QCD

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The structure of strong interaction dynamics, namely Quantum Chromodynamics, is discussed within the framework of perturbation theory. After a brief introduction to the historical developments, we will discuss in detail the role of perturbative QCD to understand the physics at various high energy colliders involving hadrons. We will discuss how certain large threshold logarithms that show up at every order in perturbation theory can be resummed to all orders. We will also discuss some of the recent advances in performing higher order perturbative corrections.

1 Introduction

The question of what constitutes the visible matter around us has been there for several centuries. There have been different answers at different times. Thanks to several experiments and theoretical studies, we could probe distances that are of the order of less than a few fermi to unravel what constitutes the matter and the dynamics that govern them. The Standard Model (SM) of particle physics is extremely successful in explaining electromagnetic, weak and strong forces within a single framework. However, there are several phenomena that we do not have explanations for within the SM and the efforts to understand them in a single framework are still going on. The Large hadron collider (LHC) at Geneva is designed not only to test the SM to unprecendental accuracy but also probe physics beyond the SM. At LHC, high energetic protons are collided to shed light on the dynamics at the smallest legnth scale possible.

Among the four forces, the force that binds the nucleons inside the nucleus of every atom is the strongest. In addition, it is blind to electric charges of the nucleons. Since the colliding particles at the LHC are hadrons, the strong interaction plays an important role. Hence, it is important to understand this force at short distances. In this article, we discuss how to apply QCD to test SM at high energies.

In order to set stage, we will give brief introduction to Quark Model that describes the structure of hadrons in terms of its constituents called quarks and anti quarks. Then, we present how observables at hadron colliders can be expressible in terms of these constituents. We exploit QCD factorisation of short and large distance physics at high energies to study these observables. We show that the key property of QCD, namely aymptotic freedom allows us to compute short distance part reliably. Often the fixed order perturbative predictions are affected by large logarithms resulting from soft glons in the threshold region. We show how resummation of such logarithms to all orders can be done to make sensible predictions. Precision measurements at the LHC demand precise predictions from QCD. The later is difficult to obtain due to prence of large number of Feynman diagrams and the corresponding multi loop and phase space integrals. We present some of the modern technique to deal with these quantities.

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2 Quark model

In the 1950s [1], large numbers of strongly interacting particles, called hadrons, were discovered. It was a challenging task not only to classify them in a systematic fashion but also to look for constituents that act as building blocks of these composite objects. The hadrons can be classified in two classes based on their spin quantum numbers. Those with half integral spins are called baryons and the ones with integral spins are called mesons. Hadrons such as the proton, neutron, Λ and Σ are baryons, while $\pi^{\pm}$, π^0, $K^{\pm}$ and K^0 are mesons.

Symmetry transformations play an important role in physics and strong interaction is no exception. One finds that the interchange of protons with neutrons does not affect the strong interaction dynamics. Hence, it was proposed that these two hadrons can form a basis for isospin transformations, in particular the isospin doublets of the continuous group $SU(2)$ and the strong interaction is said to be invariant under $SU(2)$ transformations. The symmetry group $SU(2)$ allows to accommodate other hadrons. For example, charged as well as neutral pions, $\pi^{\pm}$ and π^0, form an isotriplet of the same $SU(2)$. Similarly the Σ baryons, $\Sigma^{\pm}$ and Σ^0, form an isotriplet. Hadrons such as η, ω, Λ, etc. are simply isosinglets. Among the particles, K mesons and Λ baryons showed a peculiar behaviour, namely, they were produced in large numbers through the strong interaction, while their life times were longer and their decays proceeded through weak interactions. This led to the introduction of a new quantum number called 'strangeness', akin to the electric charge. One finds that the strong interaction preserves strangeness, while the weak interaction violates it. The symmetry transformation was found to be a U(1) symmetry and the conserved charge is called hypercharge Y, where Y is sum of baryon number and strangeness. Remarkably, the electric charge of every hadron that was observed was found to satisfy a relation, called the Gell-Mann–Nishijima relation, $Q = T_3 + Y/2$, where T_3 is the generator of $SU(2)$. In summary, both isospin $SU(2)$ and strangeness $U(1)$ symmetries enormously simplified the classification of baryons in terms of a pair of quantum numbers (T_3, Y). Gell-Mann proposed a larger symmetry group $SU(3)$, where the symmetry group $SU(2) \otimes U(1)$ is a subgroup, and arranged the mesons and baryons in a scheme called the eight-fold way. Using the higher-dimensional representations of $SU(3)$, a large number of baryons and mesons were classified and, interestingly, this led to the prediction of so far unobserved hadrons that were later discovered. While this approach was successful, the mere proliferation of the number of hadrons posed a serious challenge that was eventually resolved thanks to a proposal made by Gell-Mann and Zweig independently. It states that all known hadrons can be thought of as composite particles made of point-like spin-1/2 fermions, called quarks. The model assumes that there are three types of quarks, called up, down and strange quarks, which form a multiplet that transform in the fundamental representation of $SU(3)$. The anti-quarks are in the conjugate representation. According to this model, mesons are made up of a pair of quark and anti-quark, and they come in both singlet and octet representations, because $8 \times 8 = 1 + 8$. Similarly, the baryons are made of three types of quarks and they show up in one of the representations of $3 \times 3 \times 3 = 1 + 8 + 8 + 10$. While the quark model was enormously successful in explaining most of the hadronic phenomena, the existence of hadronic states having $J^p = 3/2^+$ posed a serious problem, as it apparently violates the spin-statistics theorem. The reason is that the wave functions of these spin-3/2 hadrons remain symmetric under the interchange of their constituents, as a result of their spins being aligned and the spatial part of the wave function being in the symmetric zero-angular-momentum state. The solution to this problem was the introduction of a new quantum number,

called color, for each of the quark states. An additional $SU_c(3)$ symmetry group corresponding to the color quantum number was introduced. Each type of quark can come in three different colors and, hence, each of them transforms in the fundamental representation of $SU_c(3)$. Before we study the role of this symmetry group in strong interaction physics, we will discuss two developments that played an important role in understanding the structure of the hadrons [2].

3 Form factors and structure functions of hadrons

Scattering experiments provide valuable information on both the structure of the particles involved and the dynamics that govern the various interactions. In the following we will discuss two distinct scattering processes involving hadrons.

3.1 Form factors

In elastic electron–proton scattering one can explore the spatial structure of the proton in terms of electric and magnetic form factors. For example, one finds, for $e^-(k) + P(P) \to e^-(k') + P(P')$,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\omega^2 \sin^4(\theta/2)} \frac{\omega'}{\omega} \left\{ A \cos^2(\theta/2) + B \sin^2(\theta/2) \right\} \quad , \tag{1}$$

where α is the fine structure constant, $A = (G_E^2 + \tau G_M^2)/(1+\tau)$ and $B = 2\tau G_M^2$. Here $G_E = \mathcal{F}_1 - \tau\kappa\mathcal{F}_2$ and $G_M = \mathcal{F}_1 + \kappa\mathcal{F}_2$, with $\tau = Q^2/4/M^2, Q^2 = -q^2$. Here $q = k' - k$ and M is the mass of the hadron. The angle θ is the scattering angle in the rest frame of the hadron. κ defines the anomalous magnetic moment of the proton, related to it by $\mu_p = (1+\kappa)e/2/M$. The form factors capture the spatial structure of the hadrons and can be extracted from experiments. The best fit to the data leads, for the proton, to $G_E = [1 + Q^2/(0.71\ \mathrm{GeV}^2)]^{-2}$ up to $Q^2 = 10\ \mathrm{GeV}^2$. There exist several fits demonstrating the non-trivial structure of the hadrons through the Q^2 dependence of the form factors.

3.2 Deep inelastic scattering

When the energy transfer in the scattering is large, the scattered electrons loose a lot of energy, leading to deep inelastic scattering events [3]. This leads to breaking up the proton into pieces, which then fragment into a bunch of hadrons in the final state. The inclusive cross section, where the hadronic final states (X) are summed over, can be used to study the structure of hadrons at very high energies. The dominant contribution arises from the scattering involving a single photon exchange and the corresponding cross section factorises into leptonic and hadronic pieces. That is,

$$d\sigma = \frac{1}{4k.P} \left[\frac{4\pi e^4}{Q^4} L_{\mu\nu}(k,q) W^{\mu\nu}(q,P) \right] \frac{d^3k'}{2\omega'(2\pi)^3} \quad , \tag{2}$$

where the lepton tensor is given by $L_{\mu\nu} = 2(k_\mu k'_\nu + k_\nu k'_\mu - Q^2/2 g_{\mu\nu})$. The hadronic tensor is found to be $W_{\mu\nu} = (8\pi M)^{-1} \sum_{X(P'),s} < P(P,s)|j_{\mu,em}(0)|X(P') >< X(P')|j_{\nu,em}(0)|P(P,s) > \times (2\pi)^4 \delta^{(4)}(P + q - P')$. Here s stands for the spin of the hadron P and subscript em indicates the current is electromagnetic. Unlike the leptonic tensor, the hadronic tensor is not calculable perturbatively, however, using Lorentz covariance and the invariance of the strong interaction under parity and time

reversal transformations, it can be decomposed as $W_{\mu\nu} = (-g_{\mu\nu} + q_\mu q_\nu/q^2)W_1(q,P) + (P_\mu - P\cdot q/q^2 q_\mu)(P_\nu - P\cdot q/q^2 q_\nu)W_2(q,P)$, where $W_i(q,P) = W_i(P\cdot q, Q^2, M^2)$, with $i = 1,2$, are unknown scalar functions, called structure functions. The inclusive cross section is found to be

$$\frac{d\sigma}{dQ^2 d\nu} = \frac{\pi\alpha^2}{4\omega^2 \sin^4(\theta/4)} \frac{1}{\omega\omega'} \left(W_2(\nu, Q^2)\cos^2(\theta/2) + 2W_1(\nu, Q^2)\sin^2(\theta/2) \right) \quad . \tag{3}$$

Since we sum over all the final state hadrons, it is straightforward to show that the structure functions are related to the hadronic matrix elements of the commutator of a pair of electromagnetic currents at different space-time points. Applying the ideas of current algebra, Björken found a remarkable scaling behaviour of the structure functions when Q^2 and $P\cdot q/M = \nu$ are taken to be infinity while keeping their ratio $x_{Bj} = Q^2/(2M\nu)$ fixed. These limits are called Björken limits. The scaling here means that the structure functions in the Björken limit ($\lim_{Bj}$) depend only on the Björken variable x_{Bj}, irrespectively of Q^2. In particular, $W_i(\nu, Q^2)$ satisfy $\lim_{Bj}[MW_1(\nu, Q^2)] = F_1(x_{Bj})$ and $\lim_{Bj}[\nu W_2(\nu, Q^2)] = F_2(x_{Bj})$. The scaling behaviour of the structure functions was confirmed in a series of deep inelastic scattering experiments.

3.3 Parton model

Let us try to find out the consequence of the scaling [4]. We find that if we compute the differential cross section $d\sigma/dQ^2 d\nu$ for the elastic scattering of an electron on a point-like object, then the elastic cross section takes exactly the same form as in Eq. (3), with $W_1(\mu, Q^2)$ replaced by $Q^2/(4M^2)\delta(\nu - Q^2/(2M))$ and $W_2(\nu, Q^2)$ by $\delta(\nu - Q^2/(2M))$. This implies that $F_2(x_{Bj}) = \int_{x_{Bj}}^1 dz F_2(z)\frac{\nu}{z}\delta(\nu - Q^2/(Mz))$. This can be interpreted as the incoherent sum of elastic scatterings on point-like objects, each one carrying a momentum fraction z of the parent hadron. Along this line, Feynman and Björken proposed a model to explain the observed Björken scaling of the structure functions in terms of point-like objects, called partons. It is called the parton model. The salient features of the parton model are easy to understand in the infinite-momentum frame of the target hadron. The model assumes that, for an observer at rest, a fast-moving hadron will appear like a collection of weakly interacting point-like particles, called partons. Inspired by the success of the quark model, one assumes that the partons have spin 1/2 and carry fractional charges like quarks. Due to time dilation, the time scale of the interactions among partons will be much longer than that of the scattering against the highly energetic probe: the partons look almost free. Hence, the inelastic scattering can be thought of as an incoherent sum of elastic scatterings on point-like particles, each one weighted by a probability function. Naively, we can write,

$$d\sigma(x_{Bj}, Q^2) = \sum_a \int_{x_{Bj}}^1 dz \hat{f}_{a/P}(z)\, d\hat{\sigma}_a \left(\frac{x_{Bj}}{z}, Q^2 \right) \quad , \tag{4}$$

where $\hat{f}_{a/P}(z)$ is the probability of finding a parton of type 'a' with a momentum fraction z of the parent hadron (i.e., the parton momentum is $p = zP$) and is called parton distribution function (PDF), while $d\hat{\sigma}_a$ represents the elastic scattering of an electron on a parton of type a. Remarkably, the hadronic cross section factorises into process-independent but target-dependent PDFs and target-independent but parton-dependent cross sections. Note that the partonic cross section depends on the parton-level Björken

variable given by $z_{Bj} = Q^2/2p \cdot q = x_{Bj}/z$. Using this parton model, one finds $2x_{Bj}F_1(x_{Bj}) = F_2(x_{Bj}) = \sum_i e_i^2 x_{Bj} f_i(x_{Bj})$, where e_i is the electric charge of the parton of type i. Measurements of these structure functions using charged- as well as neutral-current probes can be used to extract PDFs of different types for a wide range of x_{Bj}. Note that the PDFs contain the long-distance physics and hence they are not computable within the framework of perturbation theory. The PDFs satisfy various sum rules: for example, the energy momentum conservation leads to the momentum sum rule given by $\sum_i \int_0^1 dz\, z f_i(z) = 1$. Such sum rules can be used to constrain the fits of PDFs from various experiments. The parton model can be used to study other high-energy scattering processes, involving hadrons in both the initial and final states. For example, within the parton model, the cross section for the production of a pair of leptons in proton–(anti)proton scattering experiments can be expressed in terms of two PDFs as

$$\frac{d\sigma}{dQ^2} = \sum_{ab} \int_\tau^1 dx_1 \hat{f}_{a/P_1}(x_1) \int_{x_1}^1 dx_2 \hat{f}_{b/P_2}(x_2) d\hat{\sigma}_{ab}\left(\frac{\tau}{x_1 x_2}, Q^2\right) \quad . \tag{5}$$

Here the PDFs $f_{c/P_i}(x_i)$, where $c = a, b$ are the parton types, are process independent and $d\hat{\sigma}_{ab}$ are parton-level cross sections. Semi-inclusive DIS processes, where one tags a specific hadron with momentum P' in the final state, can be described in the parton model as

$$d\sigma(x_{Bj}, z_F, Q^2) = \sum_{a,b} \int_{x_{Bj}}^1 dz \hat{f}_{a/P}(z) \int_{z_F}^1 dz' \hat{D}_{P'/b}(z') d\hat{\sigma}_{ab}\left(x_{Bj}/z_1, \frac{z_F}{z_2}, Q^2\right) \quad . \tag{6}$$

Here, $D_{P'/b}(z_2)$ is called the fragmentation function and describes the probability that a parton of type b fragments into a hadron P' that carries away a fraction z_2 of the parent parton momentum. $z_F = 2P' \cdot q/Q^2$ is the scaling variable corresponding to the final-state hadron. It is worth emphasising that both PDFs and fragmentation functions are process independent and, hence, can be used to predict various observables at hadron colliders. In summary, the parton model provides an elegant framework to compute a variety of observables in high-energy hadronic scattering experiments.

4 Quantum chromodynamics

While the parton model is enormously successful, one can not ignore the effect of strong interactions among partons. Hence, the next task was to look for a suitable quantum field theory that captures the underlying dynamics of the strong interactions among the partons [5]. Gross, Wilzcek and Politzer independently found the right gauge theory that correctly describes the interaction among the partons. It is called Quantum Chromodynamics. The underlying gauge group turned out to be $SU_c(3)$, where the subscript 'c' denotes the color quantum number. It contains quark fields, $\psi_i(x)$, and anti-quarks, $\overline{\psi}_i(x)$, that come in three different colors, $i = 1, 2, 3$, and transform in the fundamental representation of $SU_c(3)$. Indicating with ψ the colum vector of components ψ_i, one finds $\psi(x) \to \psi'(x) = U(\vec{\alpha}(x))\psi(x)$ where $U(\vec{\alpha}(x)) = \exp(i\vec{\alpha}(x) \cdot \vec{T})$ is an element of the $SU_c(3)$ group and $\vec{\alpha}(x)$ is the space-time-dependent angle and $\vec{T}$ is the short-hand notation for the eight generators T^a of $SU_c(3)$. Similarly, the anti-quarks transform in the conjugate representation of $SU(3)$. The gauge fields in this theory are called gluons ($A^a_\mu, a = 1, \ldots, 8$) and transform in the adjoint representation of $SU_c(3)$. Denoting $A_\mu = A^a_\mu T^a$, one finds that under $SU_c(3)$ A_μ transforms as $A_\mu \to A'_\mu = U(\vec{\alpha})(A_\mu - i/g_s \partial_\mu)U^\dagger(\vec{\alpha})$.

These transformations are gauge transformations. The gauge-invariant Lagrangian takes the following form:

$$\mathcal{L}_{\mathrm{QCD}} = \overline{\psi}_j \left(i\not{D}_{jk} - m\delta_{jk}\right) \psi_k - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \mathrm{G.F.} \tag{7}$$

where $\not{D} = \left(I - ig_s T^a A^a_\mu\right)$ and $F_{\mu\nu} = F^a_{\mu\nu} T^a = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_s \left[A_\mu, A_\nu\right]$. The parameter g_s in the gauge transformation as well as in the covariant derivative D_μ is the strong coupling constant and m is the mass of the quark field. While the gauge symmetry provides the right framework to understand the interaction between quarks and anti-quarks, in order to remove the two unphysical degrees of freedom associated with gluons a gauge-fixing term (G.F.) is introduced. We choose the Lorenz gauge, i.e., $-1/2\xi(\partial_\mu A^{\mu a})^2$. Within the framework of perturbation theory, this introduces additional scalar fields in the formalism. These scalar fields, unlike the standard scalar field operators, do not commute, but they anti-commute like fermions and hence they are called ghost fields. They are denoted by c^a (for ghost field operators) and $\overline{c}^a$ (anti-ghost field operators). Since the ghost fields are introduced to deal with unphysical degrees of freedom of the gluons, they are merely a mathematical construction and hence do not correspond to any real physical particle. In the Lagrangian, the terms that are bilinear in the fields describe their propagation:

$$\mathcal{L}_{\mathrm{K.E.}} = \overline{\psi}_i (i\not{\partial} - m)\psi_i - \frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^{\nu a})^2 + (\partial_\mu c^a)(\partial^\nu c^a) \quad . \tag{8}$$

The terms that describe the interactions of quark, anti-quark and gluon fields in the Lagrangian are found to contain the following terms:

$$\mathcal{L}_{\mathrm{int}} = g_s A^a_\mu \overline{\psi} \gamma^\mu T^a \psi - g_s f^{abc} (\partial_\mu A^a_\nu) A^{\mu b} A^{\nu c} - g_s^2 f^{eab} f^{ecd} A^a_\mu A^b_\nu A^{\mu c} A^{\nu d} \quad . \tag{9}$$

The first term describes the vectorial interaction of quarks and anti-quarks with gluons, the second and third terms describe self interactions of three and four gluons. These are present due to the non-Abelian nature of the underlying gauge group $SU_c(3)$. The G.F. part of the Lagrangian takes the following form in the Lorenz gauge:

$$L_{\mathrm{G.F.}} = \partial \overline{c}^a \partial^\nu c^a - g_s f^{abc} (\partial_\mu \overline{c}^a) A^{\mu b} c^c \quad . \tag{10}$$

The first term describes the propagation of ghost particles and the second term their interaction with the gauge fields.

4.1 Ultraviolet renormalisation

As mentioned previously, we use perturbation theory to compute various quantities from the Lagrangian and the strong coupling constant is assumed to be small enough to treat it as an expansion parameter. We use Feynman's diagrammatic approach throughout. Feynman rules for propagators of the fields and for the interaction terms are obtained from $\mathcal{L}_{\mathrm{K.E.}}$ and $\mathcal{L}_{\mathrm{int}}$, $\mathcal{L}_{\mathrm{G.F.}}$, respectively. The standard perturbative techniques along with the Feynman rules, n-point Green's functions of the fields, on-shell amplitudes etc. can be computed in powers of the coupling constant g_s. Explicit calculations show that the Green's

functions as well as the on-shell amplitudes diverge. These divergences come from loop integrals of virtual diagrams. When the momentum circulating in the loop approaches $\pm\infty$, the corresponding loop integral develops divergences. They are called ultra-violet (UV) divergences. Consider, for example, the quark propagator at one-loop level. It is easy to show that in 4 dimensions the Feynman integral that appears in the computation is proportional to a divergent integral of the form $\int d^4k(k^2(k-p)^2)^{-1}$, which behaves as $\log(k^2)$ for large k^2 and diverges when $k \to \pm\infty$. Here p is the momentum of the external quark propagator.

The presence of UV divergences in these quantities poses a serious problem beyond leading order in perturbation theory. This can be solved using the technique of renormalisation. To apply this technique, we first need to regularise the theory so that the Feynman integrals are rendered finite. While there exist several ways to regularise the theory, dimensional regularisation (DR) turns out to be the most elegant as well as convenient one. In DR, the space-time dimension is extended to n dimensions, where n is not only a continuous parameter but also a complex number. If we parametrise n as $n = 4 + \varepsilon$, then the integrals diverge in the limit $n \to 4$, or equivalently $\varepsilon \to 0$. In $4 + \varepsilon$ dimensions, the mass dimensions of the fields will start depending on the variable ε. In addition, the dimensionless coupling constant develops non-zero mass dimension in the regularised theory. We use an arbitrary mass μ to define a dimensionless coupling constant in the regularised theory: i.e., the coupling in n dimensions, denoted by $g_{s,n}$, can be written as $\hat{g}_s/\mu^{\varepsilon/2}$, where $\hat{g}_s$ is dimensionless. Here, μ is called regularisation scale. In addition, the computations involving contraction of Lorentz indices as well as traces of Dirac matrices are performed in n dimensions.

Having regularised the theory, the next step is to renormalise the fields, coupling and the mass by performing a set of transformations involving re-scaling, as explained in the following. We first denote quark, anti-quark, gluon and ghost fields in the regularised theory collectively by $\{\Phi_\alpha\}$. Then, the renormalised fields $\{\Phi_{\alpha,R}\}$ are defined by $\Phi_\alpha = Z_{\Phi_\alpha}^{\frac{1}{2}}(\mu_R^2)\,\Phi_{\alpha,R}(\mu_R^2)$. Similarly, the coupling constant is renormalised as $g_s/\mu^{\frac{\varepsilon}{2}} S_\varepsilon^{\frac{1}{2}} = Z_{a_s}^{\frac{1}{2}}(\mu_R^2)\, g_{s,R}(\mu_R^2)/\mu_R^{\frac{\varepsilon}{2}}$ and the mass is renormalised by Z_m. The Z's are called renormalisation constants. The constant $S_\varepsilon = \exp(\varepsilon(\gamma_E - \log(4\pi))$, with γ_E is Euler–Mascheroni constant. The re-scaling can be interpreted as a factorisation of the unregularised quantity into the renormalisation constant and the renormalised quantity. Hence, renormalisation is multiplicative in nature.

Note that there is an arbitrariness in defining the renormalisation constants. The scale μ_R is introduced to quantify the arbitrariness. In other words, the choice of μ_R uniquely defines the divergent part of the renormalised term. It is called renormalisation scale. Like the coupling constant, the mass m can also be expressed in terms m_R using the mass renormalisation constant.

Finally, one can substitute these relations in the Lagrangian to write it in terms of renormalised fields, coupling and mass and their renormalisation constants. One can easily show that the Lagrangian for $4 + \varepsilon$ dimensions takes the following form:

$$\mathcal{L}_{\rm QCD}(\Phi_\alpha, g_{s,n}, m_n, \mu, \varepsilon) = \mathcal{L}_R(\Phi_{\alpha,R}, g_{s,R}, m_R, \mu_R, \varepsilon) + \mathcal{L}_{\rm CT}(\Phi_{\alpha,R}, g_{s,R}, m_R, \{Z\}, \mu_R) \quad , \quad (11)$$

where the first term on the left hand side is identical to the original Lagrangian except that it contains only the renormalised fields, coupling and mass. The second term depends, in addition, on the renormalisation constants and is called counter term (CT).

Given this Lagrangian in terms of renormalised quantities and renormalisation constants, our next task is to obtain finite n-point Green's functions and on-shell amplitudes in the limit $\varepsilon \to 0$. This is possible provided that the renormalisation constants can be adjusted in such a way that they contain all the UV divergences at every order in perturbation theory, without any introduction of new renormalisation constants. These divergences show up as poles in ε in the limit $n \to 4$. While the renormalisation scale parameterises the scale at which divergences are factored into the renormalisation constant, the arbitrariness associated with the finite part of each constant remains, and can be fixed by defining a suitable scheme, called renormalisation scheme. We choose the $\overline{MS}$ scheme, in which we combine the Euler constant γ_E and S_ε with the poles in the renormalisation constants.

Given any scheme, the renormalisation scale plays an important role in the understanding of the underlying dynamics of the quantum field theory. The unrenormalised fields, coupling and mass or, equivalently, the corresponding Green's functions or on-shell amplitudes do not depend on the renormalisation scale. For example, the n-point unrenormalised Green's function $\langle 0|T\prod_{i=1}^{n}\ \Phi_i(p_i)|0\rangle$, where T is the time-ordering operator, satisfies $\mu_R^2 d/d\mu_R^2\langle 0|T\prod_{i=1}^{n}\ \Phi_i(p_i)|0\rangle = 0$, which leads to

$$\mu_R^2\frac{d}{d\mu_R^2}\ln\langle 0|T\prod_{i=1}^{n}\ \Phi_i(p_i)|0\rangle = -\mu_R^2\frac{d}{d\mu_R^2}\ln Z_\Phi(a_s(\mu_R^2),\varepsilon) = \Gamma_\Phi(a_s(\mu_R^2)) \quad , \tag{12}$$

where $a_s = g_s^2(\mu_R^2)/16\pi^2$ and $\Gamma_\Phi(a_s(\mu_R^2))$ is called the anomalous dimension. The above equation is called the renormalisation group equation (RGE). The solution to the RGE provides a relation among the Green's functions at different scales. In addition, using RGE and the fixed-order results for Γ_Φ and $a_s(\mu_R^2)$ one can systematically sum up the large logarithms of physical scale(s) to all orders in perturbation theory. In the following we show how the coupling constant can be renormalised in the $\overline{MS}$ scheme and its consequences. The unrenormalised $\hat{a}_s = g_s^2/16\pi^2$ is related to renormalised $a_s(\mu_R^2) = g_s^2(\mu_R^2)/16\pi^2$ through $\hat{a}_s = Z_{a_s}(a_s(\mu_R^2),\varepsilon)(\mu^2/\mu_R^2)^{\varepsilon/2}S_\varepsilon^{-1}a_s(\mu_R^2)$. The fact that $\mu_R^2(d/d\mu_R^2)\hat{a}_s = 0$ leads to

$$\mu_R^2\frac{d}{\mu_R^2}a_s(\mu_R^2) = \beta(a_s(\mu_R^2),\varepsilon) = -a_s(\mu_R^2)\mu_R^2\frac{d}{d\mu_r^2}\log Z_{a_s}(a_s(\mu_R^2),\varepsilon) - \frac{\varepsilon}{2}a_s(\mu_R^2) \quad . \tag{13}$$

With the knowledge of the renormalisation constant Z_{a_s}, we can determine $\beta(a_s(\mu_R^2),\varepsilon)$ order by order in a_s. In the limit $\varepsilon \to 0$, we find that $\beta_0(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 + \ldots$. To leading order, we have $\mu_R^2(d/d\mu_R^2)a_s(\mu_R^2) = -\beta_0 a_s^2(\mu_R^2) + \mathcal{O}(a_s^3)$. The solution to the RGE for a_s is found to be

$$a_s(\mu_f^2) = \frac{a_s(\mu_i^2)}{1 + a_s(\mu_i^2)\beta_0\log\left(\frac{\mu_f^2}{\mu_i^2}\right)} + \mathcal{O}(a_s^2) \quad , \tag{14}$$

where μ_i and μ_f are initial and final scales. The constant β_0 in QCD is found out to be $\beta_0 = 11/3\ C_A - 2/3\ n_f$, where C_A is the Casimir of SU(N). For QCD, $C_A = 3$. n_f is the number flavours of quarks. Because $\beta_0 > 0$, one sees that the coupling constant falls off as the scale increases. This implies that at high energies quarks (anti-quarks), gluons and ghost particles will behave like free particles. This phenomenon is called asymptotic freedom. This property allows one to use perturbative techniques to study strong interaction at high energies. At low energies, we find that particles confine to form hadrons

through the phenomenon called confinement.

4.2 QCD improved parton model

Our next task is to apply perturbative QCD (pQCD) to study high-energy scattering processes involving hadrons. We consider the DIS process discussed already using the parton model. Recall that in the parton model the cross section can be expressed in terms of two structure functions F_i, with $i = 1, 2$, which are related to tree-level scattering cross sections of leptons off the quarks as well as the anti-quarks, convoluted with the PDFs of the respective quark or anti-quark. This is called the leading-order prediction. This result will get modified if the strong interaction dynamics are included. In pQCD, the parton-level cross section $\hat{\sigma}_a(\varepsilon)$ admits the expansion in powers of the strong coupling constant a_s: $\hat{\sigma}_a(\varepsilon) = \sum_{i=0}^{\infty} a_s^i(\mu_R^2)\hat{\sigma}_a^{(i)}(a_s(\mu_R^2), \varepsilon)$. Here the subscript a can be q , $\overline{q}$, or g. The perturbative corrections resulting from the order-a_s term are called next-to-leading-order contributions; similarly, those from the a_s^2 term are called next-to-next-to-LO (NNLO) corrections, and so on. At a_s order we encounter scattering processes, such as one-loop corrections to $e^- + q(p) \to e^- + p(p')$, and real emission processes, namely $e^- + q(p)(\overline{q}(p)) \to e^- + q(p')(\overline{q}(p')) + g(k)$ and $e^- + g(p) \to q(p_1) + \overline{q}(p_2)$. These contributions are UV finite. However, they are sensitive to infrared (IR) divergences in 4 space-time dimensions. There are two types of IR divergences: soft and collinear. The soft divergences show up in both real and virtual diagrams when the momentum of the gluon vanishes, i.e., $k_i^\mu \to 0$. Collinear divergences arise whenever two massless partons become collinear to each other.

We illustrate below the origin of soft and collinear divergences at a_s level. Let us begin with pure virtual contributions to quark/anti-quark initiated processes. We find that there will be QCD corrections to the quark–anti-quark–photon vertex as well as self-energy corrections to the quark/anti-quark legs. Let us consider, for example, the vertex correction within the dimensionally regularised set up. If we restrict ourselves to the region where the momentum of the gluon approaches zero, the leading contribution results from the integral

$$\int d^n k \frac{1}{k^2(k+p)^2(k+p')^2} \quad , \tag{15}$$

where k is the loop momentum and p and p' are the momenta of the incoming quark and the outgoing quark, respectively. Note that the above integral diverges in 4 dimensions when $k \to 0$. In addition, we observe that, due to the presence of the propagators $1/(k+p)^2$ and $1/(k+p')^2$, we encounter collinear divergences. For example when k is parallel to p or p' and in the centre of mass frame of the quarks, in 4 dimensions, the angular part of the integral, namely $\int_{-1}^{1} d\cos(\theta) 1/(1 \pm \cos(\theta))$, diverges. We observe that in $4 + \varepsilon$ dimensions the soft and collinear divergences appear as poles in ε. Often, there will be configurations in which both soft and collinear divergences appear together, giving rise to double-pole terms in ε.

Like virtual contributions, the real emission processes also develop soft and collinear divergences through the phase space integrals [6]. For example, the one corresponding to the parton-level process $e^- + q(\overline{q}) \to q(\overline{q}) + g + g$ contains an integral of the form $\int d^{n-1}k/(2k^0) \int d^{n-1}p'/(2p^0)(k-p)^{-2}(k+p')^{-2}\delta^{(4)}(q+p-p'-k)$. This integral diverges when $k \to 0$; similarly, when k is parallel to p or p' it gives rise to collinear divergences in $n = 4$ dimensions and develops poles in ε when $\varepsilon \to 0$. We

also find configurations containing both soft and collinear divergences simultaneously. They are, again, represented by double-pole terms in ε.

According to to the Kinoshita–Lee–Nauenberg (KLN) theorem, the soft divergences cancel between pure virtual corrections and those with at least one real parton emission. Similarly, the final-state collinear divergences also cancel among themselves when they are summed up. However, those resulting from configurations involving an initial-state parton do not vanish. According to the KLN theorem, if we sum over degenerate states the resulting cross section will be free of collinear divergences. In DIS, as we do not integrate inclusively over the momentum of the initial-state parton, the collinear singularities arising between initial- and final-state partons remain at every perturbative order.

In summary, the higher-order contributions to parton-level subprocesses always develop initial-state collinear divergences due to the presence of massless partons. This is consistent with the KLN theorem and hence not unexpected. In the following we discuss how these initial-state collinear divergences go away when we perform a sum over the momenta and quantum numbers of initial-state partons, as we expect from the KLN theorem. We demonstrate this using the mass factorisation theorem.

The mass factorisation theorem encapsulates the factorisation properties of parton-level cross sections that develop initial-state collinear divergences due to the presence of massless partons. As per the theorem, the collinear divergences factor out from the parton-level process in a way that only depends on the state of the incoming parton before it scatters off the electron state. The result depends on how the initial-state parton undergoes the QCD dynamics to become another parton that eventually scatters with the electron. According to the factorisation theorem, we can express in DIS a generic parton-level subprocess involving a parton type "a" as

$$\frac{\hat{\sigma}_a(z,Q^2,\varepsilon)}{z} = \hat{\sigma}^{(0)}(Q^2) \sum_{b=q,\bar{q},g} \int_z^1 \frac{dz_1}{z_1} \frac{\Delta_b\left(\frac{z}{z_1},Q^2,\mu_F^2,\varepsilon\right)}{(z/z_1)} \Gamma_{ba}(z_1,\mu_F^2,\varepsilon) \quad . \tag{16}$$

Here, Δ_a is called coefficient function, which is finite when $\varepsilon \to 0$, and Γ_{ba} is called the Altarelli–Parisi kernel, which contains the collinear divergences present in $\hat{\sigma}_a$. Note that the collinear divergences are factored out from $\hat{\sigma}_a$ at the scale μ_F^2, called factorisation scale. Γ_{ba} in the $\overline{MS}$ scheme only contains the poles in ε. Since the collinear divergences are purely due to the QCD dynamics among the partons, they are process independent. In other words, the kernels $\Gamma_{ba}(z,\mu_F^2,\varepsilon)$ do not depend on the interaction of the partons with the leptons. Both Δ_a and Γ_{ba} are normalised in such a way that they can be expanded in powers of $a_s(\mu_F^2)$: $X = \sum_{i=0}^{\infty} a_s^i(\mu_F^2) X^{(i)}(z,Q^2,\mu_F^2,\varepsilon)$, where $X = \Delta_a, \Gamma_{ba}$, with $X^{(0)} = \delta(1-z)$.

In the above expression, we have arranged the integrand in such a way that the integral is a Mellin convolution of two functions, namely $\Delta_a(x)/x$ and $f_b(x)$. The Mellin convolution of "n" functions $f_1(x), f_2(x) \ldots f_n(x)$ is defined by

$$f_1(x) \otimes f_2(x) \otimes \ldots \otimes f_n(x) = \prod_{i=1}^{n} \left(\int_0^1 dx_i f_i(x_i) \right) \delta\left(x - \prod_{i=1}^{n} x_i \right) \quad . \tag{17}$$

Note that the convolution is symmetric under interchange of the functions. In addition, under Mellin transformation the above convolution results in a simple product of Mellin moments of $f_i(x)$. If we

define the Mellin transformation by $f_N = \int_0^1 dx x^{N-1} f(x)$, then

$$\int_0^1 dx x^{N-1} f_1(x) \otimes f_2(x) \otimes \ldots \otimes f_n(x) = \prod_{i=1}^{n} f_{i,N} \quad . \tag{18}$$

Note that the right hand side contains simple products of Mellin moments of the functions $f_i(x_i)$.

In the notation of convolutions, the mass factorisation formula reads as

$$\frac{\hat{\sigma}_a(z, Q^2, \varepsilon)}{z} = \hat{\sigma}_0(Q^2, \varepsilon) \sum_{b=q,\overline{q},g} \frac{\Delta_b(z, Q^2, \mu_F^2, \varepsilon)}{z} \otimes \Gamma_{ba}(z, \mu_F^2, \varepsilon) \quad . \tag{19}$$

Expressing the parton model result for DIS in the convolution form and substituting the mass factorised result, we obtain

$$\frac{1}{x_{Bj}} d\sigma(x_{Bj}, Q^2) = \hat{\sigma}_0(Q^2, \varepsilon) \sum_{a,b=q,\overline{q},g} \hat{f}_a(x_{Bj}) \otimes \Gamma_{ba}(x_{Bj}, \mu_F^2, \varepsilon) \otimes \frac{1}{x_{Bj}} \Delta_b(x_{Bj}, Q^2, \mu_F^2, \varepsilon) \, . \tag{20}$$

Since the left hand side is finite, one expects that the convolution of $\hat{f}_a$ and Γ_{ba} should be finite. Since the convolution between f_a and Γ_{ba} sums up the initial state configurations, the collinear divergences contained in Γ_{ba} cancel against those in $\hat{f}_a$ in accordance with the KLN theorem. Hence, we can relate their convolution to a finite function, $f_b(z, \mu_F^2)$, as

$$f_b(z, \mu_F^2) = \sum_{a=q,\overline{q},g} \hat{f}_a(z) \otimes \Gamma_{ba}(z, \mu_F^2, \varepsilon) \quad . \tag{21}$$

We call f_b the mass-factorised parton distribution function, which is defined at the factorisation scale μ_F^2 and is finite when $\varepsilon \to 0$. In terms of f_b, the hadronic cross section reads

$$\frac{1}{x_{Bj}} d\sigma(x_{Bj}, Q^2) = \hat{\sigma}^{(0)}(Q^2) \sum_{a,b=q,\overline{q},g} f_a(x_{Bj}, \mu_F^2) \otimes \frac{1}{x_{Bj}} \Delta_b(x_{Bj}, Q^2, \mu_F^2) \quad . \tag{22}$$

The fact that $\hat{f}_a$ is independent of μ_F^2 leads to the renormalisation group equation in the infrared (collinear) sector of QCD:

$$\mu_F^2 \frac{d}{d\mu_F^2} f_a(z, \mu_F^2) = \frac{1}{2} P_{ab}(z, \mu_F^2) \otimes f_b(z, \mu_F^2) \quad , \tag{23}$$

where P_{ab} is the matrix element of $P(z, \mu_F^2) = \mu_F^2 d \log \Gamma(z, \mu_F^2, \varepsilon)/d\mu_F^2$ are finite and are called Altarelli–Parisi (AP) splitting functions. They are computable in perturbative QCD as: $P = \sum_{i=0}^{\infty} a_s^i(\mu_F^2) P^{(i)}(z)$. Few comments are in order: while f_b and Δ_b depend on the scale μ_F, the convolution of them is independent of μ_F, provided that the AP splitting functions P and the coefficient functions Δ_a are known to all orders in $a_s(\mu_F^2)$. Since these perturbative results are known only to few orders in a_s, the predictions will always be sensitive to μ_F. However, by varying the scale around the hard scale Q^2, we can estimate the theoretical uncertainty due to the truncation of the perturbative series. The other source of theoretical uncertainty is from ultraviolet renormalisation. Note that Δ_b is computable in perturbative QCD as a power series in $a_s(\mu_R^2)$. While Δ_b is μ_R-independent, the truncated

result will be sensitive to μ_R. Like μ_F, we can vary μ_R to estimate the error resulting from the truncation of the perturbative series of Δ_b. In addition to these dependences, the predictions will depend on the hard scale Q^2 even if the series is summed to all orders. Explicit calculations reveal that the Q^2 dependence is through certain logarithms whose coefficients are controlled by the β function as well as by AP splitting functions. Due to the presence of these logarithms, the hadronic cross section will depend on the hard scale Q^2, leading to a violation of the Björken scaling. The fact that the coefficients of these $\log(Q^2)$ terms, namely beta and the AP splitting functions, are computable order by order in a_s, one can predict the exact dependence of the hadronic cross section on Q^2. Remarkably, precise measurements of DIS cross sections at various Q^2 values confirm the predictions of perturbative QCD.

4.3 Threshold resummation

So far, we studied the factorisation of collinear divergences in the partonic cross sections and their universal/process-independent structure in terms of β and AP splitting functions. Our next task is to study the factorisation properties of $\hat{\sigma}_a$ in the threshold limit [8]. The threshold limit in DIS is defined by the limit when $z \to 1$. We restrict ourselves to quark or anti-quark initiated processes. The threshold-enhanced terms in the mass factorised cross sections $\Delta_{q(\overline{q})}$ take the following form:

$$\Delta_{q(\overline{q})}(z, a_s) = \Delta^{\delta}_{q(\overline{q})}(a_s)\,\delta(1-z) + \sum_{j=0}^{\infty}\Bigg[\Delta^{\mathcal{D}}_{q(\overline{q}),\,j}(a_s)\;\mathcal{D}_j(z) + \Delta^{\log z}_{q(\overline{q}),\,j}(a_s)\log^j(1-z)\Bigg] + \Delta^{\rm bNSV}_{q(\overline{q})}(z) \quad . \tag{24}$$

The upperscript labels δ and $\mathcal{D}$ indicate the terms proportional to $\delta(1-z)$ and to the distributions $\mathcal{D}_j$ (defined below). Both terms, which are the leading ones, are called "soft-plus-virtual" (SV). The label $\log z$ indicates the "next-to-SV" (NSV) terms, proportional to $\log^j(1-z)$, while the last term describes the remaining "beyond NSV" (bNSV) processes. The $\mathcal{D}$ terms contain the "+" distributions $\mathcal{D}_j(z) = (\log^j(1-z)/(1-z))_+$, defined by

$$\int_0^1 dz \mathcal{D}_j(z) f(z) = \int_0^1 dz \left(\frac{\log^j(1-z)}{1-z}\right)\Big(f(z) - f(1)\Big) \quad . \tag{25}$$

In the threshold region, these distributions can become dominant. In addition, at a given order "n" in a_s, these distributions will be as big as the inverse of $1/a_s^n$, resulting in order-one terms of the form $a_s\beta_0 \log \overline{N}$. These terms can spoil the reliability of the perturbative approach in the threshold region. The solution to this problem is to sum up these order-one terms in a systematic fashion, so that the modified perturbative expansion provides reliable predictions. This was achieved independently by Sterman and by Catani and Trentedue in the Mellin "N" space. The result takes the form

$$\log \Delta^{\rm res}_{q(\overline{q}),N}(a_s) \;\; = \;\; \log \tilde{g}_0(a_s) + \log(N) g_1(w) + \sum_{i=0}^{\infty} a_s^i g_{i+2}(w) \quad , \tag{26}$$

where $w = a_s\beta_0 \log(\overline{N})$. In the above equation g_0 is independent of N and they result from $\delta(1-z)$ terms in the threshold limit, while $g_i(w)$ are from $\mathcal{D}_j(z)$ terms. The SV limit in the N space corresponds

to taking the large-N limit. In the large-N limit, being a_s small, w becomes order-one and, hence, we need to rearrange the perturbative series in such a way that the sum over "w" terms is performed to all orders, as in the above equation.

In the following, see Ref. [7], we study the all-order perturbative structure of the coefficient function in terms of Q^2 and z, by setting up a Sudakov-type differential equation in the kinematic region where z is closer to the threshold limit $z = 1$. We begin with the mass factorisation of $\hat{\sigma}_q$ and restrict ourselves only to $\mathcal{D}$ and $\log z$ terms in $\hat{\sigma}_q$ and Γ_{bq}, with $a, b = q, \overline{q}, g$; then we find that in Γ_{bq} only $b = q$ will contribute. Note that Γ_{gq} does not contain any "+" distributions or terms that can lead to SV or NSV terms for Δ_q in the limit $z \to 1$. Similar arguments can easily convince one that Δ_g does not contain any SV terms. You will recall that the partonic cross section $\hat{\sigma}_q$ beyond leading order gets contributions from processes of three different sources: pure virtual, pure real emissions and real emission-virtual together. The pure virtual contribution to $\hat{\sigma}_q$ is found to be $|\hat{F}_q(\hat{a}_s, \mu^2, Q^2, \varepsilon)|^2$ where F_q is nothing but the form factor of the vector-boson–quark–anti-quark vertex. We factor out $|\hat{F}_q|^2$ from $\hat{\sigma}_q$ and define the "jet function" $\mathcal{S}_{J,q}$ as the following quotient:

$$\mathcal{S}_{J,q}(\hat{a}_s, \mu^2, q^2, z, \varepsilon) = |\hat{F}_q(\hat{a}_s, \mu^2, Q^2, \varepsilon)|^{-2} \delta(1-z) \otimes \hat{\sigma}^{\mathrm{SV+NSV}}(q^2, z, \varepsilon) \quad . \tag{27}$$

Note that $\mathcal{S}_{J,q}$ is computable order by order in a_s and is also renormalisation group invariant with respect to the scale μ_R. A great deal of understanding is provided about the infrared and UV structure of the form factors (FF) by the Sudakov "$K + G$" equation (see below), and about the AP kernels by the AP evolution equation in terms of universal anomalous dimensions. The factorisation of the IR singularity in a form factor implies that $\hat{F}_q(Q^2) = Z_{\hat{F}_q}(Q^2, \mu_s^2)\ F_{q,\mathrm{fin}}(Q^2, \mu_s^2)$, where $Z_{\hat{F}_q}$ is IR singular, while $F_{q,\mathrm{fin}}$ is IR finite, and the scale μ_s is the IR factorisation scale. The peculiar IR singularity structure of $Z_{\hat{F}_q}$ implies that the kernel defined by $K_q = 2d \log Z_{\hat{F}_q}/d \log Q^2$ is independent of Q^2 and contains only IR poles in ε, while $G_q = 2d \log \hat{F}_q/d \log Q^2$ is finite, as well as dependent on Q^2. This implies that $\hat{F}_q$ satisfies the $K + G$ equation, namely $d \log \hat{F}_q/d \log Q^2 = K_q(\mu_s^2, \varepsilon) + G_q(Q^2, \mu_s^2, \varepsilon)$. The solution to the $K + G$ equation is given by

$$\hat{F}_q(Q^2, \varepsilon) = \exp\left(\int_0^{Q^2} \frac{d\lambda^2}{\lambda^2} \Gamma_{\hat{F}_q}(\lambda^2, \varepsilon)\right) \quad , \tag{28}$$

where $\hat{F}_q(Q^2 = 0, \varepsilon) = 1$ and $\Gamma_{\hat{F}_q} = (K_q + G_q)/2$. In terms of $\mathcal{S}_J$, the mass-factorised cross section reads as

$$\Delta_q(z, Q^2, \varepsilon) = |\hat{F}_q(Q^2, \varepsilon)|^2 \delta(1-z) \otimes \mathcal{S}_{J,q}(q^2, z, \varepsilon) \otimes \Gamma_{qq}^{-1}(z, \mu_F^2, \varepsilon) \quad . \tag{29}$$

Differentiating the above equation with respect to $\log Q^2$ and using the $K+G$ equation for $\hat{F}_q$, we obtain a $(K + G)$-like equation for $\mathcal{S}_{J,q}$:

$$q^2 \frac{d\mathcal{S}_{J,q}}{dq^2} = \Gamma_{S_{J,q}}(q^2, z) \otimes \mathcal{S}_{J,q}(z, q^2) \quad , \tag{30}$$

where $\Gamma_{S_{J,q}} = q^2 d\Delta_q/dq^2 \Big(\mathcal{C} \log \Delta_q - \log |\hat{F}_q|^2 \delta(1-z)\Big)$. Here $\mathcal{C}$ means that in the perturbative ex-

pansion of $\log \Delta_q$ in powers of a_s, the product of z dependent functions should be understand as convolution of z dependent functions. We find that $\Gamma_{S_{J,q}}$ admits a remarkable structure, namely, it can be written as the sum of a q^2-independent IR-divergent term and a q^2-dependent IR-finite term. If we identify the former as a "K-type" term and the latter as a "G-type" term, then the jet function $\mathcal{S}_{J,q}$ does satisfy a $(K+G)$-type equation. This implies that the jet function $\mathcal{S}_{J,q}$ can be factorized into an IR-divergent renormalisation constant $Z_q(z,q^2,\mu_s^2,\varepsilon)$ and a finite quotient $\mathcal{S}_{J,q,\mathrm{fin}}(z,q^2,\mu_s^2,\varepsilon)$, where μ_s is the factorisation scale. That is, $\mathcal{S}_{J,q}(z,q^2,\varepsilon) = Z_q(z,q^2,\mu_s^2,\varepsilon)\ \mathcal{S}_{J,q,\mathrm{fin}}(z,q^2,\mu_s^2,\varepsilon)$. The solution to the above differential equation takes the following form:

$$\mathcal{S}_{J,q}(q^2,z,\varepsilon) = \mathcal{C}\exp\left(\int_0^{q^2}\frac{d\lambda^2}{\lambda^2}\Gamma_{\mathcal{S}_{J,q}}(\lambda^2,z,\varepsilon)\right) \quad . \tag{31}$$

The general structure of the exponent can be derived from the explicit perturbative results for the mass factorisation coefficient function Δ_q, the form factor $\hat{F}_q$ and the AP factorisation kernel Γ_{qq}, the latter being known to the third order in a_s. In particular, the divergent part of the jet function can be determined by noting that $\mathcal{S}_{J,q}$ should contain the right singularities to cancel those from the form factor and the AP kernel. You will recall that the singularity structures of the form factor and the AP kernel are controlled by universal anomalous dimensions such as A_q, B_q, f_q, C_q,D_q and the β function of QCD. We also observe that in dimensional regularisation both the form factor and Δ_q show certain structures related to transcendentality at every order in perturbation theory. Using the fact that Δ_q is finite and its transcendentality structure, we propose a solution for the jet function $\mathcal{S}_{J,q}$ to all orders:

$$\log \mathcal{S}_{J,q} = \sum_{i=1}^{\infty}\hat{a}_s^i\left(\frac{q^2(1-z)}{\mu^2}\right)^{i\frac{\varepsilon}{2}} S_\varepsilon^i\left(\frac{i\varepsilon}{2(1-z)}\right)\left(\hat{\Phi}_q^{\mathrm{SV,(i)}}(\varepsilon)+\frac{2}{i\varepsilon}(1-z)\,\hat{\varphi}_q^{(i)}(z,\varepsilon)\right). \tag{32}$$

In the above equation, $\hat{P}hi^{\mathrm{SV}}(\varepsilon)$ encodes all contributions from the pure distributons, while $\hat{\varphi}_q^{(i)}(z,\varepsilon)$ encodes z dependent next-to-SV terms. The AP kernels Γ_{qq} satisfy the AP evolution equation and, in the approximation we work with, they are controlled only by the diagonal AP slitting functions P_{qq}. Hence, the all-order solution takes the simple form:

$$\Gamma_{qq}(\mu_F^2,z,\varepsilon) = \mathcal{C}\exp\left(\frac{1}{2}\int_0^{\mu_F^2}\frac{d\lambda^2}{\lambda^2}P_{qq}(\lambda^2,z,\varepsilon)\right). \tag{33}$$

The AP splitting function is known to the third order in perturbation theory and the SV distributions and NSV logarithms present in them are controlled by universal cusp and collinear anomalous dimensions. Putting all of them together we obtain

$$\Delta_q(Q^2,z,\varepsilon) = \mathcal{C}\exp\left(\int_{\mu_F^2}^{Q^2(1-z)}\frac{d\lambda^2}{\lambda^2}P'_{qq}(a_s(\lambda^2),z)+Q_q(a_s(Q^2(1-z)),z)\right) \quad , \tag{34}$$

where

$$
\begin{aligned}
Q_q \;=& \left(\frac{1}{2(1-z)} \overline{G}^{\rm SV}_{q,J} \left(a_s(Q^2(1-z)) \right) \right)_+ + \overline{\varphi}_{f,q}(a_s(Q^2(1-z)), z) \\
& + \ln C_{0,q}(a_s(\mu_R^2), Q^2, \mu_R^2, \mu_F^2) \quad .
\end{aligned}
\tag{35}
$$

In the above equation, we have decomposed Q_q in terms of pure SV and z dependent contributions denoted by $\overline{G}^{\rm SV}_{q,J}$, next to SV $\overline{\varphi}_{f,q}$ and z independent matching term $C_{0,q}$. Expanding the exponent in powers of a_s, we can obtain both SV and NSV terms. For example, if we know the exponent to order a_s, the expansion of the exponential will provide leading SV terms $(\mathcal{D}_3, \mathcal{D}_2), (\mathcal{D}_5, \mathcal{D}_4), \ldots, (\mathcal{D}_{2i-1}, \mathcal{D}_{2i-2})$ and leading NSV terms $\log^3(1-z), \log^5(1-z), \ldots, \log^{2i-1}(1-z)$ at $a_s^2, a_s^3, \ldots, a_s^i$, respectively, for all i. Similarly, $(\mathcal{D}_2, \mathcal{D}_1), (\mathcal{D}_4, \mathcal{D}_3), \ldots, (\mathcal{D}_{2i-3}, \mathcal{D}_{2i-4})$ and leading NSV terms $\log^4(1-z), \log^6(1-z), \ldots, \log^{2i-2}(1-z)$ at $a_s^3, a_s^4, \ldots, a_s^i$, respectively, for all i. This can be generalised for an arbitrary order in a_s for the exponent. In Mellin space, after reorganising the exponent according to the logarithmic accuracy we obtain

$$
\begin{aligned}
\log \Delta_q(Q^2, N, \varepsilon) \;=& \; \log g_0^q(a_s(\mu_R^2)) + \tilde{g}_1^q(\omega) \log N + \sum_{i=0}^{\infty} a_s^i(\mu_R^2) \tilde{g}_{i+2}^q(\omega) \\
& + \; \frac{1}{N} \sum_{i=0}^{\infty} a_s(\mu_R^2) h_i^q(\omega, N) \quad ,
\end{aligned}
\tag{36}
$$

where

$$
\tilde{g}_i^q(\omega) = g_i^q(\omega) + \frac{1}{N} \overline{g}_i^q(\omega), \qquad \omega = a_s(\mu_R^2) \beta_0 \log N \tag{37}
$$

and

$$
h_0^q(\omega, N) = h_{00}^q(\omega) + h_{01}^q(\omega) \log N, \qquad h_i^q(\omega, N) = \sum_{i=0}^{\infty} h_{ik}^q \log N \quad . \tag{38}
$$

In the above equation, the result of the Mellin integrals are decomposed into z dependent SV, NSV terms and z indpendent matching terms. Using the above equation, one can predict resummed contributions to leading logarithmic (LL) accuracy, next-to-leading logarithmic accuracy etc. in a systematic fashion for the inclusive cross section at various Q^2 values.

In summary, we find that the perturbative results not only help us to make precise predictions from the theory but also unravel universal structures of the theory. The comparison of the predictions against experimental observations can put the theory on firm footing. In addition, they can put stringent bounds on the parameters of physics beyond the SM. Similarly, understanding the UV and IR structures of the theory can provide ingredients to sum up potentially important contributions from all orders and also shed light on the power corrections. For example, the resummation of threshold and next-to-threshold logarithms was possible due to the universal structure in the perturbative predictions.

4.4 Multi-leg and multi-loop Feynman diagrams

In the following we will discuss various methods of obtaining perturbative results [9]. Let us restrict ourselves to the computation of scattering cross sections at hadron colliders. The task reduces to writing down on-shell scattering amplitudes, squaring their moduli, performing loop integrals and then phase-space integrals, taking into account the constraints from experiments. One begins with setting up Feynman rules and then using the rules to write down the amplitudes. Often, one finds that the number of Feynman diagrams becomes very large as we increase the number of legs or the order of the perturbative expansion. For example, the number of Feynman diagrams is four for the tree-level process $g+g \to g+g$, while for $g+g \to 5g$ it becomes 2485, and 10525900 for $g+g \to 8g$. These numbers will increase if we include loop corrections to the tree-level amplitudes. One often resorts to computer programs to generate these diagrams and obtain the corresponding amplitudes in the analytical form. Packages such as FeynArts and QGRAF are found to be very useful for this purpose. Next, we need to compute the square of the modulus of the total amplitude. Note that the amplitudes are made up of Dirac spinors, $u_i(p,s), v_i(p,s), \overline{u}_i(p,s), \overline{v}_i(p,s)$, chains of Dirac matrices, polarisation vectors of gauge fields, $\epsilon_\mu(q,\lambda), \epsilon^*_\nu(q,\lambda)$. Here, p is the momentum, s is the spin of the Dirac particle, while q is the momentum and λ is the polarisation of the gauge field. In addition, the amplitude will contain chains of Dirac matrices and generators and structure constants of the $SU(N)$ group. Note that the index i in the Dirac spinor is due to $SU(N)$. The modulus of the sum of the amplitudes involves the computation of a large number of traces of Dirac matrices and also the simplification of the $SU(N)$ generators and structure constants. Again, one can set up computer codes to perform this task, if we have a small number of amplitudes. For processes with larger numbers of amplitudes, it is desirable to simplify the amplitudes so that the evaluation of their moduli is manageable. One can make the simplification at the amplitude level if the properties of Dirac spinors and the freedom of gauge choice are exploited. The documentation of a powerful technique using helicity amplitudes can be found in the celebrated book "The Ubiquitous photon: Helicity method for QED and QCD" by R. Gastmans and T.T. Wu. The modern versions of this approach provide a set up suitable for faster computer codes. For example, one defines simpler notations for helicity amplitudes, namely $(1\pm\gamma_5)u(k_i) \equiv u_\pm(k_i)$, $(1\pm\gamma_5)v(k_i) \equiv v_\mp(k_i)$ and $\overline{u}_\pm(k_i) = \overline{v}_\mp(k_i) \equiv \langle i^\pm| \equiv \langle k_i^\pm|$. Furthermore, we can define $|i\rangle = |i^+\rangle$ and $|i] = |i^-\rangle$ and obtain $\overline{u}_-(k_i)u_+(k_j) = \langle ijj\rangle$ and $\overline{u}_+(k_i)u_-(k_j) = [ij]$. In addition, the Gordon identity $[p\gamma^\mu p\rangle = 2p^\mu$, the Fierz identity $\langle p\gamma^\mu q|r\gamma_\mu s\rangle = 2\langle ps\rangle[rq]$ and the Schouten identity $\langle pq\rangle\langle rs\rangle + \langle pr\rangle\langle sq\rangle + \langle ps\rangle\langle qr\rangle = 0$ can be used to simplicity the expressions at the amplitude level. For gluon polarisations, one uses $\epsilon^+_\mu(p,q) = \langle q|\gamma_\mu p]/(\sqrt{2}\langle qp\rangle)$ and $\epsilon^-_\mu(p,q) = -[q|\gamma_\mu|p\rangle/(\sqrt{2}[qp])$ with $p_\mu\epsilon^\mu = 0$ and q being any light-like vector.

Like Dirac spinors and gamma matrices, the generators and structure constants of the $SU(N)$ group in the vertices of the amplitudes pose additional complexity. However, a remarkable simplification is achieved by stripping them off from each amplitude. Using the $SU(N)$ algebra $[T^a, T^b] = if^{abc}T^c$, or, equivalently, $if^{abc} = 2(Tr(T^aT^bT^c) - Tr(T^bT^aT^c))$, one can replace all the if^{abc} terms by the latter identity, to obtain a color-stripped amplitude. One finds a tree-level amplitude involving n gluons,

$$\mathcal{A}_n^{(0)}(g_1, g_2, \ldots, g_n) = g_s^{n-2} \sum_{\alpha\in S_n/Z_n} 2Tr(T^{a_{\sigma(1)}}\ldots T^{a_{\sigma(n)}})A_n^{(0)}(g_{\sigma(1)}, ..., g_{\sigma(n)}) \quad , \tag{39}$$

where $A_n^{(0)}$ are called partial amplitudes. They do not contain any $SU(N)$ factors and, in addition, each of them is gauge invariant. By repeatedly applying the trace identity one can simplify the original Feynman rules to obtain a new set of Feynman rules for the partial amplitudes. The advantage of this approach is the reduction of the number of independent amplitudes. For example, one gets 12925 amplitudes instead of 10525900. Several approaches were developed in order to improve the speed of the computation. For example, using off-shell currents, Berends and Giele constructed recursion relations, which not only give enormous simplifications, but also reduce the computation time significantly. Thanks to these approaches, the computation of tree level amplitudes is now an accessible task. In addition, the results of certain amplitudes show remarkable simplifications. For example, a certain class of amplitudes, called Parke–Taylor amplitudes, describing n gluons with specific polarisation assignments, reduces to a single term. The mysterious simplification in the Parke–Taylor amplitudes was explained by Britto, Cachazo, Feng and Witten (BCFW) through their recursion relations, which use Cauchy's residue theorem together with the analytical structure of the $SU(N)$ gauge theory amplitudes. Further progress was made by working in twistor space. In addition, there are also efforts to apply some of these methods to amplitudes with loops. Finally, it is worth mentioning that, while all these approaches provide enormous simplification, as well as insights in the theory, each one presents disadvantages when dealing with amplitudes having a large number of legs.

Having obtained the amplitudes, the next task is to perform Feynman loop and phase space integrals [10]. As we had seen in the lectures, the Feynman integrals are sensitive to UV and IR divergences. We use dimensional regularisation regulate them and then proceed to compute them. Often we need to deal with large number of multi loop and many-body phase space integrals, each of them is highly complicated to solve. The standard text book methods do not work and hence one resorts to alternate ones. We present two important developments that revolutionized the computation of Feynman diagrams. Note that loop integrals and phase integrals differ as the later contain delta functions from on-shell external legs. Replacing the delta functions by the imaginary part of the corresponding propagator, we can evaluate phase space integrals the way we evaluate loop integrals. We relate each $\delta(p_i^2 - m_j^2)$ with the imaginary part of $1/(p_i^2 - m_j^2 + i\epsilon)$ where ϵ is infinitesimally small and positive number in the Feynman prescription. This approach is called the method of reverse unitarity. The method of integration by parts (IBP) identities reduces a set of large number of Feynman integrals to fewer integrals, called master integrals (MI). The method of differential equations (DE) solves the MIs in a most efficient way. The results of these master integrals can often be expressed in terms of certain class of special functions namely classical polylogarithms, multiple polylogarithms, Nielsen integrals, generalised polylogarithms or Goncherov polylogrithms or Chen integrated integrals. One finds cases where more complicated integrals such as elliptic integrals. We give brief account on both IBP and DE in the following.

The typical L loop Feynman integral in n space time dimensions with $p_j, j = 1, \cdots, n_e$ external momenta takes the following form

$$\int \Pi_{i=1}^{L} d^n l_i \frac{\mathcal{N}(l_i, p_j)}{D_1^{\alpha_1} \cdots D_M^{\alpha_M}} \tag{40}$$

where D_i are propagators involving the momenta $\{l_i\}$ and $\{p_j\}$ and masses $\{m_k\}$. The number of scalar products here at the most is $L(n_e + L/2 - 1/2)$. Beyond one loop, this number is always greater than M

and hence we can introduce auxiliary propagators so that the extra scalar products are expressed in terms of them. This way we can express any loop integral in terms of the expanded set of N propagators where the exponent of the propagator can be negative. Since the loop integrals are invariant under the shift of the loop momenta namely $l_i \to l_i + \sum_k c_k p_k$ for some constants c_k, we can relate many integrals and reduce number of integrals that we deal with. Similarly, integrals are invariant under Lorentz transformation of external momenta, i.e $p_j^\mu \to p_j^\mu + \omega^{\mu\nu} p_{j,\nu}$, giving

$$\omega^{\mu\nu} \sum_j^{n_e} p_{j,\nu} \frac{\partial}{\partial p_j^\mu} \int \prod_{i=1}^{L} d^n l_i \frac{1}{D_1^{\alpha_1} \cdots D_N^{\alpha_N}} = 0 \tag{41}$$

for arbitrary antisymmetric constant $\omega^{\mu\nu}$

$$\sum_j^{n_e} \left(p_{j,\nu} \frac{\partial}{\partial p_j^\mu} - p_{j,\mu} \frac{\partial}{\partial p_j^\nu} \right) \int \prod_{i=1}^{L} d^n l_i \frac{1}{D_1^{\alpha_1} \cdots D_N^{\alpha_N}} = 0 \quad . \tag{42}$$

The generalisation of the above two transformations is given by $l_i \to q_i = c_{ij} l_j + d_{ij} p_j$. The invariance under this transformation gives

$$\int \prod_{i=1}^{L} d^n l_i \frac{\partial}{\partial l_{i,\mu}} \left(\frac{q_t^\mu}{D_1^{\alpha_1} \cdots D_N^{\alpha_N}} \right) = 0 \quad . \tag{43}$$

The above equation is called integration by parts identity (IBP). Since the exponents are arbitrary, we can generate infinite of IBP identities, of which most of them are redundant. One can show that a finite set of integrals can solve these identities, we call these integrals the Master Integrals. This process reduces the task of computing too many integrals.

Given the set of MIs, our next task to compute each one of them efficiently. The standard approach is to apply Feynman's trick or Schwinger parameterization to the integrands which allows to perform integration over loop momenta. However, the standard approach brings in parametric integrals which are hard to perform beyond one loop integrals. The method of DE is the alternate approach to solve MIs beyond one loop ones. Here, we first make set of Lorentz scalars out of all the external momenta and their masses. Let this set be $\{s_{ij}\} = \{x_1, \cdots, x_m\} = \vec{x}$ where m is the number of scalars $s_{ij} = (p_i + p_j)^2$ constructed out of p_j, including their masses. Then construct a set $\{\partial/\partial s_{ij}\} = \partial/\partial \vec{x}$ and apply them on all the MIs. Denoting MIs by $\vec{I}(\{s_{ij}/\varepsilon\}) = I_1(\vec{x}, \varepsilon), \cdots, I_{N_m}(\vec{,}\varepsilon$, where N_m is the number of MIs, we obtain

$$\frac{\partial}{\partial \vec{x}} \vec{I}(\vec{,}\varepsilon) = \hat{B}(\vec{x}, \varepsilon) \cdot \vec{J}(\vec{x}, \varepsilon) = \hat{A}(\vec{x}, \varepsilon) \cdot \vec{I}(\vec{x}, \varepsilon) \quad . \tag{44}$$

In the above we used IBP identities to express $\vec{J}$ in terms of $\vec{I}$ which converts $\hat{B}$ to $\hat{A}$. The coupled differential equations being first order ones are straightforward to solve provided $\vec{A}$ has fewer entries and the boundary integrals are known. Often, the later are easy to obtain for certain choice of $\vec{x} = \vec{x}_0$. There are several ways to solve the system of DEs depending on the structure of $\hat{A}$. In fact one can transform $\vec{I} \to \vec{I'} = U \cdot I$ in such a way that $\hat{A}' = U \cdot \hat{A} \cdot U^{-1} - U \cdot \partial/\partial \vec{x} \cdot U^{-1}$ takes the simple form. For the case when $\hat{A}'(\vec{x}, \varepsilon) = \varepsilon \tilde{A}(\vec{x})$, the solution demonstrates a peculiar all order structure in ε. One finds the

solution of $\vec{I}'$ is a set of iterated integrals with uniform transcendentality:

$$I'(\vec{x},\varepsilon) = \mathcal{P}\exp\left(\varepsilon\int_{\mathcal{C}} d\vec{x}\cdot\tilde{A}(\vec{x})\right)\cdot\vec{I}'(\vec{x}_0,\varepsilon) \quad , \tag{45}$$

where P is path ordering along the curve $\mathcal{C}$. If we assign transcendentality weight i for ε^{-i} and $\log^i(g(x))$, the terms in the expansion will have uniform weight.

5 Conclusion

We have demonstrated how the theory of strong interaction, namely quantum chromodynamics, plays an important role to understand physics at subatomic level in high energy experiments. We have shown that QCD can be applied in a systematic way using its factorisation properties. The perturbative methods demonstrate that various observables can be computed reliably. In addition we have discussed how large logarithms show up at the threshold region and how resummation of them to all orders can be performed. We have also discussed few modern techniques that are available to perform various computations efficiently.

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