Chapter I.2

Special relativity

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In most situations particle accelerators use beams at nearly the speed of light. Without Einstein's theory of Special Relativity, they simply wouldn't work. Since the theory is typically part of the university curriculum, we restrict ourselves to a review of the basics which are of interest for particle accelerators.

I.2.1 Introduction

Before Maxwell had published his equations (1864) the concept of space and time, conceived by Galilei and Newton, was solidly established. Space and time were independent, and time was an absolute quantity. The understanding of relativity required that the mechanical laws (Newton's laws) be the same for all inertial reference frames. They were invariant under the Galilean transformation (G-T). An object moving with v' in S' is seen in S with the velocity $v + v'$, Fig. I.2.1.

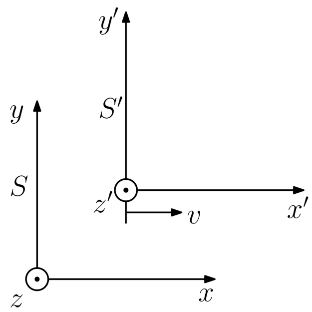

Fig. I.2.1: The frames S and S' moving relatively to each other with constant velocity v.

Maxwell's equations shattered the existing concept:

– The equations are not invariant under G-T,
– they describe waves with equal velocity (velocity of light c) in all inertial frames. That contradicted the belief that waves needed a supporting medium in which the velocity would be different than in frames moving with respect to the medium.

Numerous experiments were performed to prove the equations wrong. They all failed. Finally, in the late 1880's Heinrich Hertz confirmed the equations in a set of brilliant experiments. At about the same time

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an experiment by Michelson and Morley (1887), Fig. I.2.2, confirmed the constant velocity of light and forced people to change the Galilei-Newton concept of space and time.

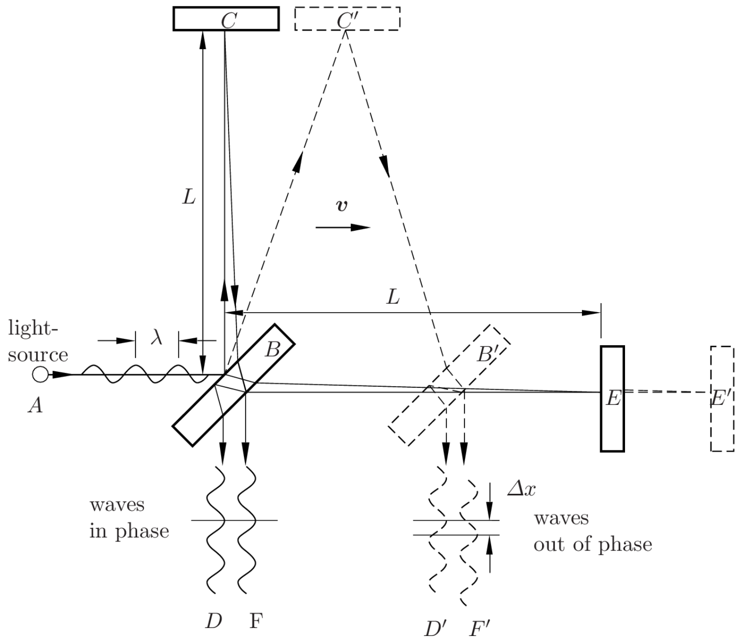


Fig. I.2.2: Interferometer of Michelson and Morley.

The interferometer splits a monochromatic wave by means of a semi-transparent mirror B into two beams. The parallel and perpendicular directed beams are reflected by mirrors C and E, respectively. Let us assume that the interferometer is moving with constant velocity v. Then, round travel times for the two beams are

$$
\begin{aligned}
T_{\|} &= \frac{L/c}{1-v/c} + \frac{L/c}{1+v/c} = \frac{2L/c}{1-(v/c)^2} \quad , \\
T_{\perp} &= \frac{2L/c}{\sqrt{1-(v/c)^2}} \quad ,
\end{aligned}
$$

i.e. $T_{\|}$ is $1/\sqrt{1-(v/c)^2}$ times larger than $T_{\perp}$. The two recombined beams D' and F' should show an interference pattern. No interference pattern is supposed to be only for $v = 0$. However, interference was never detected although the direction of the interferometer was changed, and the earth is moving in space. Clearly, the speed of light is equal in all reference frames and the Galilei-Newton concept of space and time had to be corrected.

I.2.2 Relativistic Kinematics

Poincaré and in particular Einstein based the new concept on two postulates:

1. All inertial frames of reference are equivalent w.r.t. all laws of physics.
2. The speed of light is equal in all inertial frames of reference.

As a consequence, space has to be isotropic and homogeneous, and the laws must be form-invariant under a transformation. The transformation must be linear and orthogonal

$$\begin{aligned} ct' &= a_{00}ct + a_{01}x + a_{02}y + a_{03}z \\ x' &= a_{10}ct + a_{11}x + a_{12}y + a_{13}z \\ y' &= a_{20}ct + a_{21}x + a_{22}y + a_{23}z \\ z' &= a_{30}ct + a_{31}x + a_{32}y + a_{33}z. \end{aligned} \tag{I.2.1}$$

Successive use of homogeneity and isotropy of space requires

$a_{02} = a_{03} = 0$ since events at $y = \pm y_0$ or $z = \pm z_0$ have to take place at equal times in S'

$a_{20} = a_{30} = 0$ the origin $x = y = z = 0$ has to stay on the x-axis

$a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$ axes x, x' and y, y' and z, z' should stay parallel

$a_{22} = a_{33}$ cylindrical symmetry requires equal transformation of y and z.

Further, due to the relativistic principle, the inverse transformation follows from replacing the primed variables by unprimed, the unprimed by primed and v by $-v$. Therefore,

$$y' = a_{22}(v)y \rightarrow y = a_{22}(-v)y' \quad \text{and} \quad a_{22}(v) = a_{22}(-v) = 1$$
$$\text{since for } v \rightarrow 0: \; y' = y.$$

Thus Eq. (I.2.1) simplifies to

$$ct' = a_{00}ct + a_{01}x, \quad x' = a_{10}ct + a_{11}x, \quad y' = y, \quad z' = z. \tag{I.2.2}$$

A light pulse in the origin $x' = y' = z' = 0$ at $t' = 0$ propagates on a spherical shell in S' and in S as-well (c is equal in both systems). Therefore, the space-time-interval

$$\left(ct'\right)^2 - x'^2 - y'^2 - z'^2 = (ct)^2 - x^2 - y^2 - z^2 \tag{I.2.3}$$

is invariant. Substituting Eq. (I.2.2) into Eq. (I.2.3) and comparing coefficients yields 3 equations

$$1 - a_{00}^2 + a_{10}^2 = 0, \quad 1 + a_{01}^2 - a_{11}^2 = 0, \quad a_{10}a_{11} - a_{00}a_{01} = 0. \tag{I.2.4}$$

A 4$^{\text{th}}$ equation follows from the motion of the origin ($x = y = z = 0$) in S' together with Eq. (I.2.2)

$$x' = -vt', \quad ct' = a_{00}ct, \quad x' = a_{10}ct \;\rightarrow\; a_{10} = -\frac{v}{c}a_{00}. \tag{I.2.5}$$

Equations (I.2.4), (I.2.5) are 4 equations for 4 unknowns and the final result is the Lorentz-Transform (L-T)

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \mathbf{L} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}, \tag{I.2.6}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

L-T is an affine transformation. It preserves the rectilinearity and parallelism of straight lines. The inverse transformation is obtained by replacing β by $-\beta$.

As the Michelson-Morley experiment implies, the arm of the interferometer in direction of the motion (parallel arm) appears smaller by $1/\gamma$ compared to the perpendicular arm, such that the round trip times of the beams become equal. That is in fact the case. A meter on the x'-axis, going from x'_1 to x'_2, which is measured in S has the length

$$\begin{aligned} x_2 - x_1 &= \gamma\left(x'_2 + \beta c t'_2\right) - \gamma\left(x'_1 + \beta c t'_1\right) \\ &= \gamma\left(x'_2 - x'_1\right) + \beta\gamma\left(ct'_2 - ct'_1\right) \\ &= \gamma\left(x'_2 - x'_1\right) + \beta\gamma^2\left(ct_2 - \beta x_2 - ct_1 + \beta x_1\right) \\ \left(1+\beta^2\gamma^2\right)\left(x_2 - x_1\right) &= \gamma^2(x_2 - x_1) = \gamma\left(x'_2 - x'_1\right) + \beta\gamma^2\left(ct_2 - ct_1\right) \end{aligned}$$

and since the ends have to be measured at equal times $t_2 = t_1$, it is

$$x_2 - x_1 = \frac{1}{\gamma}\left(x'_2 - x'_1\right) \quad \text{or} \quad L = \frac{L'}{\gamma}. \tag{I.2.7}$$

The lengths in parallel direction are contracted (**length contraction**) but not in perpendicular direction. Also, time intervals are modified. Let us take two events in S' at times t'_1, t'_2 both at location x'_0. In S they appear at times

$$ct_2 - ct_1 = \gamma\left(ct'_2 + \beta x'_0\right) - \gamma\left(ct'_1 + \beta x'_0\right) = \gamma\left(ct'_2 - ct'_1\right) \text{ or } T = \gamma T'. \tag{I.2.8}$$

Time intervals are dilated (**time dilation**).

A famous proof is the lifetime of muons which are created in the upper atmosphere by cosmic rays. Muons are not stable and follow the law of radioactive decay

$$n = n_0 \left(\frac{1}{2}\right)^{t'/T'_{1/2}}$$

where n_0 is the number at $t' = 0$ and $T'_{1/2} = 1.5\,\mu\text{s}$ is the half-lifetime. The muons travel almost with the speed of light $v = 0.994c$. So, going upwards in the atmosphere the intensity should double every height increase of $l' = cT'_{1/2} = 450\,\text{m}$. However, the rapidly moving reference frame of the muons increases the half-lifetime on earth by $\gamma = 9$ to $T'_{1/2} = 13.5\,\mu\text{s}$ and the doubling distance, as experimentally measured, is $l = 4\,\text{km}$.

Since distances and times change it is not surprising that velocities also change. The velocity u' of a particle in S' translates to a velocity u in S as follows

$$\begin{aligned} u_x &= \frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt} = \gamma\left(\frac{dx'}{dt'} + \beta c\right)\frac{dt'}{dt} = \gamma\left(u'_x + v\right)\frac{dt'}{dt} \\ u_y &= \frac{dy}{dt} = \frac{dy}{dt'}\frac{dt'}{dt} = u'_y\frac{dt'}{dt} \end{aligned}$$

$$u_z = \frac{dz}{dt} = \frac{dz}{dt'}\frac{dt'}{dt} = u_z'\frac{dt'}{dt}, \quad \frac{dt}{dt'} = \gamma\left(1 + \frac{\beta}{c}\frac{dx'}{dt'}\right) = \gamma\left(1 + \frac{vu_x'}{c^2}\right),$$

therefore

$$\begin{aligned} u_x &= \frac{u_x' + v}{1 + u_x' v/c^2} \\ u_y &= \frac{u_y'}{\gamma\,(1 + u_x' v/c^2)} \\ u_z &= \frac{u_z'}{\gamma\,(1 + u_x' v/c^2)}. \end{aligned} \tag{I.2.9}$$

Again, the inverted transformation is obtained by replacing the primed and unprimed variables by the unprimed and primed, respectively, and β by $-\beta$. As an example, let us assume a fast object with $u_x' = 0.9c$ in a fast train with $v = 0.9c$. In S, on the platform, the G-T would give $u_x = u_x' + v = 1.8c$, while in reality Eq. (I.2.9) gives

$$u_x = \frac{0.9c + 0.9c}{1 + 0.9^2} = 0.9945c.$$

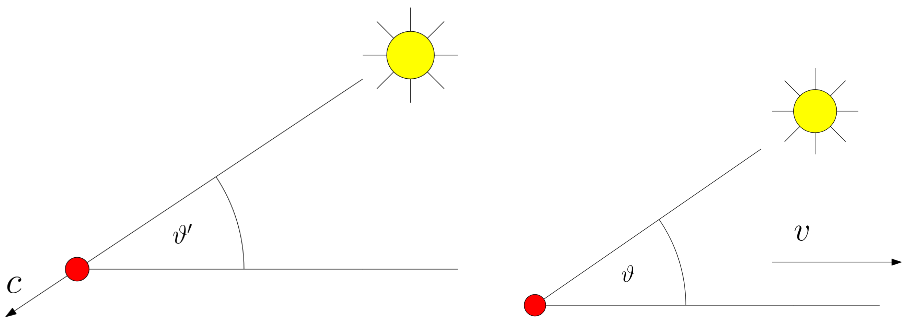
In a similar way one finds the acceleration $\boldsymbol{a}$ in S of a particle traveling with $\boldsymbol{u}'$ in S' and experiencing an acceleration $\boldsymbol{a}'$

$$\begin{aligned} a_x &= \frac{du_x}{dt} = \frac{du_x}{dt'}\frac{dt'}{dt} = \frac{d}{dt'}\left(\frac{u_x' + v}{1 + u_x' v/c^2}\right)\frac{dt'}{dt} = \frac{a_x'}{\gamma^3(1 + u_x' v/c^2)^3} \\ a_y &= \frac{a_y'}{\gamma^2(1 + u_x' v/c^2)^2} - \frac{(u_y' v/c^2)a_x'}{\gamma^2(1 + u_x' v/c^2)^3} \\ a_z &= \frac{a_z'}{\gamma^2(1 + u_x' v/c^2)^2} - \frac{(u_z' v/c^2)a_x'}{\gamma^2(1 + u_x' v/c^2)^3}. \end{aligned} \tag{I.2.10}$$

Acceleration in an inertial frame is possible.

I.2.2.1 Example: Light aberration

The position of a star appears on the moving earth under an angle smaller than its real position, Fig I.2.3. If the earth were at rest the light of the star would move with $u_x' = -c\cos\vartheta'$, $u_y' = -c\sin\vartheta'$. For the



(a) Earth at rest **(b)** Earth moving with v

Fig. I.2.3: Star seen under an angle on earth.

moving earth the velocity transforms to

$$u_x = -c\cos\vartheta = \frac{-c\cos\vartheta' - v}{1+\beta\cos\vartheta'}$$
$$u_y = -c\sin\vartheta = \frac{-c\sin\vartheta'}{\gamma(1+\beta\cos\vartheta')}$$

which becomes using $\tan\vartheta/2 = \sin\vartheta/(1+\cos\vartheta)$

$$\begin{aligned}\tan\frac{\vartheta}{2} &= \frac{\sin\vartheta'}{\gamma\left(1+\beta\cos\vartheta'\right)\left(1+\left(\cos\vartheta'+\beta\right)/\left(1+\beta\cos\vartheta'\right)\right)} \\ &= \frac{\sin\vartheta'}{1+\cos\vartheta'}\frac{1}{\gamma(1+\beta)} = \sqrt{\frac{1-\beta}{1+\beta}}\tan\frac{\vartheta'}{2} < \tan\frac{\vartheta'}{2}.\end{aligned} \tag{I.2.11}$$

I.2.2.2 Example: Doppler effect

An emitter T_x is moving with $\boldsymbol{v}$, while the receiver R_x is at rest, Fig. I.2.4.

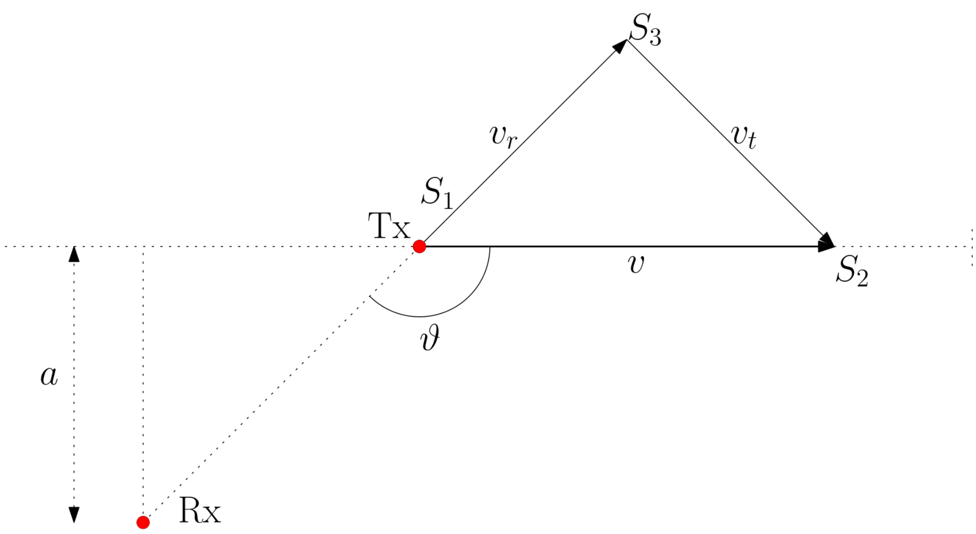


Fig. I.2.4: Radio emitter Tx moving with v and receiver Rx at rest.

Depending on the direction of $\boldsymbol{v}$ the received signal changes its frequency. A signal with frequency f_0 emitted at $t = 0$ and position S_1 travels a time

$$t_1 = \frac{1}{c}\overline{R_x S_1} = \frac{a}{c\sin\vartheta}$$

until it reaches R_x. In one RF period T_0 the emitter moves to S_2. Then the time difference between $t = 0$ and the arrival time at R_x of the signal emitted at S_2 is

$$\begin{aligned}t_2 &= \gamma T_0 + \frac{1}{c}\overline{R_x S_2} = \gamma T_0 + \frac{1}{c}\left(\overline{R_x S_1} + \gamma T_0 v\cos(180^\circ - \vartheta)\right) \\ &= \gamma T_0 + \frac{1}{c}\left(\frac{a}{\sin\vartheta} - \gamma T_0 v\cos\vartheta\right)\end{aligned}$$

where T_0 has been dilated by γ and $\overline{R_x S_2}$ approximated by $\overline{R_x S_3}$ for $vT_0 \ll \overline{R_x S_1}$. The RF period experienced by R_x is

$$T = t_2 - t_1 = \gamma(1 - \beta\cos\vartheta)T_0$$

and therefore

$$\frac{f}{f_0} = \frac{T_0}{T} = \frac{1}{\gamma(1 - \beta\cos\vartheta)} = \frac{\sqrt{1-\beta^2}}{1 - \beta\cos\vartheta}. \tag{I.2.12}$$

For $\vartheta \to \pi$ the emitter withdraws from the receiver and the frequency is decreased

$$\frac{f}{f_0} = \sqrt{\frac{1-\beta}{1+\beta}},$$

while for $\vartheta \to 0$ the T_x approaches R_x and the frequency is increased

$$\frac{f}{f_0} = \sqrt{\frac{1+\beta}{1-\beta}}.$$

In astronomy the typical situation is a withdrawing star far away, that corresponds to $\vartheta \to \pi$. The wavelength increases

$$\frac{\lambda}{\lambda_0} = \frac{f_0}{f} = \sqrt{\frac{1+\beta}{1-\beta}} = 1 + z, \quad z > 0.$$

z is called redshift and determines the escape velocity of a star

$$\beta = \frac{(1+z)^2 - 1}{(1+z)^2 + 1}. \tag{I.2.13}$$

E.g. $z = 3$ corresponds to an escape velocity of $v = 0.882c$. A typical redshifted spectrum is shown in Fig I.2.5.

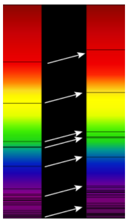


Fig. I.2.5: Redshifted spectrum of an escaping star.

I.2.2.3 Minkowski Diagram

A convenient way to show the coordinates in S' for events in S and vice versa is the Minkowski diagram. Referring to a L-T

$$x = \gamma(x' + \beta ct')$$
$$ct = \gamma(ct' + \beta x')$$

it is seen that events $E_1'(ct' = 0,\ x' = 1)$ and $E_2'(ct' = 1,\ x' = 0)$ in S' appear at $(x = \gamma,\ ct = \beta\gamma)$ and $(x = \beta\gamma,\ ct = \gamma)$ in S. These points define the position of the x', ct'-coordinated in S, Fig I.2.6.

The coordinates of S' are tilted by an angle δ, $\tan\delta = \beta$ and the scale is given by

$$\alpha = \sqrt{\gamma^2 + \beta^2\gamma^2} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}.$$

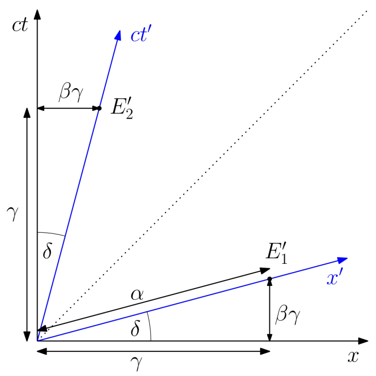


Fig. I.2.6: Minkowski diagram.

As an example let us take a light pulse at $x = ct = 2$. It expands in all directions with the velocity of light and reaches an observer at $x = 0$ and at the time $ct = 4$. Whereas an observer in a frame moving with $\beta = 0.58$ will receive the light pulse at $ct' = 2$, Fig. I.2.7.

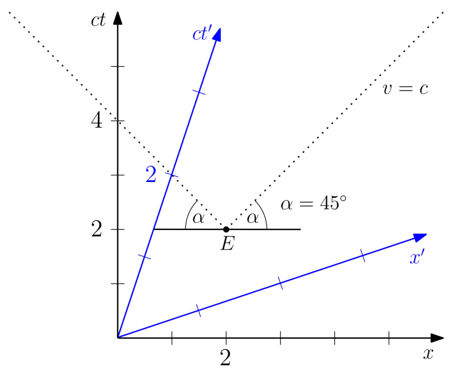


Fig. I.2.7: Light pulse emitted at $x = ct = 2$ and received at $ct = 4$ in S and at $ct' = 2$ in S', $\beta = 0.58$.

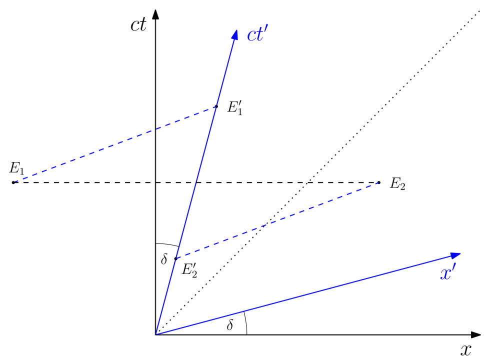


Fig. I.2.8: Two simultaneous events E_1, E_2 in S happen at different times in S'.

Another diagram treats simultaneity. Two events E_1, E_2 which are simultaneous in S, happen in S' at different times, E_1', E_2', Fig. I.2.8.

Let us consider a rocket flying with v. A light flash is emitted at the center and reaches the front and end detector at the same time, $ct_f' = ct_e' = l'/2$, Fig. I.2.9. On earth the times, when the detectors react, are

$$ct_f = \frac{l}{2} + vt_f, \quad ct_e = \frac{l}{2} - vt_e$$

i.e.

$$t_f - t_e = \beta\gamma^2 \frac{l}{c} = \beta\gamma \frac{l'}{c}. \tag{I.2.14}$$

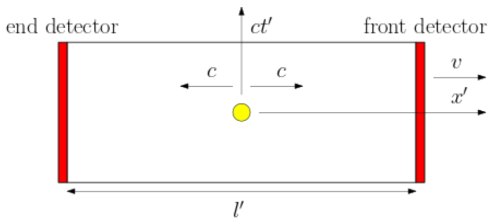
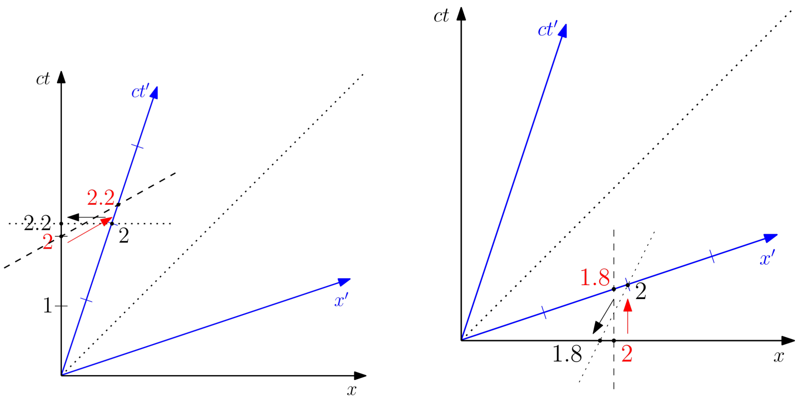


Fig. I.2.9: Light flash in the middle of a rocket chamber moving with v.

Finally, we see time dilation and length contraction between S and S' moving with $\beta = 0.42$, Fig. I.2.10.



(a) Time dilation

(b) Length contraction

Fig. I.2.10: Example for $\beta = 0.42$, $\gamma = 1.1$, $\delta = 22.8°$, $\alpha = 1.2$.

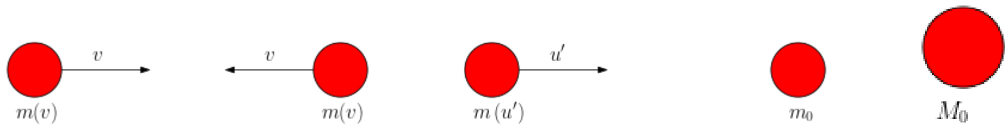
I.2.3 Relativistic dynamics

The Relativistic Dynamics is based on two principles:

1. Conservation of momentum
2. Conservation of energy

I.2.3.1 Moving mass

Because of Einstein's equation $E = mc^2$ the mass must depend on v, $m = m(v)$. The function we determine by an experiment, an inelastic collision between two identical particles, Fig. I.2.11.



(a) Before collision **(b)** In the rest frame of the right particle **(c)** After collision

Fig. I.2.11: Inelastic collision between two identical particles.

In the rest frame S' of the right particle the frame S of the left particle moves with v to the right. In S the left particle moves with v. Therefore, its velocity in S' is

$$u' = \frac{v+v}{1+(v/c)^2} = \frac{2v}{1+\beta^2}. \tag{I.2.15}$$

After collision a composite particle $M(v)$ moves to the right in the rest frame S' and conservation of momentum and energy require

$$m(u')u' = M(v)v \tag{I.2.16}$$

$$m(u')c^2 + m_0c^2 = M(v)c^2. \tag{I.2.17}$$

Combining Eqs. (I.2.15), (I.2.16), (I.2.17) yields

$$m(u') = \frac{1+\beta^2}{1-\beta^2}m_0 = \frac{m_0}{\sqrt{1-(u'/c)^2}} = \gamma_{u'}m_0. \tag{I.2.18}$$

The mass $m(u')$ increases with the relativistic factor $\gamma_{u'}$, while the mass $M(v)$ of the composite particle is

$$M(v) = \frac{1+\beta^2}{1-\beta^2}\frac{m_0}{v}\frac{2v}{1+\beta^2} = \frac{2m_0}{1-\beta^2} = \gamma M_0, \quad M_0 = 2\gamma m_0. \tag{I.2.19}$$

Obviously, rest masses are not conserved

$$M_0 - 2m_0 = 2m_0(\gamma - 1) > 0$$

and the kinetic energy is completely converted into mass

$$2E_{\text{kin}} = 2(E - E_0) = 2(\gamma m_0c^2 - m_0c^2) = (M_0 - 2m_0)c^2.$$

I.2.3.2 Momentum, force, energy

As mentioned in Eq. (I.2.16) the momentum of a particle is

$$\boldsymbol{p}(u) = m(u)\boldsymbol{u} = \gamma_u m_0\boldsymbol{u}. \tag{I.2.20}$$

To change the momentum a force is necessary

$$\begin{aligned}\boldsymbol{f} &= \frac{d\boldsymbol{p}}{dt} = m_0\frac{d\gamma_u}{dt}\boldsymbol{u} + m_0\gamma_u\frac{d\boldsymbol{u}}{dt} = m_0\boldsymbol{u}\frac{d}{dt}\frac{1}{\sqrt{1-\boldsymbol{u}\cdot\boldsymbol{u}/c^2}} + \gamma_u m_0\boldsymbol{a} \\ &= \gamma_u^3\frac{m_0}{c^2}\left(\boldsymbol{u}\cdot\boldsymbol{a}\right)\boldsymbol{u} + \gamma_u m_0\boldsymbol{a}.\end{aligned} \quad \text{(I.2.21)}$$

In general $\boldsymbol{f}$, $\boldsymbol{u}$ and $\boldsymbol{a}$ are not co-linear. The acceleration $\boldsymbol{a}$ can be derived from Eq. (I.2.21) using $\boldsymbol{f}\cdot\boldsymbol{u} = \gamma_u^3 m_0(\boldsymbol{u}\cdot\boldsymbol{a})$

$$\gamma_u m_0\boldsymbol{a} = \boldsymbol{f} - \frac{1}{c^2}\left(\boldsymbol{f}\cdot\boldsymbol{u}\right)\boldsymbol{u}. \quad \text{(I.2.22)}$$

In a **linear accelerator** (linac) with

$$\boldsymbol{u} = (u_x, 0, 0), \quad \boldsymbol{f} = (f_x, 0, 0), \quad \boldsymbol{a} = (a_x, 0, 0)$$

it follows from Eq. (I.2.21)

$$f_x = \gamma_u^3 m_0 a_x. \quad \text{(I.2.23)}$$

I.e., for a given force the acceleration decreases due to the so-called **longitudinal mass**

$$m_{\|} = \gamma_u^3 m_0.$$

In that case, the work done along a path element dx is

$$dE_{\text{kin}} = f_x dx = \gamma_u^3 m_0 a_x dx = \gamma_u^3 m_0\frac{du_x}{dt}dx = \gamma_u^3 m_0 u_x du_x$$

and the total energy provided in an acceleration from $u = 0$ to u

$$E_{\text{kin}} = m_0c^2\int_0^{\beta_u}\frac{\beta_u d\beta_u}{(1-\beta_u^2)^{\frac{3}{2}}} = \gamma_u m_0c^2 - m_0c^2 = E - E_0.$$

In the process the particle absorbs power at a rate

$$P = \frac{dE_{\text{kin}}}{dt} = \boldsymbol{f}\cdot\boldsymbol{u} \quad \text{(I.2.24)}$$

where we used Eq. (I.2.22) together with

$$\boldsymbol{f} = \frac{d\boldsymbol{p}}{dt} = \frac{dm}{dt}\boldsymbol{u} + m\frac{d\boldsymbol{u}}{dt} = \frac{1}{c^2}\frac{dE_{\text{kin}}}{dt}\boldsymbol{u} + \gamma_u m_0\boldsymbol{a}.$$

The absorbed power, Eq. (I.2.24), is the scalar product of $\boldsymbol{f}$ and $\boldsymbol{u}$ as in classical mechanics.

In a **circular accelerator** (e.g. synchrotron) the acceleration is perpendicular to the velocity, $\boldsymbol{a}\perp\boldsymbol{u}$, and Eq. (I.2.21) becomes

$$\boldsymbol{f} = \gamma_u m_0\boldsymbol{a}. \quad \text{(I.2.25)}$$

Here, one speaks of **transverse mass**

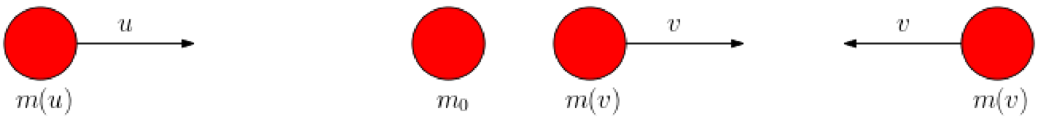
$$m_{\perp} = \gamma_u m_0.$$

I.2.3.3 Example: Collider versus fixed target machine

Let us first consider a 3.5 TeV head-on p-p collider. Figure I.2.11a shows the particles before and Fig. I.2.11c after collision. The total kinetic energy is converted to rest mass $M_0 = 2\gamma m_0$ and the center of mass energy is

$$E_{\mathrm{CM}} = M_0 c^2 = 2\gamma m_0 c^2 = 7\,\mathrm{TeV}.$$

For a fixed target machine the situation before collision is shown in Fig. I.2.12a and after a transformation into a center of mass frame S' in Fig. I.2.12b.



(a) Left particle moving with velocity u and right particle at rest

(b) Situation in the center of mass frame S'

Fig. I.2.12: Inelastic collision between two particles.

The center of mass frame S' moves to the right with v and S moves to the left with v. v follows from the transformation of velocity u in S

$$v = \frac{u - v}{1 - uv/c^2}$$

and

$$\beta_v = \frac{v}{c} = \frac{1}{\beta_u}\left(1 - \sqrt{1-\beta_u^2}\right), \quad \beta_u = \frac{u}{c}, \quad \gamma_v = \frac{1}{\sqrt{1-\beta_v^2}} = \sqrt{\frac{1}{2}\,(1+\gamma_u)}.$$

The center of mass energy is

$$E_{\mathrm{CM}} = 2\gamma_v E_0 = \sqrt{2(1+\gamma_u)}E_0 = \sqrt{2(E_0+E)E_0} = 81\,\mathrm{GeV}, \quad E_0 = 938\,\mathrm{MeV}.$$

The available center of mass energy is much lower, nearly 86 times, than in the case of a head-on collider.

I.2.3.4 Energy-momentum diagram

The expression for energy can be written as

$$E^2 = \left(mc^2\right)^2 = \left(m_0c^2\right)^2\gamma^2 = \left(m_0c^2\right)^2 \frac{1-\beta^2+\beta^2}{1-\beta^2} = E_0^2 + (\gamma m_0 v c)^2$$

$$\frac{E}{c} = \sqrt{(m_0c)^2 + p^2} \quad \text{and} \quad \frac{E}{c} = |p| \text{ for a massless particle.} \tag{I.2.26}$$

Since E is a conserved quantity, p must also be a conserved quantity. In a graph for E as a function of p the position vectors for points on the curves can be added like vectors, e.g. all interactions are allowed in which energy-momentum vectors $\boldsymbol{a}$, $\boldsymbol{b}$ add up to vector $\boldsymbol{s}$, see Fig. I.2.13

I.2.3.5 Example: Photon absorption and emission

Figure I.2.14a shows the absorption of a photon by a composite particle which goes into an excited state. In Fig. I.2.14b the inverse process is shown. An excited particle drops into a lower energy state by

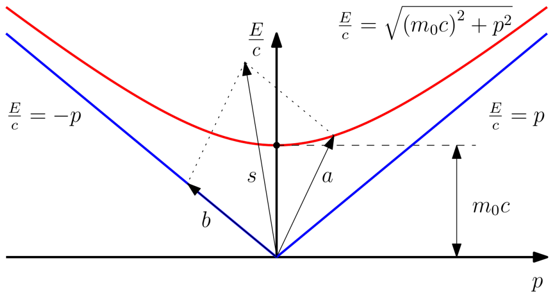
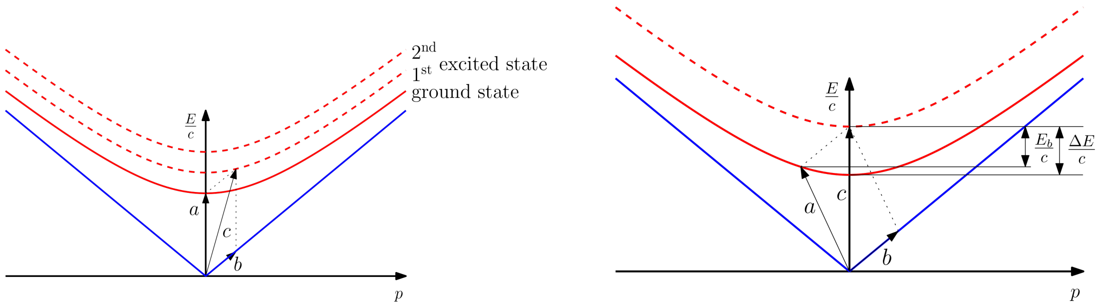


Fig. I.2.13: Interactions with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ result in $\boldsymbol{s}$.

emitting a photon. It experiences a recoil because the energy difference ΔE between the excited and ground state is larger than the difference between the excited and final state.



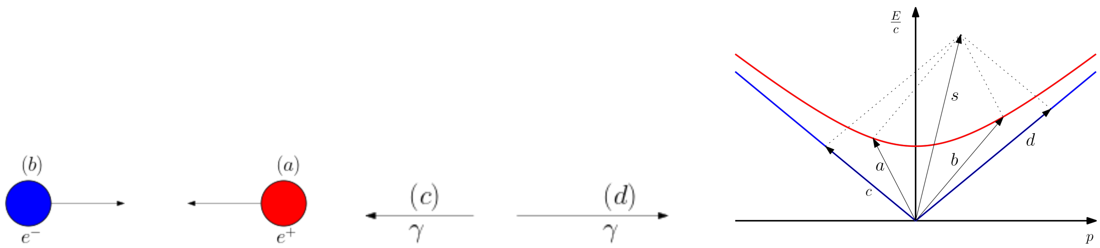
(a) Absorption of a photon

(b) Emission of a photon

Fig. I.2.14

I.2.3.6 Example: Pair annihilation

In Fig. I.2.15 a positron, position a, collides with an electron, position b. The charge is annihilated while the energy is transformed into forward (c) and backward (d) scattered photons.



(a) Before collision **(b)** After collision **(c)** Energy-momentum exchange

Fig. I.2.15: Annihilation of an electron and positron in a collision.

I.2.4 4-vectors

Normal 3-dimensional vectors are defined by a linear transformation. They are invariant against transformation and rotation of the coordinate system. In a similar way one defines 4-vectors by the linear Lorentz-Transformation (L-T):

Any quadruple which transforms with an L-T is a 4-vector.

In order to apply Einstein's summation rule contra- and covariant 4-vectors are defined

$$\begin{aligned} \mathcal{X}^\mu &= (\mathcal{X}^0, \mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3) \qquad \text{contravariant} \\ \mathcal{X}_\mu &= (\mathcal{X}_0, -\mathcal{X}_1, -\mathcal{X}_2, -\mathcal{X}_3) \qquad \text{covariant.} \end{aligned} \tag{I.2.27}$$

The scalar product is then

$$\mathcal{X}^\mu \mathcal{X}_\mu = \mathcal{X}^0\mathcal{X}_0 - \mathcal{X}^1\mathcal{X}_1 - \mathcal{X}^2\mathcal{X}_2 - \mathcal{X}^3\mathcal{X}_3 = \mathcal{X}'^\mu \mathcal{X}'_\mu$$

and it is invariant under a L-T. This can be generalized to be the case for any 4-vectors

$$\mathcal{A}^\mu \mathcal{B}_\mu = \mathcal{A}'^\mu \mathcal{B}'_\mu. \tag{I.2.28}$$

I.2.4.1 Position 4-vector

The edge of a spherical light pulse forms a sphere with radius ct

$$x^2 + y^2 + z^2 = (ct)^2.$$

It turns out that the expression

$$(ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2$$

is invariant under a L-T and that it is related to the length (space-time-interval) of the position vector in the 4-dimensional space

$$\mathcal{X}^\mu = (ct, x, y, z). \tag{I.2.29}$$

Equation (I.2.29) is the position 4-vector.

I.2.4.2 Velocity 4-Vector

In classical dynamics velocity is given by the time-derivative of the position vector. In relativistic dynamics t is not invariant and instead of t one has to find a quantity with dimension of time which is invariant.

An event which moves by (dx, dy, dz) in dt has the invariant space-time-interval ds

$$\begin{aligned} ds &= \sqrt{d\mathcal{X}^\mu \cdot d\mathcal{X}_\mu} = \sqrt{(cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2} \\ &= cdt\sqrt{1 - \frac{1}{c^2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)} = cdt\sqrt{1 - \left(\frac{v}{c}\right)^2} = \frac{cdt}{\gamma} = cd\tau. \end{aligned}$$

The new quantity τ has the dimension of time and is dilated by γ in the laboratory system S

$$dt = \gamma d\tau. \tag{I.2.30}$$

It can therefore be interpreted as the **proper time** an observer would measure in a system S' moving with v. τ is a Lorentz scalar (invariant under L-T). Using τ instead of t, one gets the velocity 4-vector

$$\begin{aligned}\mathcal{U}^\mu &= \frac{d\mathcal{X}^\mu}{d\tau} = \frac{d\mathcal{X}^\mu}{dt}\frac{dt}{d\tau} = \left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)\frac{dt}{d\tau} = \gamma(c, v_x, v_y, v_z), \\ \mathcal{U}^\mu\mathcal{U}_\mu &= \gamma^2 c^2\left[1 - \frac{1}{c^2}\left(v_x^2, v_y^2, v_z^2\right)\right] = c^2.\end{aligned} \tag{I.2.31}$$

$\mathcal{U}^\mu$ is not a measurable quantity. $d\mathcal{X}^\mu$ is the space-time-interval between two events in one frame, while $d\tau$ is the time increment in a different frame in which both events take place at the same location. But $\mathcal{U}^\mu$ helps to facilitate calculations.

To understand the situation better, put yourself on the ground when your friend is flying in an airplane. You want to know when he arrives. So, you measure position and velocity of the airplane and calculate the arrival time. Your friend, however, is interested at which time τ he will arrive. So, he asks the tower for the position of the plane at two time instants τ_1 and τ_2 and calculates velocity and arrival time. (Be aware of the negligence. Asking the tower will take time and the plan does not work. Imagine therefore no delay in the communication with the tower.)

I.2.4.3 Energy-momentum 4-vector

A particle with restmass m_0 moves in S with $\boldsymbol{u} = (u_x, 0, 0)$. Then,

$$E = \gamma_u m_0 c^2, \quad p_x = \gamma_u m_0 u_x = E\frac{u_x}{c^2}.$$

In S' its velocity, energy and momentum are

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \longrightarrow \gamma_{u'} = \frac{1}{\sqrt{1 - (u'_x/c)^2}} = \gamma_u\gamma(1 - \beta_u\beta)$$

$$\frac{E'}{c} = \gamma_{u'} m_0 c = \gamma(\gamma_u m_0 c - \beta\gamma_u m_0 u_x) = \gamma\left(\frac{E}{c} - \beta p_x\right) \tag{I.2.32}$$

$$p'_x = \gamma_{u'} m_0 u'_x = \gamma\gamma_u m_0(1 - \beta\beta_u)\frac{u_x - v}{1 - \beta\beta_u} = \gamma(\gamma_u m_0 u_x - \gamma_u m_0 c\beta) = \gamma\left(p_x - \beta\frac{E}{c}\right). \tag{I.2.33}$$

As can be seen E/c and p_x transform with an L-T and they form the energy-momentum 4-vector

$$\mathcal{P}^\mu = (\frac{E}{c}, p_x, p_y, p_z). \tag{I.2.34}$$

I.2.4.4 Example: Derivation of the energy-momentum equation

In a frame where the momentum is not zero it is

$$\mathcal{P}^\mu \mathcal{P}_\mu = \left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2 = \left(\frac{E}{c}\right)^2 - p^2$$

and in a frame with zero momentum

$$\mathcal{P}'^\mu \mathcal{P}'_\mu = \left(\frac{E_0}{c}\right)^2.$$

Making use of Eq. (I.2.28), the energy-momentum equation, Eq. (I.2.26), is easily found.

$$\mathcal{P}^\mu \mathcal{P}_\mu = \mathcal{P}'^\mu \mathcal{P}'_\mu \longrightarrow E^2 = E_0^2 + (pc)^2.$$

I.2.4.5 Example: Derivation of Planck's hypothesis

A photon with energy E' in S' travels in $-x'$ direction. Its momentum is

$$p'_x = -\frac{E'}{c}.$$

The energy in S follows from the inverse form of Eq. (I.2.32)

$$\frac{E}{c} = \gamma\left(\frac{E'}{c} + \beta p'_x\right) = \sqrt{\frac{1-\beta}{1+\beta}}\frac{E'}{c}.$$

The Doppler shifted frequency Eq. (I.2.12) is for $\vartheta = 180°$

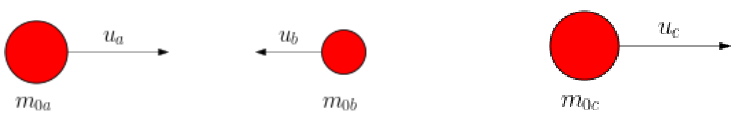
$$\nu = \sqrt{\frac{1-\beta}{1+\beta}}\nu'.$$

Dividing the two equations yields a constant ratio equal to Planck's constant.

$$\frac{E}{\nu} = \frac{E'}{\nu'} = \text{const.} = h$$

I.2.4.6 Example: Inelastic collision

Two particles with restmasses m_{0a}, m_{0b} and velocities u_a, u_b collide head-on inelastically. Figure I.2.16 shows the situation.



(a) Before **(b)** After collision

Fig. I.2.16: Inelastic collision between two particles.

Energy and momentum conservation require

$$\mathcal{P}_a^\mu + \mathcal{P}_b^\mu = \mathcal{P}_c^\mu$$

which will be multiplied with $\mathcal{P}_{a\mu} + \mathcal{P}_{b\mu} = \mathcal{P}_{c\mu}$

$$\mathcal{P}_a^\mu \mathcal{P}_{a\mu} + 2\mathcal{P}_a^\mu \mathcal{P}_{b\mu} + \mathcal{P}_b^\mu \mathcal{P}_{b\mu} = \mathcal{P}_c^\mu \mathcal{P}_{c\mu}. \tag{I.2.35}$$

In the rest frames of (a), (b), (c) it is

$$\mathcal{P}_a^\mu \mathcal{P}_{a\mu} = (m_{0a}c)^2\,, \quad \mathcal{P}_b^\mu \mathcal{P}_{b\mu} = (m_{0b}c)^2\,, \quad \mathcal{P}_c^\mu \mathcal{P}_{c\mu} = (m_{0c}c)^2\,. \tag{I.2.36}$$

Using the energy-momentum vectors in the laboratory frame

$$\begin{aligned}
\mathcal{P}_a^\mu &= (\gamma_a m_{0a} c, \gamma_a m_{0a} u_a, 0, 0) \\
\mathcal{P}_b^\mu &= (\gamma_b m_{0b} c, -\gamma_b m_{0b} u_b, 0, 0) \\
2\mathcal{P}_a^\mu \mathcal{P}_{b\mu} &= 2\gamma_a \gamma_b m_{0a} m_{0b} (c^2 + u_a u_b)
\end{aligned}$$

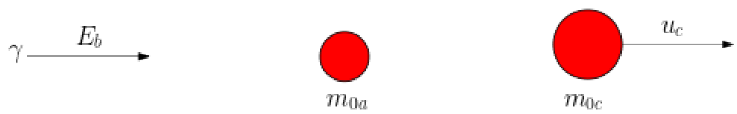
and substituting it into Eq. (I.2.35) together with Eq. (I.2.36) yields

$$m_{0c} = \sqrt{m_{0a}^2 + m_{0b}^2 + 2m_{0a}m_{0b}\gamma_a\gamma_b(1 + u_a u_b/c^2)} \geq m_{0a} + m_{0b}.$$

The rest mass of particle (c) is larger than the sum of the restmasses of (a) and (b). The increase corresponds to the difference in kinetic energy before and after the collision.

I.2.4.7 Example: Absorption of a photon by an atom at rest

The situation is shown in Fig. I.2.17.



(a) Before **(b)** After absorption

Fig. I.2.17: Absorption of a photon by an atom at rest.

With the designation (a), (b), (c) one can use Eq. (I.2.35). The energy-momentum 4-vectors in the rest frames of (a) and (c) are

$$\mathcal{P}_a^\mu = (m_{0a}c, 0, 0, 0), \quad \mathcal{P}_c^\mu = (m_{0c}c, 0, 0, 0).$$

In the laboratory frame the photon has

$$\mathcal{P}_b^\mu = \left(\frac{E_b}{c}, p_{bx}, 0, 0\right) = \left(\frac{h}{c}\nu, \frac{h}{c}\nu, 0, 0\right).$$

Then,

$$\mathcal{P}_a^\mu \mathcal{P}_{a\mu} = (m_{0a}c)^2, \quad \mathcal{P}_b^\mu \mathcal{P}_{b\mu} = 0, \quad \mathcal{P}_c^\mu \mathcal{P}_{c\mu} = (m_{0c}c)^2, \quad \mathcal{P}_a^\mu \mathcal{P}_{b\mu} = m_{0a}h\nu$$

which substituted in Eq. (I.2.35) yields

$$(m_{0a}c)^2 + 2m_{0a}c\frac{h\nu}{c} = (m_{0c}c)^2$$

or

$$m_{0c} = \sqrt{m_{0a}^2 + 2m_{0a}\frac{h\nu}{c^2}} = m_{0a}\sqrt{1 + 2\frac{h\nu}{m_{0a}c^2}}.$$

If $m_{0a}c^2 \gg h\nu$ it is

$$m_{0c}c^2 \approx m_{0a}c^2 + h\nu,$$

i.e. the rest energy of particle (c) equals the rest energy of particle (a) plus the photon energy.

I.2.4.8 Example: Inverse Compton effect (photon scattered at electron)

The energy of a photon can be increased when scattered at an incoming electron, see Fig. I.2.18.

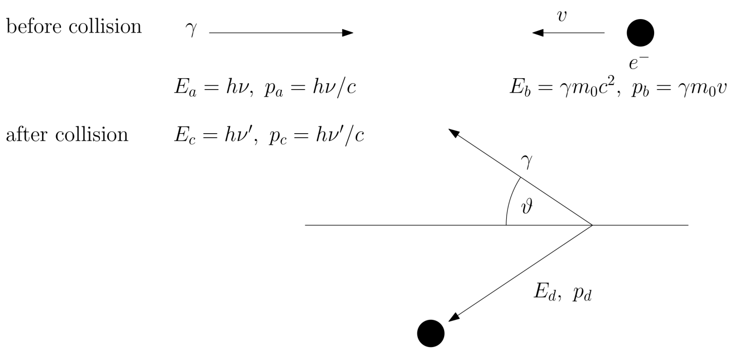


Fig. I.2.18: Scattering of a photon at an incoming electron.

The equation for energy and momentum conservation is

$$\mathcal{P}_a^\mu + \mathcal{P}_b^\mu = \mathcal{P}_c^\mu + \mathcal{P}_d^\mu. \tag{I.2.37}$$

which will be multiplied with itself

$$\mathcal{P}_a^\mu \mathcal{P}_{a\mu} + 2\mathcal{P}_a^\mu \mathcal{P}_{b\mu} + \mathcal{P}_b^\mu \mathcal{P}_{b\mu} = \mathcal{P}_c^\mu \mathcal{P}_{c\mu} + 2\mathcal{P}_c^\mu \mathcal{P}_{d\mu} + \mathcal{P}_d^\mu \mathcal{P}_{d\mu}. \tag{I.2.38}$$

The products for the photons are

$$\mathcal{P}_a^\mu \mathcal{P}_{a\mu} = \mathcal{P}_c^\mu \mathcal{P}_{c\mu} = 0$$

and for the electron they are equal in the rest frames of (b), (d)

$$\mathcal{P}_b^\mu \mathcal{P}_{b\mu} = \mathcal{P}_d^\mu \mathcal{P}_{d\mu}.$$

Equation (I.2.38) simplifies to

$$\mathcal{P}_a^\mu \mathcal{P}_{b\mu} = \mathcal{P}_c^\mu \mathcal{P}_{d\mu}. \tag{I.2.39}$$

Next, we multiply Eq. (I.2.37) with $\mathcal{P}_{c\mu}$ and use Eq. (I.2.39)

$$\mathcal{P}_a^\mu \mathcal{P}_{c\mu} + \mathcal{P}_b^\mu \mathcal{P}_{c\mu} = \mathcal{P}_d^\mu \mathcal{P}_{c\mu} = \mathcal{P}_a^\mu \mathcal{P}_{b\mu}. \tag{I.2.40}$$

In the laboratory frame it is

$$\mathcal{P}_a^\mu = \left(\frac{h}{c}\nu, \frac{h}{c}\nu, 0, 0\right), \quad \mathcal{P}_b^\mu = (\gamma m_0 c, -\gamma m_0 v, 0, 0), \quad \mathcal{P}_c^\mu = \left(\frac{h}{c}\nu', -\frac{h}{c}\nu' \cos\vartheta, \frac{h}{c}\nu' \sin\vartheta, 0\right)$$

which after substitution into Eq. (I.2.40) yields

$$\left(\frac{h}{c}\right)^2 \nu\nu'(1+\cos\vartheta) + \gamma m_0 h\nu'(1-\beta\cos\vartheta) = \gamma m_0 h\nu(1+\beta)$$

$$\frac{\nu'}{\nu} = \frac{1+\beta}{1-\beta\cos\vartheta + (1+\cos\vartheta)\dfrac{E_a}{E_b}}. \tag{I.2.41}$$

If the electron is at rest, $\beta = 0$ and $\vartheta = 180° - \varphi$ (forward scattering), one gets the **Compton effect**

$$\frac{\nu'}{\nu} = \frac{1}{1 + (1-\cos\varphi)h\nu/m_0c^2}$$

or with $\nu = c/\lambda$ the Compton equation

$$\lambda' - \lambda = \frac{h}{m_0 c}(1-\cos\varphi)$$

where $h/m_0c = 2.42 \times 10^{-12}\,\mathrm{m}$ is the **Compton wavelength**.

On the other hand, if the electron has a high energy the photon energy, i.e. frequency, can be shifted upwards very strongly. As an example let us take the microwave background radiation $E_a = 1 \times 10^{-3}\,\mathrm{eV}$ and a high energy electron $\gamma \gg 10^8$ as it happens in star explosions. The photon will be back-scattered

$$\vartheta \approx 0, \quad 1+\beta \approx 2, \quad 1-\beta \approx \frac{1}{2\gamma^2}$$

and from Eq. (I.2.41)

$$\frac{\nu'}{\nu} \approx \frac{4\gamma^2}{1+4\gamma^2 E_a/E_b} \approx \frac{E_b}{E_a} = \frac{\gamma E_{e0}}{h\nu}$$

$$E_e = h\nu' \approx E_b = \gamma E_{e0} = \gamma\, 511\,\mathrm{keV}.$$

The photon energy is dramatically increased by γ.

I.2.4.9 Acceleration 4-vector

As for the velocity 4-vector one has to use the proper time τ. Then, the acceleration 4-vector is obtained from the time derivative of the velocity 4-vector.

$$\begin{aligned} \mathcal{A}^\mu &= \frac{d\mathcal{U}^\mu}{d\tau} = \frac{d\mathcal{U}^\mu}{dt}\frac{dt}{d\tau} = \gamma_u\left(\frac{d\gamma_u}{dt}(c,\boldsymbol{u}) + \gamma_u\frac{d}{dt}(c,\boldsymbol{u})\right) \\ &= \frac{\gamma_u^4}{c^2}\left(\boldsymbol{u}\cdot\boldsymbol{a}\right)(c,\boldsymbol{u}) + \gamma_u^2(0,\boldsymbol{a}) \end{aligned} \tag{I.2.42}$$

with

$$\mathcal{A}^\mu\mathcal{A}_\mu = -\frac{\gamma_u^6}{c^2}\left(\boldsymbol{u}\cdot\boldsymbol{a}\right)^2 - \gamma_u^4 a^2 \qquad \boldsymbol{a} = \frac{d\boldsymbol{u}}{dt}. \tag{I.2.43}$$

Velocity and acceleration 4-vectors are perpendicular

$$\mathcal{U}^\mu\mathcal{A}_\mu = 0.$$

Often, one needs the **proper acceleration** α, i.e. the acceleration in an instantaneous rest frame S'

$$\boldsymbol{u}' = 0, \quad \gamma_{u'} = 1, \quad \mathcal{A}^\mu = (0,\boldsymbol{\alpha}), \quad \mathcal{A}^\mu\mathcal{A}_\mu = -\alpha^2. \tag{I.2.44}$$

For **linear acceleration**, $\boldsymbol{u} \parallel \boldsymbol{a}$, the proper acceleration follows from Eqs. (I.2.43), (I.2.44) as

$$-\alpha^2 = -\gamma_u^6\beta_u^2 a^2 - \gamma_u^4 a^2 = -\gamma_u^6 a^2 \longrightarrow \alpha = \gamma_u^3 a,$$

the same result as in Eq. (I.2.10) for $u_x' = 0$, $v = u$

$$a_x = \frac{a_x'}{\gamma_u^3} \longrightarrow a_x' = \alpha = \gamma_u^3 a_x.$$

In the case of **circular motion**, $\boldsymbol{u} \perp \boldsymbol{a}$, the proper acceleration is

$$-\alpha^2 = -\gamma^4 a^2 \longrightarrow \alpha = \gamma^2 a.$$

which is obtained also from Eq. (I.2.10) in an instantaneous rest frame $u_x' = u_y' = 0$, $v = u$ and reference is made to Fig. I.2.19.

$$a_y = \frac{a_y'}{\gamma_u^2} \longrightarrow a_y' = \alpha = \gamma_u^2 a_y$$

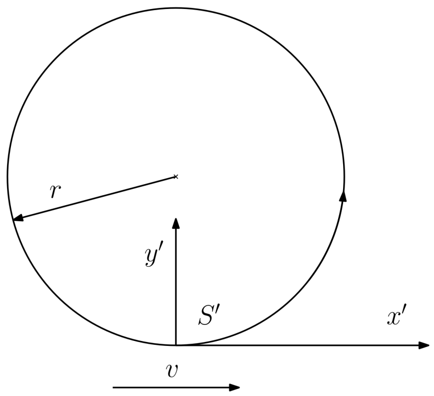


Fig. I.2.19: Instantaneous rest frame S' in a circular motion.

I.2.4.10 Frequency-wavenumber 4-vector

A plane wave in free space propagates in direction $\boldsymbol{e}_k$

$$\boldsymbol{E} = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$$

where $\boldsymbol{k} = k\boldsymbol{e}_k = (\omega/c)\boldsymbol{e}_k$ and $\boldsymbol{r}$ the position vector. The phase of the wave must be the same for all reference frames because the velocity of light is the same

$$\phi = \omega t - \boldsymbol{k}\cdot\boldsymbol{r} = \omega t - k_x x - k_y y - k_y y = \phi'.$$

ϕ can be written as

$$\phi = \mathcal{K}^\mu \mathcal{X}_\mu = \left(\frac{\omega}{c}, k_x, k_y, k_z\right) \cdot (ct, -x, -y, -z) \tag{I.2.45}$$

with the frequency-wavenumber 4-vector

$$\mathcal{K}^\mu = \left(\frac{\omega}{c}, k_x, k_y, k_z\right) \tag{I.2.46}$$

and the position 4-vector $\mathcal{X}^\mu$. Obviously, the frequency and wavenumber transform with an L-T. Both effects were already shown as light aberration and Doppler effect in the examples I.2.2.1 and I.2.2.2. Here, they can be rapidly derived using the frequency-wavenumber 4-vector. A transmitter in the moving frame S' is emitting a wave with frequency ω' under an angle ϑ'. In the laboratory frame S it is

$$\mathcal{K}^\mu = \left(\frac{\omega}{c}, k_x, k_y, 0\right) = \mathbf{L}^{-1}\left(\frac{\omega'}{c}, k'_x, k'_y, 0\right)$$

$$\frac{\omega}{c} = \gamma\left(1 + \beta\cos\vartheta'\right)\frac{\omega'}{c} \tag{I.2.47a}$$

$$\frac{\omega}{c}\cos\vartheta = \gamma(\beta + \cos\vartheta')\frac{\omega'}{c} \tag{I.2.47b}$$

$$\frac{\omega}{c}\sin\vartheta = \frac{\omega'}{c}\sin\vartheta'. \tag{I.2.47c}$$

Equation (I.2.47a) is the Doppler effect. Please note that ϑ' is used instead of ϑ in Eq. (I.2.12). Equations (I.2.47b) and (I.2.47c) result in Eq. (I.2.11) when using $\tan\vartheta/2 = \sin\vartheta/(1+\cos\vartheta)$.

I.2.4.11 Charge-current 4-vector

When going from one reference frame to another, charge must be conserved, i.e. a deferentially small amount of charge must be invariant

$$\rho_0 dxdydz = \rho'_u dx'dy'dz'$$

and since $dx' = dx/\gamma_u$, $dy' = dy$, $dz' = dz$, the charge density of the moving charge is

$$\rho'_u = \gamma_u \rho_0. \tag{I.2.48}$$

Similar to non-relativistic dynamics where the current density is given by the charge density times velocity

$$\boldsymbol{J} = \rho \boldsymbol{u} = \gamma_u \rho_0 \boldsymbol{u}$$

one has in the relativistic case

$$\mathcal{J}^\mu = \rho_0 \mathcal{U}^\mu = \gamma_u \rho_0 \left(c, u_x, u_y, u_z\right) = \left(\rho c, j_x, j_y, j_z\right), \tag{I.2.49}$$

where the rest charge density had to be used. Equation (I.2.49) is the charge-current 4-vector because $\mathcal{U}^\mu$ is a 4-vector and ρ_0 a Lorentz scalar.

I.2.4.12 Potential 4-vector

The starting point are Maxwell's equations

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{J} + \mu\varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \tag{I.2.50a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{I.2.50b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon} \tag{I.2.50c}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{I.2.50d}$$

Equation (I.2.50d) allows to express $\boldsymbol{B}$ through a vector potential $\boldsymbol{A}$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

which substituted into Eq. (I.2.50b) yields

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t},$$

with ϕ being the scalar potential. Using the ansatz for $\boldsymbol{B}$ and $\boldsymbol{E}$ in the leftover equations (I.2.50a), (I.2.50c) wave equations for the potentials can be derived

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J}$$
$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\varepsilon}, \tag{I.2.51}$$

after making use of the Lorentz-Gauge

$$\nabla \cdot \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}.$$

With the charge-current 4-vector Eq. (I.2.49) one can write Eq. (I.2.51) as

$$\Box^2 \mathcal{Q}^\mu = -\mu \mathcal{J}^\mu \tag{I.2.52}$$

where the Lorentz scalar $\Box^2$ is the 4-dimensional Laplace operator

$$\Box^2 = \nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2},$$

and

$$\mathcal{Q}^\mu = \left(\frac{\phi}{c}, \boldsymbol{A}\right) \tag{I.2.53}$$

is the potential 4-vector.

I.2.4.13 Power-force 4-vector (Minkowski force)

Similar to velocity and acceleration 4-vector the power-force 4-vector is the derivative of the energy-momentum 4-vector with respect to the proper time

$$\mathcal{F}^\mu = \frac{d\mathcal{P}^\mu}{d\tau} = \frac{d\mathcal{P}^\mu}{dt}\frac{dt}{d\tau} = \gamma_u \left(\frac{1}{c}\frac{dE}{dt}, \frac{dp_x}{dt}, \frac{dp_y}{dt}, \frac{dp_z}{dt}\right) \tag{I.2.54}$$

which becomes using Eq. (I.2.24)

$$\mathcal{F}^\mu = \gamma_u \left(\frac{1}{c}\boldsymbol{f} \cdot \boldsymbol{u}, f_x, f_y, f_z\right). \tag{I.2.55}$$

With $\mathcal{A}^\mu$, Eq. (I.2.42), and $\mathcal{F}^\mu$ the relativistic 2$^{\text{nd}}$ law of Newton is

$$\mathcal{F}^\mu = m_0 \mathcal{A}^\mu. \tag{I.2.56}$$

As an example, let us derive again the results for linear and circular motion.

In **linear motion**, $\boldsymbol{u} = (u_x, 0, 0)$, $\boldsymbol{f} = (f_x, 0, 0)$, Eqs. (I.2.55) and (I.2.42) give

$$\mathcal{F}^1 = m_0 \mathcal{A}^1 \longrightarrow \gamma_u f_x = m_0 \gamma_u^4 a_x \quad \text{or} \quad f_x = \gamma_u^3 m_0 a_x. \tag{I.2.57}$$

Whereas in the rest frame of the motion

$$\mathcal{F}^{1'} = m_0 \mathcal{A}^{1'} \longrightarrow f'_x = m_0 a'_x.$$

In case of **circular motion**, $\boldsymbol{u} = u(-\sin\varphi, \cos\varphi, 0)$, $\boldsymbol{a} = u^2(-\cos\varphi, -\sin\varphi, 0)/R$ and for e.g. $\varphi = 0$ it is

$$\mathcal{F}^2 = m_0\mathcal{A}^2 \longrightarrow \gamma_u f_y = m_0\gamma_u^2 a_y \quad \text{or} \quad f_y = \gamma_u m_0 a_y.$$

In the instantaneous rest frame S' one has

$$\mathcal{F}^{2'} = m_0\mathcal{A}^{2'} \longrightarrow f'_y = m_0 a'_y.$$

Both linear and circular motions correspond to Eqs. (I.2.23), (I.2.25) and also to the results given in the chapter I.2.4.9 for the acceleration 4-vector.

I.2.5 Transformation of electromagnetic fields

4-vectors do not exist for the electric and magnetic field, only for potentials. Therefore, we choose for the transformation of the fields a detour via the transformation of the power-force 4-vector. Substituting the Lorentz force

$$\boldsymbol{f} = q\,(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) \tag{I.2.58}$$

into Eq. (I.2.55)

$$\mathcal{F} = \gamma_u\left(\frac{1}{c}\boldsymbol{f}\cdot\boldsymbol{u}, f_x, f_y, f_z\right) = \gamma_u q\left(\frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{u}, \boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}\right) \tag{I.2.59}$$

and in components

$$\begin{bmatrix}\mathcal{F}^0\\ \mathcal{F}^1\\ \mathcal{F}^2\\ \mathcal{F}^3\end{bmatrix} = \frac{q}{c}\begin{bmatrix}0 & E_x & E_y & E_z\\ E_x & 0 & cB_z & -cB_y\\ E_y & -cB_z & 0 & cB_x\\ E_z & cB_y & -cB_x & 0\end{bmatrix}\begin{bmatrix}\gamma_u c\\ \gamma_u u_x\\ \gamma_u u_y\\ \gamma_u u_z\end{bmatrix} = \frac{q}{c}\mathbf{T}\mathcal{U}^\mu. \tag{I.2.60}$$

With the L-T from S to S' one obtains

$$\mathcal{F}'^\mu = \mathbf{L}\mathcal{F}^\mu = \frac{q}{c}\mathbf{L}\mathbf{T}\mathbf{L}^{-1}\mathcal{U}'^\mu = \frac{q}{c}\mathbf{T}'\mathcal{U}'^\mu \tag{I.2.61}$$

where

$$\mathbf{T}' = \begin{bmatrix}0 & E_x & \gamma(E_y - vB_z) & \gamma(E_z + vB_y)\\ E_x & 0 & \gamma(cB_z - \beta E_y) & -\gamma(cB_y + \beta E_z)\\ \gamma(E_y - vB_z) & -\gamma(cB_z - \beta E_y) & 0 & cB_x\\ \gamma(E_z + vB_y) & \gamma(cB_y + \beta E_z) & -cB_x & 0\end{bmatrix}.$$

Comparing $\mathbf{T}'$ in Eq. (I.2.61) with

$$\begin{bmatrix}\mathcal{F}'^0\\ \mathcal{F}'^1\\ \mathcal{F}'^2\\ \mathcal{F}'^3\end{bmatrix} = \frac{q}{c}\begin{bmatrix}0 & E'_x & E'_y & E'_z\\ E'_x & 0 & cB'_z & -cB'_y\\ E'_y & -cB'_z & 0 & cB'_x\\ E'_z & cB'_y & -cB'_x & 0\end{bmatrix}\begin{bmatrix}\gamma_{u'} c\\ \gamma_{u'} u'_x\\ \gamma_{u'} u'_y\\ \gamma_{u'} u'_z\end{bmatrix} = \frac{q}{c}\mathbf{T}\mathcal{U}'^\mu$$

one finds

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma(E_y - vB_z) & B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\ E'_z &= \gamma(E_z + vB_y) & B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{aligned}$$

which can be written in form of longitudinal and transverse field components

$$\begin{aligned} \boldsymbol{E}'_{\parallel} &= \boldsymbol{E}_{\parallel} & \boldsymbol{B}'_{\parallel} &= \boldsymbol{B}_{\parallel} \\ \boldsymbol{E}'_{\perp} &= \gamma\left(\boldsymbol{E}_{\perp} + \boldsymbol{v} \times \boldsymbol{B}_{\perp}\right) & \boldsymbol{B}'_{\perp} &= \gamma\left(\boldsymbol{B}_{\perp} - \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E}_{\perp}\right). \end{aligned} \tag{I.2.62}$$

It is interesting to evaluate the expressions

$$\begin{aligned} \boldsymbol{E} \cdot \boldsymbol{B} &= \boldsymbol{E}' \cdot \boldsymbol{B}' \\ E^2 - c^2 B^2 &= E'^2 - c^2 B'^2 \end{aligned} \tag{I.2.63}$$

which are invariant (exercise: I.2.6.11). In case of plane waves, the expressions are particularly simple. Since, $\boldsymbol{E}$ is perpendicular to $\boldsymbol{B}$ and $E = cB$, the expressions are zero in all frames.

A plane wave remains a plane wave in all frames. Only the amplitudes change.

I.2.5.1 Example: Plane wave traveling in x-direction

In the laboratory frame

$$\begin{aligned} E_y &= E_0 & B_z &= \frac{E_0}{c} \\ E_x &= E_z = 0 & B_x &= B_y = 0. \end{aligned}$$

The transformed fields Eq. (I.2.62) are

$$E'_x = E'_z = B'_x = B'_y = 0, \qquad E'_y = \sqrt{\frac{1-\beta}{1+\beta}}E_0, \qquad B'_z = \sqrt{\frac{1-\beta}{1+\beta}}\frac{E_0}{c}.$$

With increasing v the field amplitudes decrease until for $v \to c$ the wave vanishes.

I.2.5.2 Example: Uniformly moving point charge

A point charge at rest in S' has a purely radial electric field, which in components is

$$\begin{aligned} E'_x &= \frac{q}{4\pi\varepsilon_0}\frac{x'}{(\rho'^2 + x'^2)^{\frac{3}{2}}}, & \rho'^2 &= y'^2 + z'^2 \\ E'_\rho &= \frac{q}{4\pi\varepsilon_0}\frac{\rho'}{(\rho'^2 + x'^2)^{\frac{3}{2}}}, & \boldsymbol{B}' &= \boldsymbol{0}. \end{aligned}$$

In S the charge is moving with v and the transformed fields are

$$
\begin{aligned}
\boldsymbol{E}_{\parallel} &= \boldsymbol{E}'_{\parallel} && \longrightarrow E_x = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x}{(\rho^2+\gamma^2x^2)^{\frac{3}{2}}}\\
\boldsymbol{B}_{\parallel} &= \boldsymbol{B}'_{\parallel} && \longrightarrow B_x = 0\\
\boldsymbol{E}_{\perp} &= \gamma\left(\boldsymbol{E}'_{\perp} - \boldsymbol{v}\times\boldsymbol{B}'_{\perp}\right) && \longrightarrow E_\rho = \frac{q}{4\pi\varepsilon_0}\frac{\gamma \rho}{(\rho^2+\gamma^2x^2)^{\frac{3}{2}}}\\
\boldsymbol{B}_{\perp} &= \gamma\left(\boldsymbol{B}'_{\perp} + \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}'_{\perp}\right) && \longrightarrow B_\varphi = \frac{q}{4\pi\varepsilon_0 c}\frac{\beta\gamma \rho}{(\rho^2+\gamma^2x^2)^{\frac{3}{2}}}
\end{aligned}
$$

where

$$\rho' = \rho, \quad \rho^2 = y^2 + z^2, \quad x = x'/\gamma.$$

The field components can be written as

$$
\begin{aligned}
E_r &= \sqrt{E_x^2 + E_\rho^2} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma\sqrt{x^2+\rho^2}}{(\rho^2+\gamma^2x^2)^{\frac{3}{2}}} = \frac{q}{4\pi\varepsilon_0 r^2}\frac{1}{\gamma^2\left(1-\beta^2\sin^2\vartheta\right)^{\frac{3}{2}}}\\
B_\varphi &= \frac{\mu_0 q}{4\pi r^2}\frac{v\sin\vartheta}{\gamma^2\left(1-\beta^2\sin^2\vartheta\right)^{\frac{3}{2}}}
\end{aligned}
$$

with ϑ the angle between r and the x-axis. In forward and backward direction, $\vartheta \approx 0, \pi$, the fields are reduced by $1/\gamma^2$, whereas in perpendicular direction, $\vartheta \approx \pi/2$, the fields are increased by γ, Fig. I.2.20. For very large γ the fields are confined within an angle of $\vartheta \approx \pi/2 \pm 2/\gamma$.

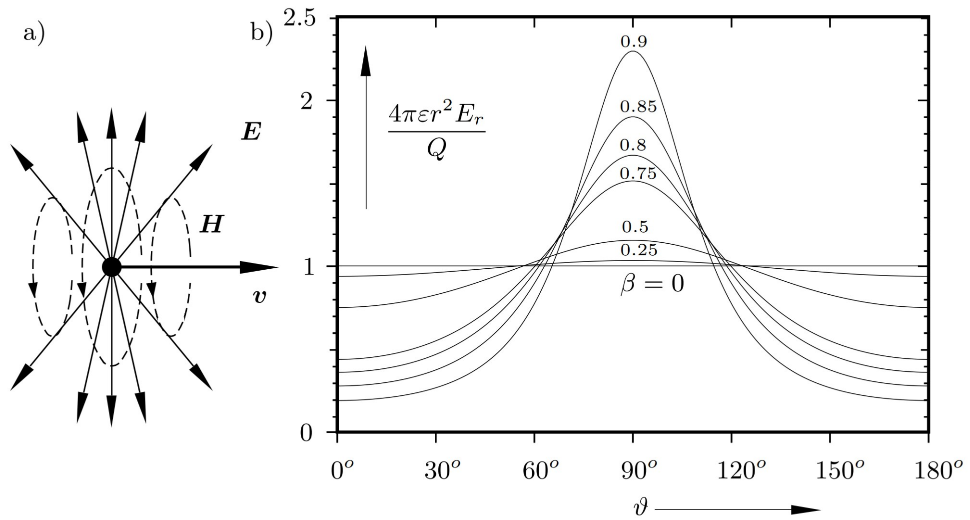


Fig. I.2.20: Uniformly moving point charge. a) Field plot. b) Magnitude of the electric field as a function of ϑ.

I.2.6 Exercises

I.2.6.1 Exercise 1: Prove that the scalar product of any two 4-vectors is Lorentz invariant

I.2.6.2 Exercise 2: Prove the relativistic 2nd law $\mathcal{F}^\mu = m_0\mathcal{A}^\mu$ by comparing the components

I.2.6.3 Exercise 3: A spacecraft travels away from earth with $\beta = 0.8$. At a distance $d = 2.16\cdot 10^8$ km a radio signal from earth is transmitted to the spacecraft. How long does the signal need to reach the spacecraft in the system of the earth? (solve it by using Minkowski diagram)

I.2.6.4 Exercise 4: A charge q is at rest. At $t = 0$ an electric field E_x is turned on. Calculate the velocity of the charge in two different ways: 1) In the laboratory frame, 2) Using the proper acceleration $\alpha = qE_x/m_0$ in the instantaneous rest frame S'.

I.2.6.5 Exercise 5: A particle moves in S with velocity $u = (0, u_y, 0)$ and experiences a force $f = (0, f_y, 0)$. What is the force in S', which moves with v?

I.2.6.6 Exercise 6: An electron, velocity u, collides inelastically with a proton at rest. After collision, a photon is emitted, and the electron and proton are moving together with u'. What is the energy of the emitted photon?

I.2.6.7 Exercise 7: A point charge moves with velocity v parallel to a current carrying wire. Calculate the force on the charge in its rest frame S' by transforming the electro-magnetic fields.

I.2.6.8 Exercise 8: A charged particle, velocity u, is injected into a homogeneous magnetic field B_0 pointing in z-direction. Calculate its trajectory.

I.2.6.9 Exercise 9: A proton synchrotron has an injection energy of $E_i = 2$ GeV and a final energy of $E_f = 4$ GeV. What is the required change of the magnetic field for a constant radius?

I.2.6.10 Exercise 10: Prove that the velocity 4-vector is perpendicular to the acceleration 4-vector.

I.2.6.11 Exercise 11: Prove that the two expressions $E\cdot B = E'\cdot B'$, $E^2 - c^2B^2 = E'^2 - c^2B'^2$ are invariant.

I.2.7 Solutions to the exercises

I.2.7.1 Solution to exercise 1

Proof:

$$\mathcal{A}^\mu = \left(\mathcal{A}^0, \mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3\right), \qquad \mathcal{B}^\mu = \left(\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3\right)$$
$$\mathcal{A}^\mu\mathcal{B}_\mu = \mathcal{A}^0\mathcal{B}_0 - \mathcal{A}^1\mathcal{B}_1 - \mathcal{A}^2\mathcal{B}_2 - \mathcal{A}^3\mathcal{B}_3$$

L-T of $\mathcal{A}^\mu$, $\mathcal{B}^\mu$

$$\begin{aligned}
\mathcal{A}'^\mu &= \left(\gamma\left(\mathcal{A}^0 - \beta\mathcal{A}^1\right), \gamma\left(-\beta\mathcal{A}^0 + \mathcal{A}^1\right), \mathcal{A}^2, \mathcal{A}^3\right) \\
\mathcal{B}'^\mu &= \left(\gamma\left(\mathcal{B}^0 - \beta\mathcal{B}^1\right), \gamma\left(-\beta\mathcal{B}^0 + \mathcal{B}^1\right), \mathcal{B}^2, \mathcal{B}^3\right) \\
\mathcal{A}'^\mu\mathcal{B}'_\mu &= \gamma^2\left(\mathcal{A}^0\mathcal{B}_0 - \beta\mathcal{A}^0\mathcal{B}_1 - \beta\mathcal{A}^1\mathcal{B}_0 + \beta^2\mathcal{A}^1\mathcal{B}_1 - \beta^2\mathcal{A}^0\mathcal{B}_0 + \beta\mathcal{A}^0\mathcal{B}_1 + \beta\mathcal{A}^1\mathcal{B}_0 - \mathcal{A}^1\mathcal{B}_1\right) \\
&\quad - \mathcal{A}^2\mathcal{B}_2 - \mathcal{A}^3\mathcal{B}_3
\end{aligned}$$

$$= \gamma^2 \left(1-\beta^2\right) \mathcal{A}^0\mathcal{B}_0 - \gamma^2 \left(1-\beta^2\right) \mathcal{A}^1\mathcal{B}_1 - \mathcal{A}^2\mathcal{B}_2 - \mathcal{A}^3\mathcal{B}_3$$
$$= \mathcal{A}^0\mathcal{B}_0 - \mathcal{A}^1\mathcal{B}_1 - \mathcal{A}^2\mathcal{B}_2 - \mathcal{A}^3\mathcal{B}_3 = \mathcal{A}^\mu\mathcal{B}_\mu$$

I.2.7.2 Solution to exercise 2

It is:

$$\mathcal{F}^\mu = \gamma_u \left[\frac{1}{c}\boldsymbol{f}\cdot\boldsymbol{u}, \boldsymbol{f}\right] \tag{I.2.64}$$

$$\mathcal{A}^\mu = \gamma_u^2 \left[\gamma_u^2 \left(\boldsymbol{\beta}_u \cdot \boldsymbol{a}\right), \gamma_u^2 \left(\boldsymbol{\beta}_u \cdot \boldsymbol{a}\right)\boldsymbol{\beta}_u + \boldsymbol{a}\right] \tag{I.2.65}$$

$$\boldsymbol{f} = \gamma_u m_0 \left[\gamma_u^2 \left(\boldsymbol{\beta}_u \cdot \boldsymbol{a}\right)\boldsymbol{\beta}_u + \boldsymbol{a}\right] \tag{I.2.66}$$

$$\boldsymbol{f}\cdot\boldsymbol{\beta}_u = \gamma_u m_0 \left(\boldsymbol{\beta}_u \cdot \boldsymbol{a}\right)\left[\gamma_u^2\beta_u^2 + 1\right] = \gamma_u^3 m_0 \left(\boldsymbol{\beta}_u \cdot \boldsymbol{a}\right) \tag{I.2.67}$$

Substituting Eqs. (I.2.66), (I.2.67) in Eq. (I.2.64) and comparing with Eq. (I.2.66) gives

$$\mathcal{F}^\mu = m_0\mathcal{A}^\mu .$$

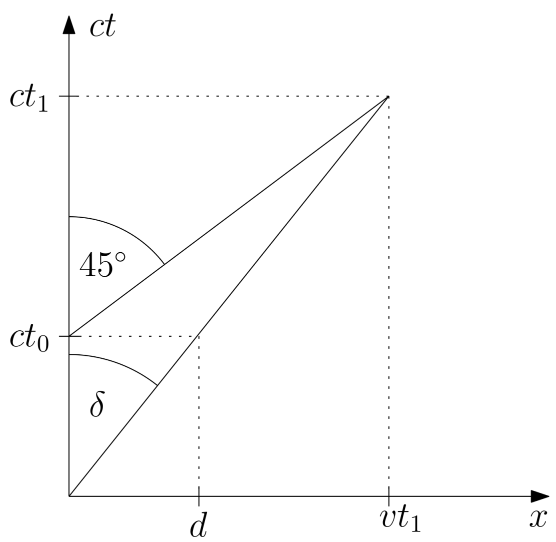
I.2.7.3 Solution to exercise 3

The spacecraft travels on the time line ct' in its rest frame S'. The time line is under an angle δ. Relative to the earth it has travelled a distance d in the time ct_0.

$$\tan\delta = \beta = \frac{d}{ct_0} \longrightarrow \delta = 38.7^\circ, \qquad t_0 = \frac{d}{\beta c} = 0.9\times 10^3\,\text{s}.$$

At ct_0 the radio signal is emitted and travels under an angle of 45°. It crosses the time line of the spacecraft at ct_1

$$ct_1 = ct_0 + \frac{vt_1}{\tan 45^\circ} \longrightarrow t_1 = \frac{t_0}{1-\beta} = 5t_0, \qquad\qquad t_1 - t_0 = 4t_0 = 3600\,\text{s} = 1\,\text{h}.$$



I.2.7.4 Solution to exercise 4

1) In the laboratory frame S

$$f_x = \frac{dp_x}{dt} = m_0 \frac{d}{dt} \frac{u_x}{\sqrt{1-(u_x/c)^2}} = qE_x$$

and after integration

$$\frac{u_x/c}{\sqrt{1-(u_x/c)^2}} = \frac{q}{m_0 c} E_x t = \frac{\alpha}{c} t$$

$$\frac{u_x}{c} = \frac{(\alpha/c)t}{\sqrt{1+(\alpha t/c)^2}}.$$

2) In the instantaneous rest frame S'

$$\boldsymbol{u}' = (0,0,0), \quad \boldsymbol{a}' = (\alpha,0,0)$$

where α is the proper acceleration. In S the acceleration is

$$\boldsymbol{\alpha} = (\alpha/\gamma^3, 0, 0).$$

Since the relative velocity between S and S' is the charge velocity u_x, the acceleration in S is

$$a_x = \frac{du_x}{dt} = \alpha \left[1 - \left(\frac{u_x}{c}\right)^2\right]^{\frac{3}{2}}$$

$$\frac{du_x/c}{[1-(u_x/c)^2]^{\frac{3}{2}}} = \frac{\alpha}{c} dt \longrightarrow \frac{u_x}{c} = \frac{(\alpha/c)t}{\sqrt{1+\left(\frac{\alpha}{c}t\right)^2}}.$$

I.2.7.5 Solution to exercise 5

Power-force 4-vector

$$\mathcal{F}^\mu = \gamma_u \left(\frac{1}{c}\boldsymbol{f}\cdot\boldsymbol{u}, f_x, f_y, f_z\right) = \gamma_u \left(\frac{1}{c} f_y u_y, 0, f_y, 0\right).$$

In S' the velocity $\boldsymbol{u}'$ appears as

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} = -v, \quad u'_y = \frac{u_y}{\gamma(1-u_x v/c^2)} = \frac{u_y}{\gamma}, \quad u'_z = 0$$

and $\gamma_{u'}$ becomes

$$\gamma_{u'} = \frac{1}{\sqrt{1-(u'^2_x + u'^2_y)/c^2}} = \frac{1}{\sqrt{1-(v/c)^2-(u_y/\gamma c)^2}} = \frac{\gamma}{\sqrt{\gamma^2(1-\beta^2)-\beta_u^2}} = \gamma\gamma_u. \quad \text{(I.2.68)}$$

L-T of $\mathcal{F}^\mu$

$$\begin{aligned} \mathcal{F}'^1 &= \gamma\left(\mathcal{F}^1 - \beta\mathcal{F}^0\right) & & \gamma_{u'} f'_x = -\beta\gamma\gamma_u \frac{1}{c} f_y u_y \\ \mathcal{F}'^2 &= \mathcal{F}^2 & \longrightarrow\ & \gamma_{u'} f'_y = \gamma_u f_y \\ \mathcal{F}'^3 &= \mathcal{F}^3 & & \gamma_{u'} f'_z = 0. \end{aligned} \tag{I.2.69}$$

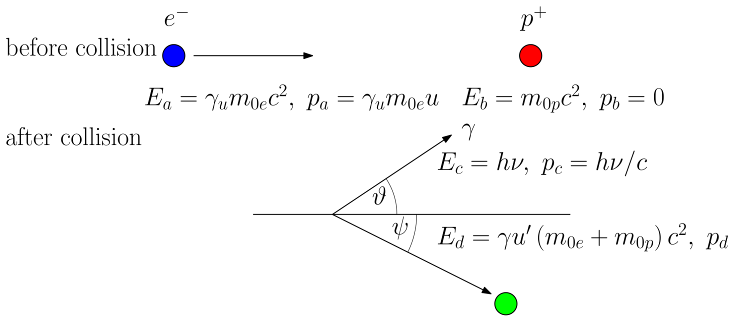
Substituting Eq. (I.2.68) into (I.2.69) gives a virtual force in x'-direction, due to the motion of S'

$$f'_x = -\beta\beta_u f_y$$

and a real force in y'-direction

$$f'_y = \frac{f_y}{\gamma}, \quad f'_z = 0.$$

I.2.7.6 Solution to exercise 6



Using the energy-momentum 4-vector

$$\mathcal{P}^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

the energy and momentum conservation requires

$$\mathcal{P}_a^\mu + \mathcal{P}_b^\mu = \mathcal{P}_c^\mu + \mathcal{P}_d^\mu \tag{I.2.70}$$

which becomes after squaring

$$\mathcal{P}_a^\mu \mathcal{P}_{a\mu} + 2\mathcal{P}_a^\mu \mathcal{P}_{b\mu} + \mathcal{P}_b^\mu \mathcal{P}_{b\mu} = \mathcal{P}_c^\mu \mathcal{P}_{c\mu} + 2\mathcal{P}_c^\mu \mathcal{P}_{d\mu} + \mathcal{P}_d^\mu \mathcal{P}_{d\mu}. \tag{I.2.71}$$

In the laboratory frame S it is

$$\begin{aligned} \mathcal{P}_a^\mu &= \gamma_u \frac{E_{0e}}{c}(1, \beta_u, 0, 0), & \mathcal{P}_b^\mu &= \frac{E_{0p}}{c}(1, 0, 0, 0) \\ \mathcal{P}_c^\mu &= \frac{h\nu}{c}(1, \cos\vartheta, \sin\vartheta, 0), & \mathcal{P}_d^\mu &= \gamma_{u'} \frac{1}{c}(E_{0e} + E_{0p})(1, \beta_{u'}\cos\psi, -\beta_{u'}\sin\psi, 0) \\ \mathcal{P}_a^\mu \mathcal{P}_{b\mu} &= \gamma_u \frac{1}{c^2} E_{0e} E_{0p}, & \mathcal{P}_a^\mu \mathcal{P}_{c\mu} &= \gamma_u \frac{1}{c^2} E_{0e} h\nu (1 - \beta_u \cos\vartheta), \quad \mathcal{P}_b^\mu \mathcal{P}_{c\mu} = E_{0p} h\nu / c^2 \end{aligned} \tag{I.2.72}$$

and in the rest frame of (a), (b), (c)

$$\begin{aligned} \mathcal{P}_a^\mu \mathcal{P}_{a\mu} &= (E_{0e}/c)^2, & \mathcal{P}_b^\mu \mathcal{P}_{b\mu} &= (E_{0p}/c)^2 \\ \mathcal{P}_d^\mu \mathcal{P}_{d\mu} &= (E_{0p}+E_{0e})^2/c^2, & \mathcal{P}_c^\mu \mathcal{P}_{c\mu} &= 0. \end{aligned} \tag{I.2.73}$$

Substituting Eqs. (I.2.72), (I.2.73) into Eq. (I.2.71) gives

$$\mathcal{P}_c^\mu \mathcal{P}_{d\mu} = (\gamma_u - 1) E_{0e} E_{0p}/c^2 \tag{I.2.74}$$

and multiplying Eq. (I.2.70) with $\mathcal{P}_{c\mu}$

$$\mathcal{P}_a^\mu \mathcal{P}_{c\mu} + \mathcal{P}_b^\mu \mathcal{P}_{c\mu} = \mathcal{P}_c^\mu \mathcal{P}_{c\mu} + \mathcal{P}_d^\mu \mathcal{P}_{c\mu}. \tag{I.2.75}$$

Using Eqs. (I.2.72), (I.2.73), (I.2.74) in Eq. (I.2.75) one obtains the energy of the photon

$$h\nu = \frac{(\gamma_u - 1) E_{0e} E_{0p}}{\gamma_u (1 - \beta \cos\vartheta) E_{0e} + E_{0p}}.$$

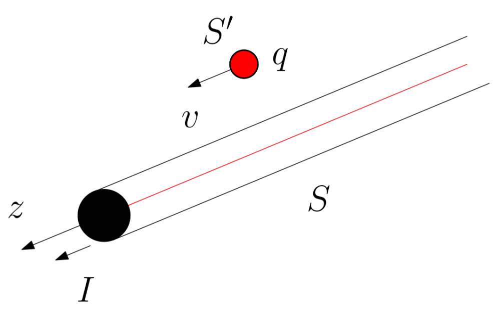
Since $E_{0e} \ll E_{0p}$ it is for γ_u small

$$h\nu \approx (\gamma_u - 1) E_{0e}.$$

For very large γ_u, $\beta \approx 1$, the photon is scattered in forward direction, $\vartheta \approx 0$, and has a high energy

$$h\nu \approx \frac{E_{0p}}{1 - \cos\vartheta}.$$

I.2.7.7 Solution to exercise 7



In the laboratory frame S the fields and the force are

$$\begin{aligned} &\boldsymbol{E} = (0,0,0), \quad \boldsymbol{B} = (0, B_\varphi, 0), \quad B_\varphi = \frac{\mu_0 I}{2\pi\rho} \\ &\boldsymbol{f}_e = q\boldsymbol{E} = 0, \quad \boldsymbol{f}_m = q\boldsymbol{v} \times \boldsymbol{B} = (-qvB\varphi, 0, 0). \end{aligned}$$

The Lorentz transformed fields and forces are

$$\begin{aligned} \boldsymbol{E}'_\perp &= \gamma\left(\boldsymbol{E}_\perp + \boldsymbol{v}\times\boldsymbol{B}_\perp\right) = (-\gamma v B_\varphi, 0, 0) \\ \boldsymbol{B}'_\perp &= \gamma\left(\boldsymbol{B}_\perp - \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}_\perp\right) = (0, \gamma B_\varphi, 0) \\ \boldsymbol{f}'_e &= q\boldsymbol{E}'_\perp = (-\gamma q v B_\varphi, 0, 0) = \gamma \boldsymbol{f}_m \\ \boldsymbol{f}'_m &= q\boldsymbol{v}'\times\boldsymbol{B}'_\perp = (0,0,0) \quad \text{since } v' = 0. \end{aligned}$$

The magnetic force in S transforms into an electric force in S'.

I.2.7.8 Solution to exercise 8

With the Lorentz force

$$\boldsymbol{f} = q\boldsymbol{u}\times\boldsymbol{B}_0$$

the acceleration becomes

$$\boldsymbol{a} = \frac{1}{\gamma m_0}\left(\boldsymbol{f} - \frac{1}{c^2}\left(\boldsymbol{f}\cdot\boldsymbol{u}\right)\boldsymbol{u}\right) = \frac{1}{\gamma m_0}\boldsymbol{f} = \frac{d\boldsymbol{u}}{dt}$$

or decomposed in cartesian components

$$\frac{du_x}{dt} = \omega_z u_y, \quad \frac{du_y}{dt} = -\omega_z u_x, \quad \frac{du_z}{dt} = 0 \tag{I.2.76}$$

where

$$\omega_z = \frac{qB_0}{\gamma m_0}, \quad \gamma = \frac{1}{\sqrt{1-(u/c)^2}}$$

is called **cyclotron frequency**. From Eq. (I.2.76)

$$\begin{aligned} \frac{d^2u_x}{dt^2} &= \omega_z\frac{du_y}{dt} = -\omega_z^2 u_x \\ u_x &= u_{x0}\cos\omega_z t + a\sin\omega_z t, & u_x(t=0) = u_{x0} \\ u_y &= \frac{1}{\omega_z}\frac{du_x}{dt} = -u_{x0}\sin\omega_z t + u_{y0}\cos\omega_z t, & u_y(t=0) = u_{y0} \\ u_z &= u_{z0} \end{aligned}$$

and after integration

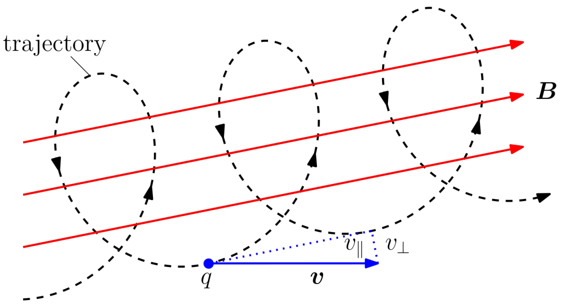
$$x = \frac{u_{x0}}{\omega_z}\sin\omega_z t - \frac{u_{y0}}{\omega_z}\cos\omega_z t, \quad y = \frac{u_{x0}}{\omega_z}\cos\omega_z t - \frac{u_{y0}}{\omega_z}\sin\omega_z t, \quad z = u_{z0}t. \tag{I.2.77}$$

Squaring and adding Eq. (I.2.77) one can eliminate cos and sin

$$x^2 + y^2 = \frac{u_{x0}^2 + u_{y0}^2}{\omega_z^2} = \frac{u_{\perp 0}^2}{\omega_z^2} = R^2 \to R = \frac{u_{\perp 0}}{\omega_z} = \gamma\frac{m_0 u_{\perp 0}}{qB_0}.$$

R is the so-called **Larmor radius**. The charge moves in a helical trajectory with radius R and a pitch of $2\pi u_{z0}/\omega_z$ in z-direction. On R the centrifugal force equals the centripetal force

$$\gamma m_0 \frac{u_{\perp 0}^2}{R} = qB_0 u_{\perp 0}.$$



I.2.7.9 Solution to exercise 9

$$R = \frac{\gamma m_0 u}{qB_0} = \frac{m_0 c}{qB_0}\gamma\beta = \frac{m_0 c}{qB_0}\sqrt{\gamma^2 - 1}.$$

In order to keep R constant B_0 has to change as

$$B = B_0\sqrt{\gamma^2 - 1}$$

and therefore with $E_{0p} = 938\,\mathrm{MeV}$, $\gamma = E/E_{0p}$

$$\frac{\Delta B}{B} = \frac{\sqrt{\gamma_f^2 - 1} - \sqrt{\gamma_i^2 - 1}}{\sqrt{\gamma_i^2 - 1}} = 1.20.$$

I.2.7.10 Solution to exercise 10

Velocity 4-vector

$$\mathcal{U}^\mu = \gamma c(1, \boldsymbol{\beta})$$

Acceleration 4-vector

$$\mathcal{A}^\mu = \gamma^2(\gamma^2(\boldsymbol{\beta}\cdot\boldsymbol{a}), \gamma^2(\boldsymbol{\beta}\cdot\boldsymbol{a})\boldsymbol{\beta} + \boldsymbol{a})$$

$$\mathcal{U}^\mu \mathcal{A}_\mu = c\gamma^3\left[\gamma^2(\boldsymbol{\beta}\cdot\boldsymbol{a}) - \gamma^2(\boldsymbol{\beta}\cdot\boldsymbol{a})\beta^2 - (\boldsymbol{\beta}\cdot\boldsymbol{a})\right] = c\gamma^3\left[(1-\beta^2)\gamma^2(\boldsymbol{\beta}\cdot\boldsymbol{a}) - (\boldsymbol{\beta}\cdot\boldsymbol{a})\right] = 0$$

I.2.7.11 Solution to exercise 11

From the transformed fields

$$\begin{aligned}\boldsymbol{E}'\cdot\boldsymbol{B}' &= [E_x\boldsymbol{e}_x + \gamma(E_y - vB_z)\boldsymbol{e}_y + \gamma(E_z + vB_y)\boldsymbol{e}_z]\cdot\left[B_x\boldsymbol{e}_x + \gamma\left(B_y + \frac{v}{c^2}E_z\right)\boldsymbol{e}_y + \gamma\left(B_z - \frac{v}{c^2}E_y\right)\boldsymbol{e}_z\right]\\ &= E_xB_x + E_yB_y + E_zB_z = \boldsymbol{E}\cdot\boldsymbol{B}\end{aligned}$$

$$
\begin{aligned}
E'^2 - c^2 B'^2 &= E_x^2 + \gamma^2 (E_y - vB_z)^2 + \gamma^2 (E_z + vB_y)^2 - c^2 \left[B_x^2 + \gamma^2 \left(B_y + \frac{v}{c^2} E_z \right)^2 + \gamma^2 \left(B_z - \frac{v}{c^2} E_y \right)^2 \right] \\
&= E_x^2 + E_y^2 + E_z^2 - c^2 \left(B_x^2 + B_y^2 + B_z^2 \right) \\
&= E^2 - c^2 B^2 .
\end{aligned}
$$

Although the fields $\boldsymbol{E}$, $\boldsymbol{B}$ are not 4-vectors the products given above are invariant. Plane waves remain plane waves when going into another frame.

Literature

R.P. Feynman, R.B. Leighton, M. Sands, Lectures on physics. Vol. I, Addison & Wesley (1963).

A.P. French, Special relativity, W.W. Norton & Company (1966).

J. Freund, Special relativity for beginners, World Scientific (2008).