2[c_R^f]^2\,(1-\cos\theta)^2\,, \tag{171}$$

$$|M_{LR}|^2 = g_Z^4|P(z)|^2[c_L^e]^2[c_R^f]^2\,(1-\cos\theta)^2\,, \tag{172}$$

$$|M_{RL}|^2 = g_Z^4|P(z)|^2[c_R^e]^2[c_L^f]^2\,(1-\cos\theta)^2\,. \tag{173}$$

For *unpolarised* $e^\pm$ beams we have to average over all possible initial helicity combinations. Integration over the scattering angle θ gives the total cross-section, which (in the narrow-width) approximation can be rewritten as

$$\sigma(ee\to Z\to ff) = \frac{12\pi s}{m_Z^2}\frac{\Gamma_{ee}\Gamma_{ff}}{(s-m_Z)^2+m_Z^2\Gamma_Z^2} \overset{\sqrt{s}=m_Z}{\Longrightarrow} \sigma^0_{ff} \equiv \frac{12\pi}{m_Z^2}\frac{\Gamma_{ee}\Gamma_{ff}}{\Gamma_Z^2}. \tag{174}$$

Here the partial width $\Gamma_{ff}\equiv\Gamma(Z\to ff)$ is given (at the tree-level) in terms of $c^f_{L,R}$ as

$$\Gamma(Z\to f\bar{f}) = \frac{g_Z^2 m_Z}{24\pi}\left([c_L^f]^2+[c_R^f]^2\right), \tag{175}$$

and Γ_Z represents the total Z width. From Eq. (174) one can see that the maximal value of the cross-section corresponds to $\sqrt{s}=m_Z$, so the position of the peak allows us to measure the mass of the Z boson. In addition, the fact that (full-width-at-half-maximum - FWHM)

$$\sigma(\sqrt{s}=m_Z\pm\Gamma_Z/2) = \sigma^0_{ff}/2,$$

allows us to extract Γ_Z directly from the energy dependence of the cross-section. Moreover, assuming *lepton universality*, we can experimentally determine $\Gamma_{ee}\simeq\Gamma_{\mu\mu}\simeq\Gamma_{\tau\tau}$ from $\sigma^0_{\mu\mu}$:

$$(12\pi)\Gamma_{ee}^2 = \sigma^0_{\mu\mu}\Gamma_Z^2 m_Z^2. \tag{176}$$

In the same way, by considering $e^+e^-\to$ hadrons one can extract the partial width for Z decaying into hadrons

$$(12\pi)\Gamma_{\text{hadrons}} = \sigma^0_{\text{hadrons}}\Gamma_Z^2 m_Z^2/\Gamma_{ee}. \tag{177}$$

Finally, assuming that $\Gamma_{\nu\nu}=\Gamma_{\nu\nu}^{SM}$ is *calculated* by means of Eq. (175) and Table 3, the number of neutrino (with $m_\nu<m_Z/2$) can be determined via

$$N_\nu = (\Gamma_Z-3\Gamma_{ee}-\Gamma_{hadrons})/\Gamma^{SM}_{\nu\nu}\simeq 2.98. \tag{178}$$

Clearly, this is consistent with three fermion generations predicted by the SM.

We can have additional constraints on the SM parameters from measurements of various Z-pole *asymmetries* (see also Fig. 26). One example of such kind of observables is the Forward-Backward asymmetry A^f_{FB}, e.g.,

$$A^\mu_{FB} = \frac{\sigma_F-\sigma_B}{\sigma_F+\sigma_B} = \frac{3}{4}\mathcal{A}_e\mathcal{A}_\mu,\quad \mathcal{A}_f = \frac{(c_L^f)^2-(c_R^f)^2}{(c_L^f)^2+(c_R^f)^2} = \frac{v_f/a_f}{1+(v_f/a_f)^2}, \tag{179}$$

$$\sigma_F = 2\pi\int_0^1\frac{d\sigma}{d\Omega}d(\cos\theta),\quad \sigma_B = 2\pi\int_{-1}^0\frac{d\sigma}{d\Omega}d(\cos\theta). \tag{180}$$

Measurements of the asymmetry parameters $\mathcal{A}_f$ for leptons at LEP and SLC indicate that albeit being slightly different they are consistent with *universality* hypothesis $\mathcal{A}_e \simeq \mathcal{A}_\mu \simeq \mathcal{A}_\tau \simeq 0.15$. In addition, we can directly measure $\sin^2\theta_W$, since for leptons $v_l/a_l = 1 - 4\sin\theta_W^2$.

It is worth pointing here that our reasoning in this section was based on the *tree-level* amplitudes. Of course, to confront theory with high-precision experiment we have to take into account various quantum corrections. Moreover, one should perform various re-summations, e.g., to account for initial state radiation (ISR), which distorts the Breit-Wigner form of the distribution. Since the topic is quite involved, we will not go into further detail (see, e.g., Ref. [9]) here but give some other arguments regarding the importance of the quantum corrections in the SM.

5.3 On the importance of radiative corrections

At the *tree* level one can write the following relations between the parameters given in Table 4:

$$\begin{aligned} \alpha_s &= \frac{g_s^2}{4\pi}, \quad (4\pi)\alpha = g^2 g'^2/(g^2+g'^2), \quad M_Z^2 = \frac{(g^2+g'^2)v^2}{4}, \\ G_F &= \frac{1}{\sqrt{2}v^2}, \quad M_h^2 = 2\lambda v^2 = 2|m_\Phi|^2, \quad m_f = y_f v/\sqrt{2} \quad . \end{aligned} \tag{181}$$

At higher orders in PT, the relations are modified and perturbative corrections turn out to be mandatory if one wants to confront theory predictions [29–31] with high-precision experiments. A simple example to demonstrate this fact comes from the tree-level "prediction" for the W-mass M_W. From Eq. (160) and Eq. (181) we can derive

$$\frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2M_W^2(1 - M_W^2/M_Z^2)}. \tag{182}$$

Plugging the Particle Data Group (PDG) [23] values

$$\alpha^{-1} = 137.035999139(31), \; M_Z = 91.1876(21) \text{ GeV}, \; G_F = 1.1663787(6)\times 10^{-5}\text{ GeV}^{-2}, \tag{183}$$

in Eq. (182), one predicts

$$M_W^{tree} = 80.9387(25) \text{ GeV}, \tag{184}$$

where only uncertainties due to the input parameters, Eq. (183), are taken into account. Comparing M_W^{tree} with the measured value $M_W^{exp} = 80.379(12)$ GeV, one sees that our naive prediction is off by about 47σ! Of course, this is not the reason to abandon the SM. We just need to take quantum corrections into account (see, e.g., Fig. 22).

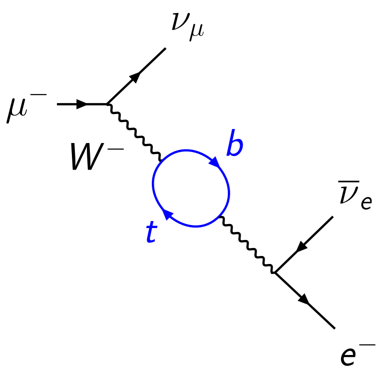


Fig. 22: An example of loop corrections to the muon decay, which give rise to the modification of the tree-level relation in Eq. (182).

The radiative corrections allows one to *relate* phenomena at different scales in the context of a single model. For example, we can study scale dependence of primary parameters, e.g., gauge couplings, and calculate high-order contribution to the corresponding beta-functions. At the one-loop order, we have the following general expression for the gauge-coupling RGE

$$\frac{d\alpha}{d\log\mu^2} = \beta\alpha^2 + \mathcal{O}(\alpha^3), \quad \alpha = \frac{g^2}{4\pi}, \quad \beta = -\frac{1}{4\pi}\left[\frac{11}{3}C_2 - \frac{2}{3}\sum_F T_F - \frac{1}{3}\sum_S T_S\right]. \tag{185}$$

Here $C_2 = N$ for the $SU(N)$ group, the sum goes over (Weyl) fermions (F) and scalars (S) coupled to the gauge field, and $T_F = T_S = \frac{1}{2}$. Figure 23 illustrates the scale dependence of the gauge couplings g, g' and g_s in the SM. It is worth pointing out that it was obtained by taking into account three-loop contributions to Eq. (185) and other SM couplings, which are also depicted for convenience. One can see that the gauge couplings tend to converge to a single value at about 10^{13-15} GeV, thus providing a hint for grand unification.

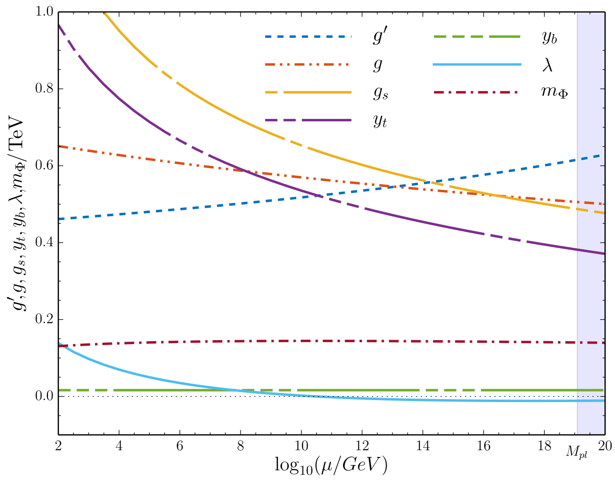


Fig. 23: Scale dependence of the SM parameters obtained by means of `mr` package [32].

Another important consequence of this kind of studies is related to the EW vacuum (meta)stability (see, e.g., Ref. [33]). In Fig. 23, it manifests itself at the scale $\mu \simeq 10^{10}$ GeV, at which the self-coupling λ becomes negative, making the tree-level potential unbounded from below. The two key parameters here are the top-quark mass M_t and the mass of the Higgs bosons M_h. According to Eq. (181) they can be related to the (boundary) values of λ and the top Yukawa y_t at the EW scale. The latter significantly influence self-coupling running, since (*cf.* Fig. 4)

$$(4\pi)^2\frac{d\lambda}{d\log\mu^2} = 12\lambda - 3{y_t}^4 + \ldots \quad . \tag{186}$$

A more elaborated analysis of the vacuum stability problem is based on the effective potential and gives rise to the well-known phase diagram in the $M_t - M_h$ plane (see, e.g., Fig. 24). One can see that the measured values of M_t and M_h lie just near the boundary between absolute stability (the EW vacuum is the true vacuum) and metastability (there exists a deeper minimum, but the tunneling time is much larger then the age of the Universe). This fact triggered many discussions about the fate of the EW vacuum in theoretical community. Without going into details, we just want to indicate again the importance of high-order corrections in the analysis: it is the next-to-next-to-leading (NNLO) effects (two-loop corrections to Eq. (181) and three-loop RGE) that "move" the absolute stability boundary just below the point corresponding to the experimentally measured values.

A modern way to obtain the values of the SM parameters is to perform a global fit to confront state-of-the-art SM predictions with high-precision experimental data. Due to quantum effects, we can

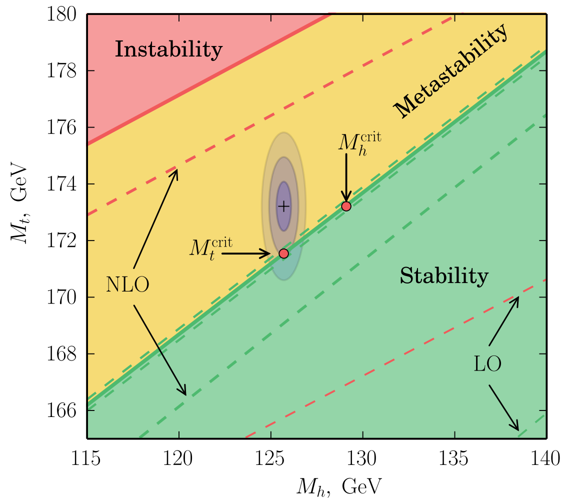


Fig. 24: Vacuum-stability phase diagram at three loops (NNLO). One can notice how the phase boundaries move upon switching from the leading-order (LO) one-loop evolution to NLO and NNLO running.

even probe new physics that can contribute to the SM processes at low energies via virtual states. Indeed, LEP precision measurements interpreted in the context of the SM were used in a multidimensional parameter fits to predict the mass of the top quark M_t ("new physics"), prior to its discovery at the Tevatron. After M_t was measured it was included in the fit as an additional constraint, and the same approach led to the prediction of a *light* Higgs boson. In Fig. 25, the famous *blue-band* plot by the LEP Electroweak Working Group (LEPEWWG) is presented [34]. It was prepared a couple of months before the official announcement of the Higgs-boson discovery. One can see that the best-fit value corresponding to $\Delta\chi^2_{min} = 0$ lies just about 1σ below the region *not* excluded by LEP and LHC.

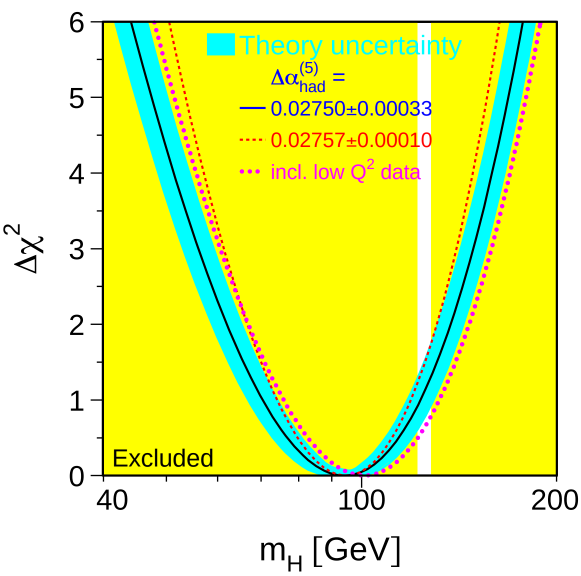
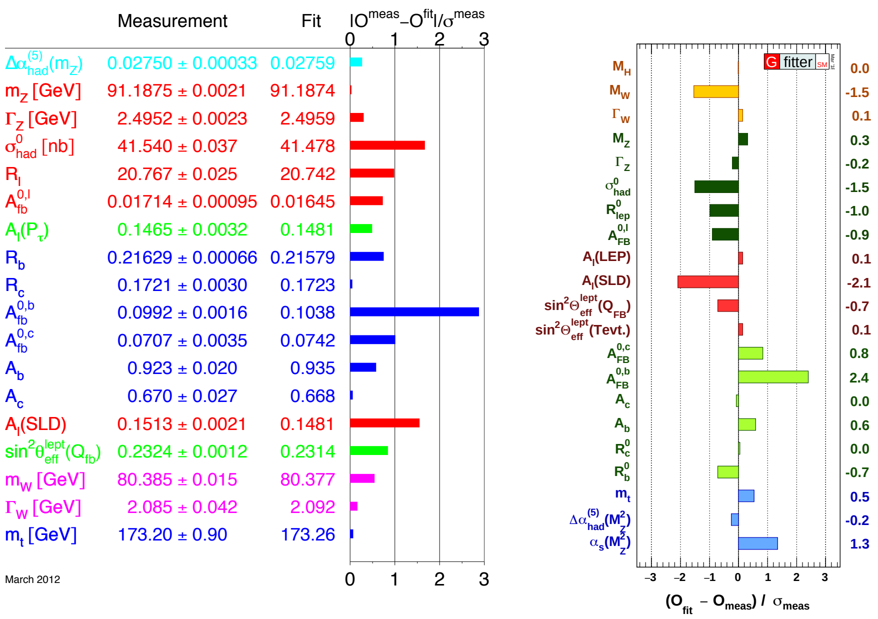


Fig. 25: The dependence of $\Delta\chi^2_{\rm min}(M_H^2) = \chi^2_{\rm min}(M_H^2) - \chi^2_{\rm min}$ on the value of M_H. The width of the shaded band around the curve shows the theoretical uncertainty. Exclusion regions due to LEP and LHC are also presented.

Obviously, at the moment the global EW fit is *over constrained* and can be used to test overall consistency of the SM. In Fig. 26 we present the comparison between measurements of different (pseudo)observables $O^{\rm meas}$ and the SM predictions $O^{\rm fit}$ corresponding to the best-fit values of fitted parameters. Although there are several quantities where *pulls*, i.e., deviations between the theory and experiment, reach more than two standard deviations, the average situation should be considered as extremely good. A similar conclusion can be drawn from the recent Figs. 27 and 28, in which experimental results for various cross-sections measured by ATLAS and CMS are compared with the SM predictions.



(a) (b)

Fig. 26: Pulls of various (pseudo)observables due to (a) LEPEWWG [34] and (b) Gfitter [35].

6 Conclusions

Let us summarize and discuss briefly the pros and cons of the SM. The model has many nice features:

– it is based on symmetry principles: Lorentz + $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry;
– it is renormalizable and unitary;
– the structure of all interactions is fixed (but not all couplings are tested experimentally);
– it is an anomaly-free theory;
– it can account for rich flavour physics (see Ref. [28]);
– three generations allow $\mathcal{CP}$-violation (see Ref. [28]);
– it can be extended to incorporate neutrino masses and mixing (see Ref. [20]);
– it allows making systematic predictions for a wide range of phenomena at different scales;
– all predicted particles have been discovered experimentally;
– it survives stringent experimental tests.

Due to this, the SM is enormously successful (*Absolutely Amazing Theory of Almost Everything*). Since it works so well, *any* new physics should reproduce it in the low-energy limit. Unfortunately, contrary to the Fermi-like non-renormalizable theories, the values of the SM parameters do not give us obvious

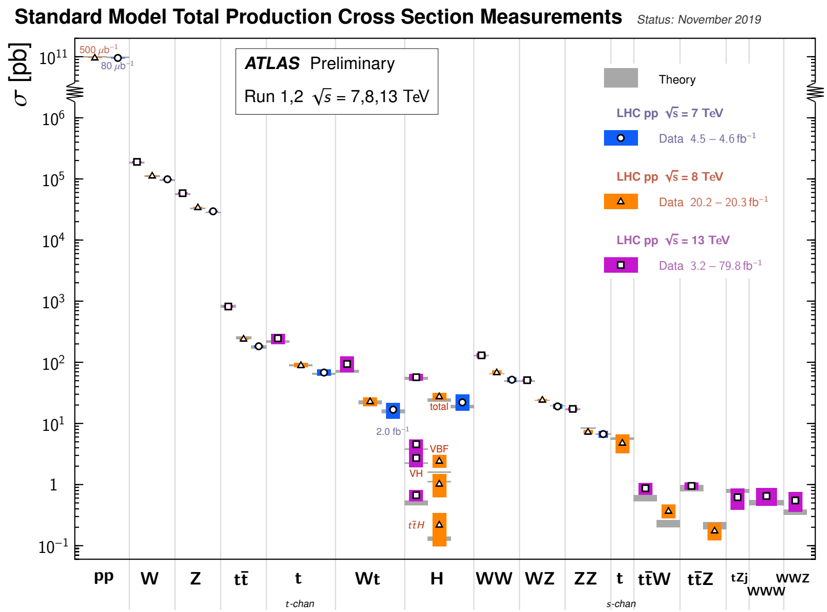


Fig. 27: ATLAS results of the SM cross-section measurements.

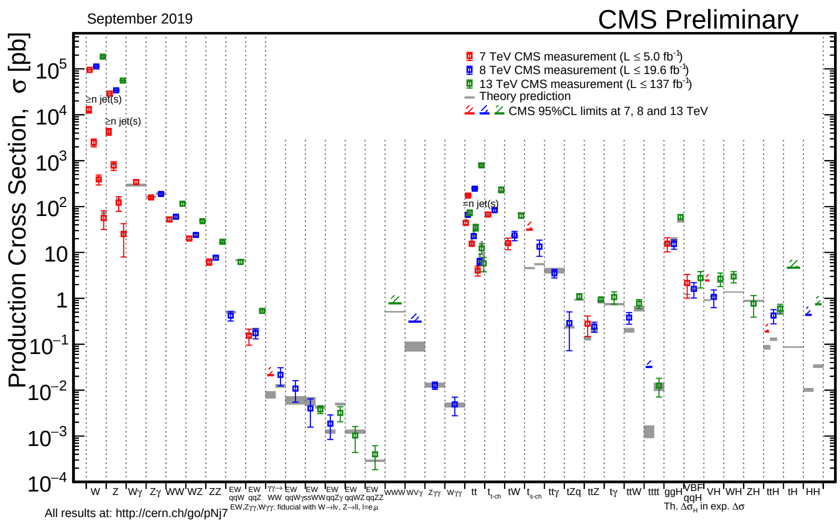


Fig. 28: SM processes at CMS.

hints for a new physics scale. But why do we need new physics if the model is so perfect? It turns out that we do not *understand*, why the SM works so well. For example, one needs to clarify the following:

– What explains the pattern behind flavour physics (hierarchy in masses and mixing, 3 generations)?
– Is there a symmetry behind the SM (electric) charge assignment?
– What is the origin of the Higgs potential?
– What is the origin of accidental baryon and lepton number symmetries?
– Why is there no CP-violation in the strong interactions (strong CP problem)? [22]
– Why is the Higgs-boson mass so low? (Hierarchy/naturalness problem, see Ref. [24])
– Is it possible to unify all the interactions, including gravity?

[22] The SM Gauge group allows such a term in the SM Lagrangian, $\mathcal{L} \ni \theta_{CP} \frac{1}{16\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}$, but it turns out that $\theta_{CP} = 0$.

In addition, there are phenomenological problems that are waiting for solutions and probably require introduction of some new physics:

– Origin of neutrino masses (see Ref. [20]);
– Baryon asymmetry (see Ref. [36]);
– Dark matter, dark energy, inflation (see Ref. [36]);
– Tension in $(g-2)_\mu$, $b \to s\mu\mu$, $b \to cl\nu$;
– Possible problems with lepton universality of EW interactions (see Refs. [28, 37]).

In view of the above-mentioned issues we believe that the SM is not an ultimate theory (see Ref. [37]) and enormous work is ongoing to prove the existence of some new physics. In the absence of a direct signal a key role is played by *precision* measurements, which can reveal tiny, yet significant, deviations from the SM predictions. The latter should be accurate enough (see, e.g., Ref. [38]) to compete with modern and future experimental precision [39].

To conclude, one of the most important *tasks* in modern high-energy physics is to find the scale at which the SM breaks down. There is a big chance that some new physical phenomena will eventually manifest themselves in the ongoing or future experiments, thus allowing us to single out viable model(s) in the enormous pool of existing NP scenarios.

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