Chapter I.1

Electromagnetism

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Electromagnetic fields are at the heart of accelerators. They accelerate, focus and guide charged particles and they are responsible for the stability as well as the instability of particle beams. Their range goes from constant fields up to very fast changing fields with frequencies of many GHz. Since electromagnetism is part of the university curriculum, we restrict ourselves to a review of some basics which are important to deal with problems in particle accelerators.

I.1.1 Introduction

Long ago, electricity and magnetism were well known separate phenomena. The birth of electromagnetism began with the discovery of Oestedt (1820) that an electric current is always associated with a magnetic field. Later on, Faraday (1831) discovered the electromagnetic induction, the creation of electric fields by a changing magnetic field. Electromagnetism was born. Maxwell (1864) extended and completed this work with the four equations, which relate the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$, together with the electromagnetic Lorentz force. The four equations are

$$\begin{aligned}
\oint \boldsymbol{H}(\boldsymbol{r},t)\cdot d\boldsymbol{s} &= \iint \boldsymbol{J}(\boldsymbol{r},t)\cdot d\boldsymbol{A} + \frac{d}{dt}\iint \boldsymbol{D}(\boldsymbol{r},t)\cdot d\boldsymbol{A} \quad ,\\
\oint \boldsymbol{E}(\boldsymbol{r},t)\cdot d\boldsymbol{s} &= -\frac{d}{dt}\iint \boldsymbol{B}(\boldsymbol{r},t)\cdot d\boldsymbol{A} \quad ,\\
\oiint \boldsymbol{D}(\boldsymbol{r},t)\cdot d\boldsymbol{A} &= \iiint \rho(\boldsymbol{r},t)\, dV \quad ,\\
\oiint \boldsymbol{B}(\boldsymbol{r},t)\cdot d\boldsymbol{A} &= 0 \quad ,
\end{aligned} \tag{I.1.1}$$

with

- $\boldsymbol{E}, \boldsymbol{H}$ the electric and magnetic fields,
- $\boldsymbol{D}, \boldsymbol{B}$ the electric displacement and the magnetic induction, which are responsible for the effects of material on the fields,
- $\boldsymbol{J}$ the electric current density,
- ρ the electric charge density.

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CERN Yellow Reports: School Proceedings, CERN-2024-003, DOI: 10.23730/CYRSP-2024-003, p. 3.

The term $\iint \boldsymbol{J}(\boldsymbol{r},t)\cdot d\boldsymbol{A}$ includes all currents going through the area A, which may consist of

$\boldsymbol{J}_c(\boldsymbol{r},t)=\kappa\boldsymbol{E}(\boldsymbol{r},t)$	the conduction current (Ohm's law),
$\boldsymbol{J}_{cv}(\boldsymbol{r},t)=\rho(\boldsymbol{r},t)\,\boldsymbol{v}(\boldsymbol{r},t)$	the convection current,
$\boldsymbol{J}_i(\boldsymbol{r},t)$	an impressed current.

The term $\iiint \rho(\boldsymbol{r},t)\,dV$ includes all charges in the volume V. Current and charge may have different distributions, e.g. point-like, lines, surfaces, volumes.

In most cases Maxwell's equations (I.1.1) are used in their differential form for which an extensive set of tools is available. They can be derived easily from Eq. (I.1.1) using Stokes' and Gauss' theorem

$$\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}+\frac{\partial\boldsymbol{D}}{\partial t}\quad, \tag{I.1.2}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}\quad, \tag{I.1.3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho\quad, \tag{I.1.4}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0\quad. \tag{I.1.5}$$

It should be noted that Maxwell's theory is a continuum theory. All quantities must be continuous. Discontinuities must be excluded but can be taken into account by means of the integral form Eq. (I.1.1).

In case of time-harmonic fields it is convenient to reduce the equations to time-independent equations. Let us take an example, the electric field

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_0(\boldsymbol{r})\cos(\omega t+\varphi)$$

can be written as

$$\boldsymbol{E}(\boldsymbol{r},t)=\mathrm{Re}\left\{\boldsymbol{E}_0(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{\mathrm{i}\omega t}\right\}=\mathrm{Re}\left\{\tilde{\boldsymbol{E}}(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\omega t}\right\}\quad, \tag{I.1.6}$$

where $\tilde{\boldsymbol{E}}$ is called **phasor**. The advantages are

- the time derivative $\partial/\partial t$ is replaced by $\mathrm{i}\omega$,
- phasors are vectors in a coordinate system rotating with ωt,
- $\mathrm{e}^{\mathrm{i}\omega t}$ cancels out in the equations.

In the following the tilde belonging to phasors will be dropped for convenience whenever time-harmonic fields are treated and/or the situation is sufficiently clear.

The fields $\boldsymbol{D}$ and $\boldsymbol{B}$ describe the effect of $\boldsymbol{E}$ and $\boldsymbol{H}$ on matter. As simple examples we take an isotropic and linear dielectric effect. A local electric field displaces the charge centers of an atom and creates an elementary dipole with a moment $p_e = qx$, Fig. I.1.1 An averaging procedure over the effect of the macroscopic field $\boldsymbol{E}$ on the microscopic dipoles (Clausius-Mossotti) leads to a macroscopic polarization field

$$\boldsymbol{P}=\varepsilon_0\chi_e\boldsymbol{E}$$

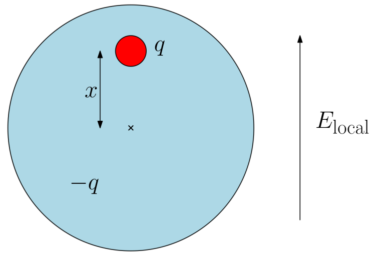


Fig. I.1.1: A local electric field displaces the charge centers of an atom and creates a dipole.

superimposed to $\boldsymbol{E}$

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \varepsilon_o \left(1 + \chi_e\right) \boldsymbol{E} = \varepsilon_r \varepsilon_0 \boldsymbol{E} \quad , \tag{I.1.7}$$

$\varepsilon_0 = 8.854 \times 10^{-12}\,\mathrm{As/Vm}$, permittivity of the vacuum. The relative permittivity ε_r and therefore $\boldsymbol{D}$ describe the influence of an electric field on matter. There are different microscopic effects: atoms or molecules which are polarized or displaced or even rotating. They happen at different frequencies and are connected with losses, which are represented by an imaginary part of ε_r

$$\varepsilon_r = \varepsilon_r' - \mathrm{i}\varepsilon_r''.$$

The real and imaginary part of ε_r are schematically shown in Fig. I.1.2

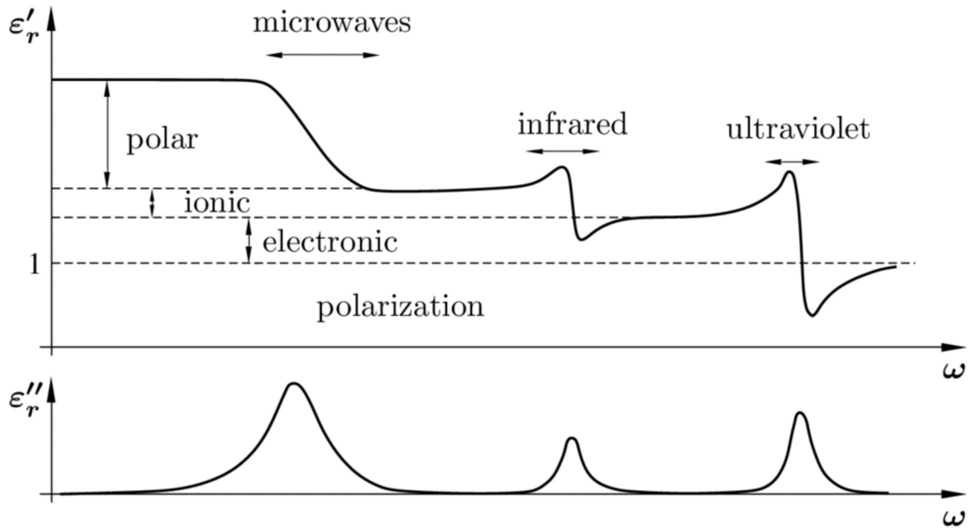


Fig. I.1.2: Principal behavior of real and imaginary part of ε_r over frequency.

The reaction of matter on magnetic fields is due the particle spins, which can be simulated by an elementary current with magnetic momentum $p_m = \pi r_e^2 I_e$, Fig. I.1.3. Again, an averaging procedure leads to a macroscopic magnetization field $\boldsymbol{M}$

$$\boldsymbol{M} = \chi_m \boldsymbol{H}$$

superimposed to $\boldsymbol{H}$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M} = \mu_0 \left(1 + \chi_m\right) \boldsymbol{H} = \mu_r \mu_0 \boldsymbol{H}, \tag{I.1.8}$$

$\mu_0 = 4\pi \times 10^{-7}\mathrm{Vs/Am}$, permeability of the vacuum.

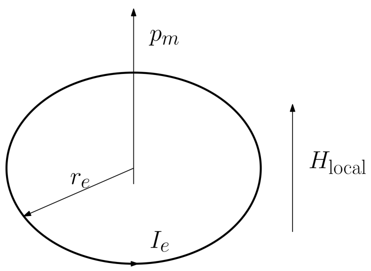


Fig. I.1.3: A local magnetic field causes elementary currents with a dipole moment p_m.

The relative permeability μ_r and therefore $\boldsymbol{B}$ describe the influence of a magnetic field on matter. In most technical materials μ_r is close to 1. An important exception are ferromagnetic materials where the relation between the external field and the magnetization is non-linear and depends on history (hysteresis), Fig. I.1.4.

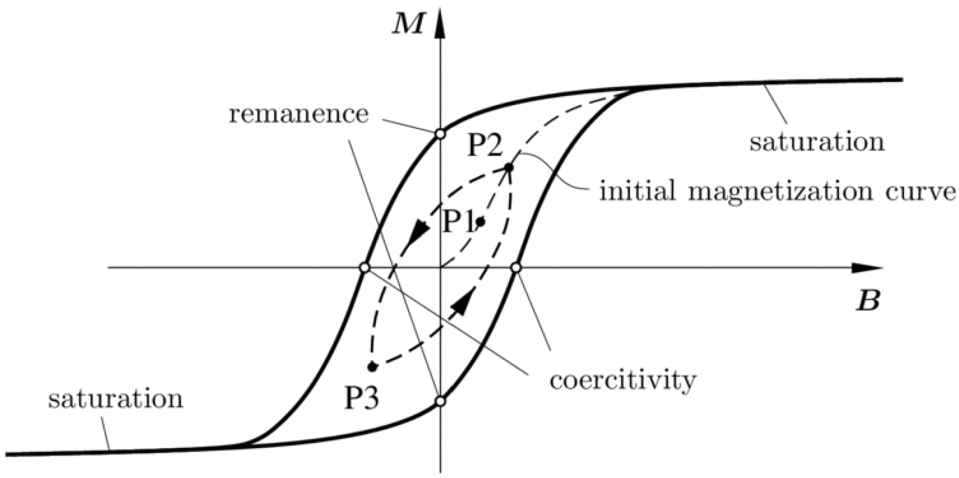


Fig. I.1.4: Typical behavior of $B(M)$ for ferromagnetic materials.

In many materials the relations $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{E})$ and $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{B})$ are linear. In some materials, however, they are non-linear or anisotropic and depend on time or frequency.

Typically they have losses due to radiation and/or friction between elementary dipoles. In technical literature the losses are expressed by a loss angle δ

$$\begin{aligned} \varepsilon &= \varepsilon' - \mathrm{i}\varepsilon'' = \varepsilon' \left(1 - \mathrm{i}\tan\delta_\varepsilon\right), \quad \tan\delta_\varepsilon = \varepsilon''/\varepsilon', \\ \mu &= \mu' - \mathrm{i}\mu'' = \mu' \left(1 - \mathrm{i}\tan\delta_\mu\right), \quad \tan\delta_\mu = \mu''/\mu'. \end{aligned} \tag{I.1.9}$$

In most dielectrics it is

$$\tan\delta_\varepsilon \ll 1, \quad \tan\delta_\mu \approx 0$$

and in good metallic conductors

$$|\boldsymbol{J}| \gg \left|\frac{\partial \boldsymbol{D}}{\partial t}\right| \longrightarrow \kappa \gg \omega\varepsilon_r\varepsilon_0$$

$$\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} = (\kappa + \mathrm{i}\omega\varepsilon_r\varepsilon_0)\,\boldsymbol{E} = \mathrm{i}\omega\varepsilon_r\varepsilon_0\left(1 - \mathrm{i}\frac{\kappa}{\omega\varepsilon_r\varepsilon_0}\right)\boldsymbol{E} = \mathrm{i}\omega\varepsilon_c\boldsymbol{E}, \quad \varepsilon_c \approx \frac{\kappa}{\mathrm{i}\omega}.$$

Finally, as mentioned above, Maxwell's theory is a continuum theory. It requires continuous, double differentiable functions. If the problem contains regions with different material, one derives general solutions for each region and matches the solutions on the interface between the regions. The matching process fixes the unknowns in the general solutions. This process uses boundary or continuity equations derived from Eq. (I.1.1). To get conditions for tangential field components a small rectangle, small compared to the distance in which the field changes, is chosen across the interface, Fig. I.1.5.

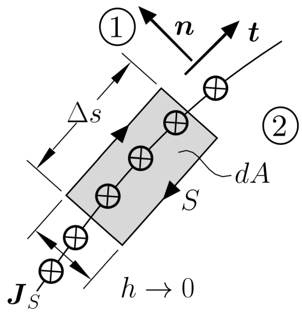


Fig. I.1.5: Integration path along a small rectangle across the interface.

Then, the first equation in Eqs. (I.1.1) gives

$$H_{t1}\Delta s - H_n h - H_{t2}\Delta s + H_n h = J_s\Delta s + \frac{\partial}{\partial t}\iint_{\Delta A} \boldsymbol{D}\cdot d\boldsymbol{A}$$

which for $h \to 0$ becomes

$$H_{t1} - H_{t2} = J_s. \tag{I.1.10}$$

Here a surface current density $\boldsymbol{J}_s$ was assumed. In a similar way the second equation in Eqs. (I.1.1) gives

$$E_{t1} - E_{t2} = 0. \tag{I.1.11}$$

If medium 2 is perfectly electric conducting, the fields vanish and Eq. (I.1.10), Eq. (I.1.11) become

$$H_{t1} = J_s, \quad E_{t1} = 0.$$

Conditions for the normal field components follow from the 3$^{\text{rd}}$ and 4$^{\text{th}}$ equations in Eq. (I.1.1). In that case one chooses a small cylinder, Fig. I.1.6.

The 3$^{\text{rd}}$ equation in Eq. (I.1.1) becomes

$$D_{n1}\Delta A - D_{n2}\Delta A + \iint_{\Delta A_{\text{cyl}}} \boldsymbol{D}\cdot d\boldsymbol{A} = \rho_s\Delta A$$

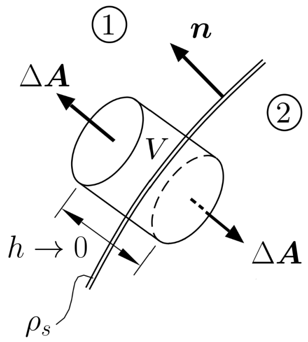


Fig. I.1.6: Integration surface of a small cylinder across the interface.

and with $h \to 0$

$$D_{n1} - D_{n2} = \rho_s, \tag{I.1.12}$$

equivalently for the 4$^{\text{th}}$ equation in Eq. (I.1.1)

$$B_{n1} - B_{n2} = 0. \tag{I.1.13}$$

Here a surface charge ρ_s was assumed. If medium 2 is perfectly electric conducting Eq. (I.1.12), Eq. (I.1.13) become

$$D_{n1} = \rho_s, \quad B_{n1} = 0.$$

After these general remarks we will treat some simplifications of Maxwell's equations and in particular wave solutions.

I.1.2 Electrostatic fields

Electrostatics describe situations where $\boldsymbol{H} = 0$ and $\partial/\partial t = 0$. Maxwell's equations Eqs. (I.1.2)-(I.1.5) simplify to

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0 \tag{I.1.14}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho. \tag{I.1.15}$$

Due to the identity $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = 0$, one can derive the electric field in Eq. (I.1.14) from a scalar potential ϕ

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi. \tag{I.1.16}$$

The negative sign is thereby chosen such that work has to be done when moving a positive charge against the field. Substituting Eq. (I.1.16) into Eq. (I.1.15) one gets a **Poisson** equation for the potential

$$\boldsymbol{\nabla}^2\phi = -\frac{\rho}{\varepsilon}. \tag{I.1.17}$$

As an example we consider an electrostatic lens consisting of two circular tubes with different potential, Fig. I.1.7.

The structure is cylindrically symmetric with no free charges and the Poisson equation (I.1.17) becomes

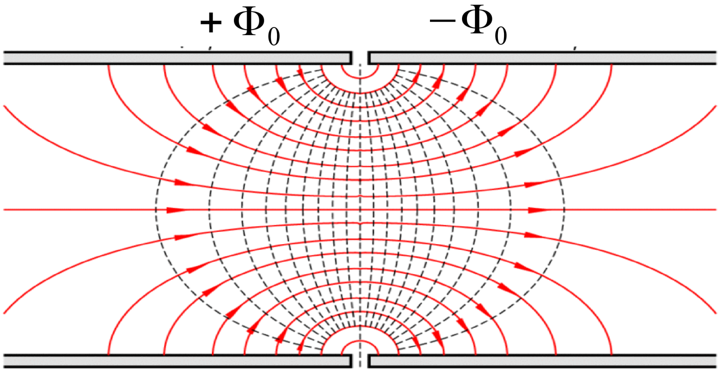


Fig. I.1.7: Two circular metallic tubes with different potential.

a circular symmetric **Laplace** equation

$$\boldsymbol{\nabla}^2\phi = \frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

Using a Bernoulli ansatz

$$\phi\left(\rho, z\right) = R\left(\rho\right) Z(z)$$

it can be written as

$$\frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \tag{I.1.18}$$

The last term is independent of ρ and must be a constant, here k_z^2. Then

$$\frac{d^2Z}{dz^2} - k_z^2 Z = 0$$

with solutions

$$Z(z) = \begin{cases} C_0 + D_0 z & k_z = 0 \\ C\mathrm{e}^{k_z z} + D\mathrm{e}^{-k_z z} & k_z \neq 0. \end{cases}$$

For $z \to \pm\infty$ ϕ must be finite, i.e. $C = D_0 = 0$ and Z becomes

$$Z(z) = \begin{cases} C_0 & k_z = 0 \\ D\mathrm{e}^{-k_z|z|} & k_z \neq 0. \end{cases} \tag{I.1.19}$$

The leftover equation (I.1.18) is **Bessel's** differential equation

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + k_z^2 R = 0$$

with Bessel J_0 and Neumann N_0 functions as solutions

$$R\left(\rho\right) = \begin{cases} A_0 + B_0 \ln\left(\frac{\rho}{\rho_0}\right) & k_z = 0 \\ AJ_0\left(k_z\rho\right) + BN_0\left(k_z\rho\right) & k_z \neq 0. \end{cases}$$

Since ϕ must stay finite for $\rho \to 0$ the constants B_0 and B must be zero. Further, for $\rho = a$ it is

$$\phi = \begin{cases} -\phi_0 & z > 0 \\ +\phi_0 & z < 0 \end{cases}$$

and therefore

$$A_0 C_0 = -\operatorname{sgn}(z)\phi_0$$
$$J_0(k_z a) = 0 \longrightarrow k_{zn} a = j_{0n}.$$

Using above conditions, ϕ simplifies to

$$\phi = -\operatorname{sgn}(z)\phi_0 + \sum_{n=1}^{\infty} A_n J_0\left(j_{0n}\frac{\rho}{a}\right) \mathrm{e}^{-j_{0n}|z|/a}. \tag{I.1.20}$$

Due to the antisymmetric behavior in z ϕ must be zero for $z = 0$ and Eq. (I.1.20) becomes

$$\phi_0 = \sum_{n=1}^{\infty} A_n J_0\left(j_{0n}\frac{\rho}{a}\right). \tag{I.1.21}$$

The coefficients A_n will be determined via a **Fourier-Bessel expansion**. One multiplies Eq. (I.1.21) with $\rho J_0(j_{0m}\rho/a)$ and integrates over ρ

$$\phi_0 \int_0^a J_0\left(j_{0m}\frac{\rho}{a}\right)\rho d\rho = \sum_{n=1}^{\infty} A_n \int_0^a J_0\left(j_{0n}\frac{\rho}{a}\right) J_0\left(j_{0m}\frac{\rho}{a}\right)\rho d\rho$$
$$\phi_0 \frac{a^2}{j_{0m}} J_1(j_{0m}) = A_m \frac{a^2}{2} J_1^2(j_{0m}).$$

With A_m given above the final result for the potential Eq. (I.1.20) is

$$\phi(\rho, z) = -\operatorname{sgn}(z)\phi_0 + 2\sum_{n=1}^{\infty} \frac{J_0\left(j_{0n}\frac{\rho}{a}\right)}{j_{0n} J_1(j_{0n})} \mathrm{e}^{-j_{0n}|z|/a}.$$

A field plot is shown in Fig. I.1.7.

I.1.3 Stationary currents

Stationary currents describe situations with $\partial/\partial t = 0$ and $\boldsymbol{H} \neq 0$. Maxwell's equations become in that case

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} \tag{I.1.22}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0. \tag{I.1.23}$$

Because of Eq. (I.1.23) one derives the electric field again from a scalar potential as in Eq. (I.1.16). Taking the divergence of Eq. (I.1.22) and substituting $\boldsymbol{J} = \kappa \boldsymbol{E}$ one gets a Laplace equation for ϕ

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) = 0 = \boldsymbol{\nabla} \cdot \boldsymbol{J} = \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{E}) = -\kappa \boldsymbol{\nabla}^2 \phi.$$

The situation is similar to electrostatics but boundary or continuity conditions are different. For instance in an example, Fig. I.1.8, the normal current density component has to be zero on the boundary

$$\boldsymbol{J}_n = \kappa \boldsymbol{E}_n = -\kappa \frac{d\phi}{dn} \boldsymbol{n} = 0.$$

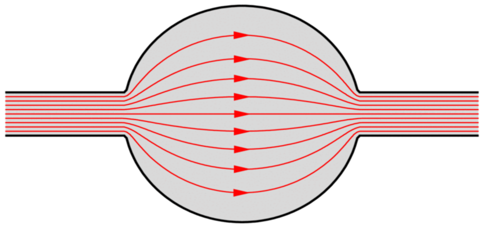


Fig. I.1.8: Constant current flow in a metallic object.

I.1.4 Magnetostatic fields

In the magnetostatic case one takes $\partial/\partial t = 0$ and $\boldsymbol{E} = 0$, but allows for impressed currents. Then, Maxwell's equations are

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{J} \tag{I.1.24}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{I.1.25}$$

Due to the identity $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$ one derives the magnetic field from a vector potential $\boldsymbol{A}$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{I.1.26}$$

Substituting Eq. (I.1.26) into Eq. (I.1.24) yields

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A} = \mu \boldsymbol{J}. \tag{I.1.27}$$

Here, $\boldsymbol{A}$ is not fully determined. A gauge transformation $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\psi$ does not change the magnetic field Eq. (I.1.26). One reduces the degree of freedom and imposes the condition $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ on Eq. (I.1.27)

$$\boldsymbol{\nabla}^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

or in Cartesian components

$$\boldsymbol{\nabla}^2 A_i = -\mu J_i, \quad i = x, y, z. \tag{I.1.28}$$

A simple way to solve Eq. (I.1.28) is to use the well known field and potential of a point charge q

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon r^2} \boldsymbol{e}_r, \quad \phi = \frac{q}{4\pi\varepsilon r}$$

which are solutions of the inhomogeneous Poisson equation

$$\boldsymbol{\nabla}^2 \phi = -\frac{q}{\varepsilon}\delta(r).$$

Taking ϕ as a **Green's function** one can calculate the potential of a charge distribution ρ with the **Coulomb integral**

$$\phi\left(\boldsymbol{r}\right) = \frac{1}{4\pi\varepsilon}\iiint \frac{\rho\left(\boldsymbol{r}'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}dV'.$$

The solution of Eq. (I.1.28) is obtained by replacing

$$\phi \rightarrow A_i, \quad \frac{1}{\varepsilon} \rightarrow \mu, \quad \rho \rightarrow J_i$$

$$\boldsymbol{A}\left(\boldsymbol{r}\right) = \frac{\mu}{4\pi}\iiint \frac{\boldsymbol{J}\left(\boldsymbol{r}'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}dV'. \tag{I.1.29}$$

Figure I.1.9 shows the corresponding geometrical quantities. In regions where $\boldsymbol{J} = \boldsymbol{0}$ one gets from Eq. (I.1.24) again a Laplace equation

$$\boldsymbol{\nabla}\times\boldsymbol{B} = 0 \longrightarrow \boldsymbol{B} = -\boldsymbol{\nabla}\varphi, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \longrightarrow \boldsymbol{\nabla}^2\varphi = 0. \tag{I.1.30}$$

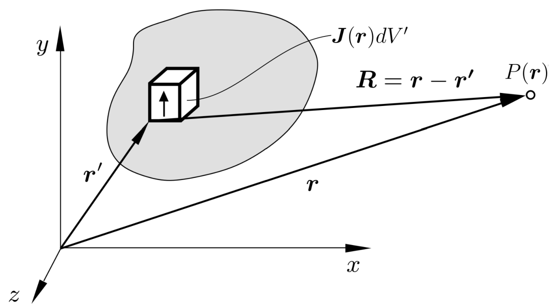


Fig. I.1.9: Geometry referring to Eq. (I.1.29).

I.1.5 Quasi-stationary fields

Still another simplification of Maxwell's equations is possible in materials with very high conductivity and no free charges, such that

$$\left|\boldsymbol{J}\right| = \kappa\left|\boldsymbol{E}\right| \gg \left|\frac{d\boldsymbol{D}}{dt}\right| = \omega\varepsilon\left|\boldsymbol{E}\right|, \quad \rho = 0.$$

Then,

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu\boldsymbol{J} \tag{I.1.31}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \tag{I.1.32}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0. \tag{I.1.33}$$

Due to Eq. (I.1.33) one derives $\boldsymbol{B}$ from a vector potential, $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, and substitutes it into Eq. (I.1.32)

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\boldsymbol{A} = -\boldsymbol{\nabla}\times\frac{\partial\boldsymbol{A}}{\partial t} \rightarrow \boldsymbol{\nabla}\times\left(\boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t}\right) = \boldsymbol{0},$$

which allows an ansatz for $\boldsymbol{E}$

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial\boldsymbol{A}}{\partial t}.$$

Substituting $\boldsymbol{B}$ and $\boldsymbol{E}$ into Eq. (I.1.31)

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A} = -\mu\kappa\left(\boldsymbol{\nabla}\phi + \frac{\partial\boldsymbol{A}}{\partial t}\right). \tag{I.1.34}$$

Here again, $\boldsymbol{A}$ and ϕ are not fully determined since the replacements

$$\boldsymbol{A}\rightarrow\boldsymbol{A}+\boldsymbol{\nabla}\psi, \quad \phi\rightarrow\phi-\frac{\partial\psi}{\partial t}$$

do not change $\boldsymbol{B}$ and $\boldsymbol{E}$. One imposes the gauge

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = -\mu\kappa\phi$$

and obtains the **vectorial diffusion equation**

$$\boldsymbol{\nabla}^2\boldsymbol{A} - \mu\kappa\frac{\partial\boldsymbol{A}}{\partial t} = 0. \tag{I.1.35}$$

For time-harmonic processes the equation is equal to the **Helmholtz** equation, which we will treat later.

I.1.6 Poynting's theorem

One starts with the full set of Maxwell's equations (I.1.2)–(I.1.5) and a gedanken experiment. The fields move a blob of charge ρdV by a distance $\delta\boldsymbol{s}$ in a time interval δt. The work done by the fields is

$$d\frac{\delta W}{\delta t} = d\boldsymbol{f}\cdot\frac{\delta\boldsymbol{s}}{\delta t} = \rho dV\,(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{v} = \boldsymbol{E}\cdot\rho\boldsymbol{v}dV = \boldsymbol{E}\cdot\boldsymbol{J}dV.$$

The term $\boldsymbol{E}\cdot\boldsymbol{J}$ will be expressed by means of Maxwell's equations. One multiplies Eq. (I.1.2) with $\boldsymbol{E}$ and Eq. (I.1.3) with $\boldsymbol{H}$ and subtracts the equations

$$\begin{aligned}\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{H}) &= \boldsymbol{E}\cdot\boldsymbol{J} + \boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}\\ \boldsymbol{H}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) &= -\boldsymbol{H}\cdot\frac{\partial\boldsymbol{B}}{\partial t}\end{aligned} \tag{I.1.36}$$

$$-\boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{H}) = \boldsymbol{E}\cdot\boldsymbol{J} + \boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{H}\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{E}\cdot\boldsymbol{J} + \frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D}+\frac{1}{2}\boldsymbol{H}\cdot\boldsymbol{B}\right).$$

Integration over a volume V and application of Gauss' law results in **Poynting's** theorem

$$-\oint\!\!\!\oint_A(\boldsymbol{E}\times\boldsymbol{H})\cdot d\boldsymbol{A} = \iiint_V \boldsymbol{E}\cdot\boldsymbol{J}dV + \frac{\partial}{\partial t}\iiint\left(\frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D}+\frac{1}{2}\boldsymbol{H}\cdot\boldsymbol{B}\right)dV, \tag{I.1.37}$$

$$\begin{aligned}
\boldsymbol{S} &= \boldsymbol{E} \times \boldsymbol{H} && \text{Poynting vector (radiation flux),} \\
p_d &= \boldsymbol{E} \cdot \boldsymbol{J} && \text{dissipated power density,} \\
w_e &= \frac{1}{2} \boldsymbol{E} \cdot \boldsymbol{D} && \text{electric energy density,} \\
w_m &= \frac{1}{2} \boldsymbol{H} \cdot \boldsymbol{B} && \text{magnetic energy density.}
\end{aligned}$$

Poynting's theorem states: The energy radiated into the volume V equals the dissipation and the increase of stored electric and magnetic energy in V, see Fig. I.1.10.

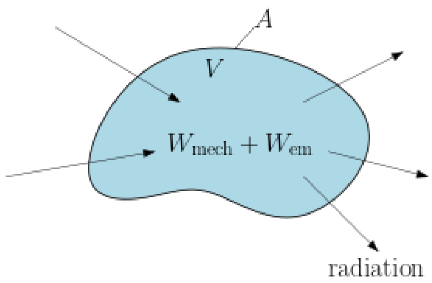


Fig. I.1.10: Radiation balance for a lossy volume with stored electromagnetic energy.

The above theorem deals with an arbitrary time-dependence. But many applications have time-harmonic fields and one is not interested in the momentaneous values but in the values averaged over one period.

Similar to Eq. (I.1.36), one takes the first two Maxwell's equations but in phasor notation

$$\begin{aligned}
\boldsymbol{\nabla} \times \tilde{\boldsymbol{H}}^* &= \tilde{\boldsymbol{J}}^* - \mathrm{i}\omega \tilde{\boldsymbol{D}}^* \\
\boldsymbol{\nabla} \times \tilde{\boldsymbol{E}} &= -\mathrm{i}\omega \tilde{\boldsymbol{B}},
\end{aligned}$$

where the star indicates complex conjugate quantities. One multiplies the first equation with $\tilde{\boldsymbol{E}}/2$, the second equation with $\tilde{\boldsymbol{H}}^*/2$ and takes the difference

$$-\frac{1}{2} \boldsymbol{\nabla} \cdot \left(\tilde{\boldsymbol{E}} \times \tilde{\boldsymbol{H}}^* \right) = \frac{1}{2} \tilde{\boldsymbol{E}} \cdot \tilde{\boldsymbol{J}}^* + \mathrm{i}2\omega \left(\frac{1}{4} \tilde{\boldsymbol{H}} \cdot \tilde{\boldsymbol{B}}^* - \frac{1}{4} \tilde{\boldsymbol{E}} \cdot \tilde{\boldsymbol{D}}^* \right).$$

Integration over the volume V and application of Gauss' law gives **Poynting's theorem** for time-harmonic fields

$$-\oint\!\!\!\oint_A \boldsymbol{S}_c \cdot d\boldsymbol{A} = \iiint_V \overline{p}_d dV + \mathrm{i}2\omega \iiint \left(\overline{w}_m - \overline{w}_e \right) dV, \tag{I.1.38}$$

where

$$\begin{aligned}
\boldsymbol{S}_c &= \frac{1}{2} \tilde{\boldsymbol{E}} \times \tilde{\boldsymbol{H}}^* && \text{complex, time-averaged radiation flux,} \\
\overline{p}_d &= \frac{1}{2} \tilde{\boldsymbol{E}} \cdot \tilde{\boldsymbol{J}}^* && \text{time-averaged dissipation,} \\
\overline{w}_m &= \frac{1}{4} \tilde{\boldsymbol{H}} \cdot \tilde{\boldsymbol{B}}^*, \ \overline{w}_e = \frac{1}{4} \tilde{\boldsymbol{E}} \cdot \tilde{\boldsymbol{D}}^* && \text{time-averaged magnetic and electric energy density.}
\end{aligned}$$

The real part of Eq. (I.1.38) is the time-averaged active power and the imaginary part the reactive power. It is interesting to note that in a resonator, where $\overline{w}_m = \overline{w}_e$, the reactive power is zero.

I.1.7 Electromagnetic waves

The simplest electromagnetic wave is a **plane wave**. It depends only on one space variable (direction of propagation) and on the time, e.g.

$$\boldsymbol{E} = \boldsymbol{E}(z,t), \quad \boldsymbol{H} = \boldsymbol{H}(z,t).$$

Substituted into the first two Maxwell's equations

$$\nabla \times \boldsymbol{H} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t}, \quad \nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t}$$

results in two sets of uncoupled equations

$$\begin{aligned} -\frac{\partial H_y}{\partial z} &= \varepsilon \frac{\partial E_x}{\partial t}, & \frac{\partial E_x}{\partial z} &= -\mu \frac{\partial H_y}{\partial t} \\ \frac{\partial H_x}{\partial z} &= \varepsilon \frac{\partial E_y}{\partial t}, & -\frac{\partial E_y}{\partial z} &= -\mu \frac{\partial H_x}{\partial t}. \end{aligned} \tag{I.1.39}$$

In the following we will treat the first set only. The second is obtained by replacing H_y by $-H_x$ and E_x by E_y. Eliminating H_y in Eq. (I.1.39) gives the **wave equation** for E_x

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial E_x}{\partial t^2} = 0, \quad c = \frac{1}{\sqrt{\mu\varepsilon}} \tag{I.1.40}$$

with d'Alembert's solutions

$$\begin{aligned} E_x &= f(z-ct) + g(z+ct) = E_x^+ + E_x^- \\ ZH_y &= f(z-ct) - g(z+ct) = ZH_y^+ - ZH_y^-. \end{aligned} \tag{I.1.41}$$

$c = 1/\sqrt{\mu\varepsilon}$ is the velocity of light and $Z = \sqrt{\mu/\varepsilon} = 377\,\Omega$ the free-space **wave impedance**. The function f belongs to a field propagating in $+z$-direction and g to a field propagating in $-z$-direction. The fields have the properties

$\boldsymbol{E}^\pm \perp \boldsymbol{H}^\pm$,
$\boldsymbol{E}^\pm, \boldsymbol{H}^\pm$ are perpendicular to the direction of propagation,
$\boldsymbol{E}^\pm/\boldsymbol{H}^\pm = \pm Z$,
$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ Poynting vector, energy flow in direction of propagation.

For simplicity, only the time-harmonic case, $\partial/\partial t = \mathrm{i}\omega$, will be treated. Then, Eq. (I.1.40), Eq. (I.1.41) become

$$\begin{aligned} &\frac{\partial E_x}{\partial z^2} + k^2 E_x = 0, \qquad k = \omega\sqrt{\mu\varepsilon} \\ &E_x = A\mathrm{e}^{\mathrm{i}(\omega t - kz)} + B\mathrm{e}^{\mathrm{i}(\omega t + kz)}, \\ &ZH_y = A\mathrm{e}^{\mathrm{i}(\omega t - kz)} - B\mathrm{e}^{\mathrm{i}(\omega t + kz)}. \end{aligned} \tag{I.1.42}$$

k is the **wave number**. In loss-free material it is

$$k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

and in lossy material

$$k = \omega\sqrt{\mu\varepsilon} = \beta - \mathrm{i}\alpha, \tag{I.1.43}$$

where β is the **phase constant** and α the **attenuation constant**. The real physical field is, e.g.

$$E_x^+ = \mathrm{Re}\left\{A\mathrm{e}^{\mathrm{i}(\omega t - kz)}\right\} = A\cos(\omega t - \beta z)\mathrm{e}^{-\alpha z}.$$

It has a time-harmonic factor and an exponential decay. Fig. I.1.11 shows the field pattern and the real field behavior.

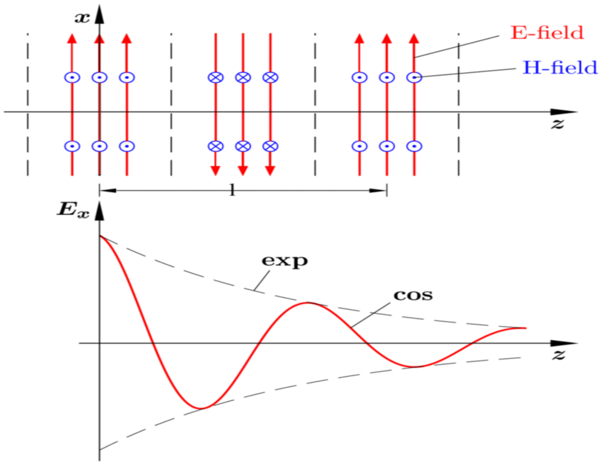


Fig. I.1.11: Field pattern and behavior of the real field E_x^+.

In case of lossy dielectrics $\varepsilon = (\varepsilon_r' - \mathrm{i}\varepsilon_r'')\,\varepsilon_0$ it is

$$\frac{\beta}{k_0} = \sqrt{\frac{1}{2}\varepsilon_r' + \frac{1}{2}\varepsilon_r'\sqrt{1 + (\varepsilon_r''/\varepsilon_r')^2}}, \quad \frac{\alpha}{k_0} = \sqrt{-\frac{1}{2}\varepsilon_r' + \frac{1}{2}\varepsilon_r'\sqrt{1 + (\varepsilon_r''/\varepsilon_r')^2}}.$$

Most of the dielectrics have low losses, $\varepsilon_r'' \ll \varepsilon_r'$, then

$$\beta \approx \sqrt{\varepsilon_r'}k_0, \quad \alpha \approx \frac{1}{2}\frac{\varepsilon_r''}{\sqrt{\varepsilon_r'}}k_0, \quad Z \approx \frac{Z_0}{\sqrt{\varepsilon_r'}}\left(1 + \frac{\mathrm{i}}{2}\frac{\varepsilon_r''}{\varepsilon_r'}\right).$$

An example is polyamide (nylon)

$$\kappa = 10^{-8}\Omega^{-1}\,\mathrm{m}^{-1},\ \varepsilon_r' = 3,\ f = 10\,\mathrm{MHz}$$

with an attenuation of $11\,\%$ in $100\,\mathrm{km}$. Very good conductors (metallic) have $\varepsilon_r'' \approx \mathrm{i}\kappa/\omega \gg \varepsilon_r'$ and

$$\beta \approx \alpha \approx \sqrt{\frac{1}{2}\omega\mu\kappa} = \frac{1}{\delta_s}, \quad Z \approx (1+\mathrm{i})\frac{\alpha}{\kappa},$$

where δ_s, called **skin depth**, is the distance in which the fields have decayed by $1/\mathrm{e}$

$$\mathrm{e}^{-\alpha\delta_s} = \frac{1}{\mathrm{e}}, \quad \delta_s = \sqrt{\frac{2}{\omega\mu\kappa}}. \tag{I.1.44}$$

In general, β is a function of ω and is called **dispersion relation**. Mostly it is a smooth function and can be developed around a frequency ω_0

$$\beta(\omega) = \beta(\omega_0) + \left.\frac{\partial\beta}{\partial\omega}\right|_{\omega_0} d\omega + \mathcal{O}\left((d\omega)^2\right). \tag{I.1.45}$$

The $\mathcal{O}$-order approximation yields the **phase velocity** v_{ph} with which the phase φ of the wave propagates

$$\begin{aligned} \varphi = \omega t \mp \beta(\omega_0) z \rightarrow \frac{d\varphi}{dt} = 0 = \omega \mp \beta(\omega_0)\frac{dz}{dt} = \omega \mp \beta(\omega_0) v_{\mathrm{ph}} \\ v_{\mathrm{ph}} = \pm\frac{\omega}{\beta(\omega_0)}. \end{aligned} \tag{I.1.46}$$

v_{ph} has no physical meaning, since monochromatic waves carry no information. For instance, v_{ph} can be larger than the velocity of light. If the wave is modulated, i.e. carries a signal, the signal propagates with the **group velocity**. An example is beating between two plane waves with ω_1 and ω_2

$$\begin{aligned} \omega_1 &= \omega_0 + \delta\omega, \quad \beta = \beta_0 + \delta\beta \\ \omega_2 &= \omega_0 - \delta\omega, \quad \beta_2 = \beta_0 - \delta\beta. \end{aligned}$$

The resulting physical field

$$\begin{aligned} \mathrm{Re}\{\mathrm{e}^{\mathrm{i}(\omega_1 t - \beta_1 z)} + \mathrm{e}^{\mathrm{i}(\omega_2 t - \beta_2 z)}\} &= \mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}(\omega_0 t - \beta_0 z)}\left(\mathrm{e}^{\mathrm{i}(\delta\omega t - \delta\beta z)} + \mathrm{e}^{-\mathrm{i}(\delta\omega t - \delta\beta z)}\right)\right\} \\ &= 2\cos(\omega_0 t - \beta_0 z)\cos(\delta\omega t - \delta\beta z) \end{aligned}$$

has a high frequency part, which propagates with the phase velocity $v_{\mathrm{ph}} = \omega_0/\beta_0$, and the envelope part (beating), which propagates with the group velocity

$$v_g = \frac{\delta\omega}{\delta\beta} \rightarrow v_g = \left.\frac{d\omega}{d\beta}\right|_{\omega_0}. \tag{I.1.47}$$

Signals with a small bandwidth $2\delta\omega$ propagate with v_g. Large bandwidth signals require higher order terms $\mathcal{O}\left((d\omega)^2\right)$ in Eq. (I.1.45). Apart from phase and group velocity, there is the velocity with which energy is transported. As shown in Fig. I.1.12, the energy transported a distance Δz in time Δt equals the energy radiated through ΔA

$$\frac{\overline{W}\Delta A \Delta z}{\Delta t} = \overline{W}\Delta A v_e = S_c \Delta A$$

or

$$v_e = \frac{S_c}{\overline{\overline{W}}}. \tag{I.1.48}$$

In case of plane waves

$$S_c = \frac{1}{2}\boldsymbol{E}\times\boldsymbol{H}^* = \frac{1}{2}\frac{A^2}{Z},$$
$$\overline{W} = \frac{1}{4}\boldsymbol{E}\cdot\boldsymbol{D}^* + \frac{1}{4}\boldsymbol{H}\cdot\boldsymbol{B}^* = \frac{1}{2}\varepsilon A^2$$

the energy velocity equals the velocity of light

$$v_e = \frac{1}{\varepsilon Z} = \frac{1}{\sqrt{\mu\varepsilon}} = c.$$

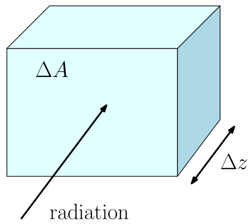


Fig. I.1.12: Radiation per unit time through an area ΔA.

I.1.8 Cylindrical, ideal conducting waveguides

This chapter treats waveguides with constant cross-section, homogeneous filling and ideal conducting walls. The full set of the time-harmonic Maxwell's equations is required

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{H} &= \mathrm{i}\omega\varepsilon\boldsymbol{E}\\ \boldsymbol{\nabla}\times\boldsymbol{E} &= -\mathrm{i}\omega\mu\boldsymbol{H}\\ \boldsymbol{\nabla}\cdot\boldsymbol{E} &= 0\\ \boldsymbol{\nabla}\cdot\boldsymbol{H} &= 0.\end{aligned}\tag{I.1.49}$$

It is possible to eliminate $\boldsymbol{H}$ or $\boldsymbol{E}$ from the first two equations and derive directly a wave equation. The problem is, that the fields will not necessarily satisfy the divergence-free equations. Therefore, an ansatz which fulfills the equations is chosen

$$\begin{aligned}\boldsymbol{E}^{\mathrm{TE}} &= \boldsymbol{\nabla}\times\boldsymbol{A}^{\mathrm{TE}}, \quad \boldsymbol{A}^{\mathrm{TE}} = A^{\mathrm{TE}}\boldsymbol{e}_z, \quad \text{TE-waves}\\ \boldsymbol{H}^{\mathrm{TM}} &= \boldsymbol{\nabla}\times\boldsymbol{A}^{\mathrm{TM}}, \quad \boldsymbol{A}^{\mathrm{TM}} = A^{\mathrm{TM}}\boldsymbol{e}_z, \quad \text{TM-waves.}\end{aligned}\tag{I.1.50}$$

A^{TE}, A^{TM} are the two independent functions, which are necessary. TE-waves are called **transverse electric** and TM **transverse magnetic** waves. The further procedure is shown for TE-waves. For TM-waves it is similar. Substituting $\boldsymbol{E}^{\mathrm{TE}}$ into the first equation (I.1.49)

$$\boldsymbol{\nabla}\times\left(\boldsymbol{H}^{\mathrm{TE}} - \mathrm{i}\omega\varepsilon\boldsymbol{A}^{\mathrm{TE}}\right) = 0 \rightarrow \boldsymbol{H}^{\mathrm{TE}} = \boldsymbol{\nabla}\varphi + \mathrm{i}\omega\varepsilon\boldsymbol{A}^{\mathrm{TE}}.$$

From the second equation (I.1.49) follows

$$\boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \boldsymbol{A}^{\mathrm{TE}}\right) = \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \boldsymbol{A}^{\mathrm{TE}}\right) - \boldsymbol{\nabla}^2 \boldsymbol{A}^{\mathrm{TE}} = -\mathrm{i}\omega\mu\boldsymbol{\nabla}\varphi + k^2\boldsymbol{A}^{\mathrm{TE}}, \quad k = \omega\sqrt{\mu\varepsilon}. \tag{I.1.51}$$

$\boldsymbol{A}^{\mathrm{TE}}$ and φ determine not fully $\boldsymbol{E}$ and $\boldsymbol{H}$. The freedom is used to impose the **Lorenz gauge**

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}^{\mathrm{TE}} = -\mathrm{i}\omega\mu\varphi. \tag{I.1.52}$$

Then, from Eq. (I.1.51)

$$\boldsymbol{\nabla}^2 \boldsymbol{A}^{\mathrm{TE}} + k^2 \boldsymbol{A}^{\mathrm{TE}} = 0$$

and since $\boldsymbol{A}$ has only a z-component one gets a scalar **Helmholtz equation** from TE and in a similar way for TM

$$\boldsymbol{\nabla}^2 A^{\mathrm{P}} + k^2 A^{\mathrm{P}} = 0, \quad \mathrm{P} = \left\{ \begin{array}{c} \mathrm{TE} \\ \mathrm{TM} \end{array} \right\}. \tag{I.1.53}$$

I.1.8.1 Circular waveguide

For a circular waveguide, Fig. I.1.13, circular cylinder coordinates are the natural choice and Eq. (I.1.53) becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A}{\partial\varphi^2} + \frac{\partial^2 A}{\partial z^2} + k^2 A = 0. \tag{I.1.54}$$

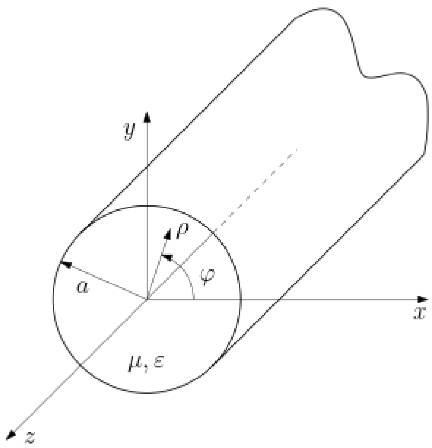


Fig. I.1.13: Geometry for circular waveguide.

Substituting a Bernoulli ansatz

$$A(\rho, \varphi, z) = R(\rho)\Phi(\varphi)Z(z) \tag{I.1.55}$$

into Eq. (I.1.54) yields

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\varphi^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} + k^2 = 0. \tag{I.1.56}$$

The term for Z is independent of ρ and φ and must be constant, written as $-k_z^2$. Then,

$$\frac{d^2Z}{dz^2} + k_z^2 Z = 0 \to Z(z) = C_1 \mathrm{e}^{-\mathrm{i}k_z z} + C_2 \mathrm{e}^{\mathrm{i}k_z z}.$$

The first term is for a wave propagating in $+z$-direction. The backward traveling wave is simply obtained replacing k_z by $-k_z$. Next, we substitute Z in Eq. (I.1.56)

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} + \rho^2\left(k^2 - k_z^2\right) = 0. \tag{I.1.57}$$

Now, the second term is independent of ρ and must be constant, chosen as $-k_\varphi^2$. Then

$$\frac{d^2\Phi}{d\varphi^2} + k_\varphi^2\Phi = 0 \to \Phi(\varphi) = C_3\cos(k_\varphi\varphi) + C_4\sin(k_\varphi\varphi)$$

and since Φ is 2π-periodic, k_φ must be an integer m. In addition, we rotate the coordinate system until $C_4 = 0$. Finally, one substitutes Φ into Eq. (I.1.57) and obtains **Bessel's** equation

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left[k_c^2 - \frac{m^2}{\rho^2}\right]R = 0, \quad k_c = \sqrt{k^2 - k_z^2} \tag{I.1.58}$$

with solutions

$$R(\rho) = C_5 J_m(k_c\rho) + C_6 N_m(k_c\rho). \tag{I.1.59}$$

On the axis, $\rho = 0$, R must be finite and therefore $C_6 = 0$. Thus, the vector potential Eq. (I.1.55) is

$$A(\rho,\varphi,z) = C_m\cos m\varphi J_m(k_c\rho)\mathrm{e}^{-\mathrm{i}k_z z}. \tag{I.1.60}$$

The last unknown constant k_c (or k_z) is determined by the boundary conditions for $\rho = a$.

In case of **TE-waves**, $\boldsymbol{E} = \boldsymbol{\nabla}\times(A\boldsymbol{e}_z)$, the boundary conditions are

$$E_\varphi = -\frac{\partial A}{\partial\rho} \sim J'_m(k_c a)$$
$$E_\varphi(\rho = a) = 0 \to J'_m(k_c a) = 0, \quad k_{cmn}a = j'_{mn}$$

with j'_{mn} being the n$^{\text{th}}$ non-vanishing zero of J'_m. Then the fields are

$$\begin{aligned} E_\rho &= \frac{1}{\rho}\frac{\partial A}{\partial\varphi} = -\frac{m}{\rho}C_{mn}\sin m\varphi J_m\left(j'_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-\mathrm{i}k_{zmn}z} \\ E_\varphi &= -\frac{\partial A}{\partial\rho} = -\frac{j'_{mn}}{a}C_{mn}\cos m\varphi J'_m\left(j'_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-\mathrm{i}k_{zmn}z} \end{aligned} \tag{I.1.61}$$

$-\mathrm{i}\omega\mu\boldsymbol{H} = \boldsymbol{\nabla} \times \boldsymbol{E}$

$$
\begin{aligned}
H_\rho &= \frac{k_{zmn}}{\omega\mu}\frac{j'_{mn}}{a}C_{mn}\cos m\varphi J'_m\left(j'_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-\mathrm{i}k_{zmn}z} \\
H_\varphi &= -\frac{k_{zmn}}{\omega\mu}\frac{m}{\rho}C_{mn}\sin m\varphi J_m\left(j'_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-\mathrm{i}k_{zmn}z} \\
H_z &= \frac{-1}{\mathrm{i}\omega\mu}\left(\frac{j'_{mn}}{a}\right)^2 C_{mn}\cos m\varphi J_m\left(j'_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-\mathrm{i}k_{zmn}z}.
\end{aligned}
\tag{I.1.62}
$$

For **TM-waves**, $\boldsymbol{H} = \boldsymbol{\nabla} \times (A\boldsymbol{e}_z)$, the boundary conditions are

$$
\begin{aligned}
E_z &= \frac{k_c^2}{\mathrm{i}\omega\varepsilon}A \sim J_m(k_c\rho) \\
E_z(\rho = a) &= 0 \to J_m(k_c a) = 0, \quad k_{cmn}a = j_{mn}
\end{aligned}
$$

with j_{mn} the n$^{\text{th}}$ non-vanishing zero of J_m. Then

$$
\begin{aligned}
H_\rho &= -\frac{m}{\rho}D_{mn}\sin m\varphi J_m\left(j_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-\mathrm{i}k_{zmn}z} \\
H_\varphi &= -\frac{j_{mn}}{a}D_{mn}\cos m\varphi J'_m\left(j_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-\mathrm{i}k_{zmn}z}
\end{aligned}
\tag{I.1.63}
$$

$\mathrm{i}\omega\varepsilon\boldsymbol{E} = \boldsymbol{\nabla} \times \boldsymbol{H}$

$$
\begin{aligned}
E_\rho &= -\frac{k_{zmn}}{\omega\varepsilon}\frac{j_{mn}}{a}D_{mn}\cos m\varphi J'_m\left(j_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-\mathrm{i}k_{zmn}z} \\
E_\varphi &= \frac{k_{zmn}}{\omega\varepsilon}\frac{m}{\rho}D_{mn}\sin m\varphi J_m\left(j_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-\mathrm{i}k_{zmn}z} \\
E_z &= \frac{1}{\mathrm{i}\omega\varepsilon}\left(\frac{j_{mn}}{a}\right)^2 D_{mn}\cos m\varphi J_m\left(j_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-\mathrm{i}k_{zmn}z}.
\end{aligned}
\tag{I.1.64}
$$

In the equations (I.1.61)-(I.1.64) the propagation constant is

$$
\begin{aligned}
k_{zmn} &= \sqrt{k^2 - k_{cmn}^2} \\
k_{zmn} &= \begin{cases} \text{real} & k > k_{cmn} \text{ propagation} \\ 0 & k = k_{cmn} \\ \text{imaginary} & k < k_{cmn} \text{ attenuation} \end{cases}.
\end{aligned}
\tag{I.1.65}
$$

with the critical wavenumber

$$
k_{cmn} = \begin{cases} j'_{mn}/a & \text{TE-waves} \\ j_{mn}/a & \text{TM-waves} \end{cases},
$$

and

$$
\begin{aligned}
f_{cmn} &= ck_{cmn}/2\pi && \text{cut-off frequency,} \\
\lambda_{cmn} &= 2\pi/k_{cmn} && \text{cut-off wavelength,} \\
\lambda_{zmn} &= 2\pi/k_{zmn} = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_{cmn})^2}} && \text{waveguide wavelength,} \\
\lambda & && \text{free space wavelength.}
\end{aligned}
$$

The ratio of the transverse field components is a constant with the dimension of an impedance and is called **field (wave) impedance**

$$Z_F = \frac{E_\rho}{H_\varphi} = -\frac{E_\varphi}{H_\rho} = \begin{cases} Z_F^{\mathrm{TE}} = \frac{\omega\mu}{k_{zmn}} \\ Z_F^{\mathrm{TM}} = \frac{k_{zmn}}{\omega\varepsilon} \end{cases} . \tag{I.1.66}$$

From the complex Poynting vector follows that

$$S_{cz} = \frac{1}{2}\left(\boldsymbol{E}\times\boldsymbol{H}^*\right)_z = \frac{1}{2}Z_F\left(|H_\rho|^2+|H_\varphi|^2\right) = \begin{cases} \text{real power flux} & k > k_{cmn} \\ 0 & k = k_{cmn} \\ \text{imaginary power flux} & k < k_{cmn} \end{cases} . \tag{I.1.67}$$

Every field with indices mn in Eqs. (I.1.61)-(I.1.64) is a particular **(eigen-)mode**. Therefore, the general field is expressed by the linear combination of all modes

$$\boldsymbol{E} = \sum_m\sum_n\left(\boldsymbol{E}_{mn}^{\mathrm{TE}}+\boldsymbol{E}_{mn}^{\mathrm{TM}}\right), \quad \boldsymbol{H} = \sum_m\sum_n\left(\boldsymbol{H}_{mn}^{\mathrm{TE}}+\boldsymbol{H}_{mn}^{\mathrm{TM}}\right). \tag{I.1.68}$$

Normally, modes are sorted referring to their cut-off frequency.

type	m	n	**(f_c/GHz)(a/cm)**
TE	1	1	8.78
TM	0	1	11.46
TE	2	1	14.56
TE/TM	0/1	1/1	18.29
TE	3	1	20.05

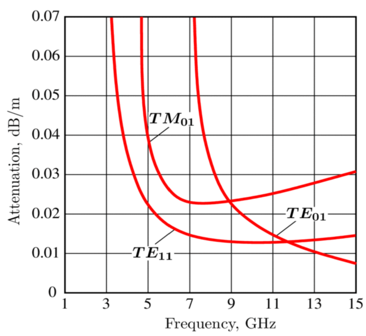


Fig. I.1.14: Attenuation for some low-order modes.

In Fig. I.1.14 the attenuation for some modes is given. Particularly interesting is the TE_{01}-mode because the attenuation decreases with $\omega^{-3/2}$. It is used for low-loss transmission, although there are three modes with lower cut-off frequency and special measures are necessary to suppress these modes.

Some field plots of modes with low cut-off frequency are shown in Fig. I.1.15.

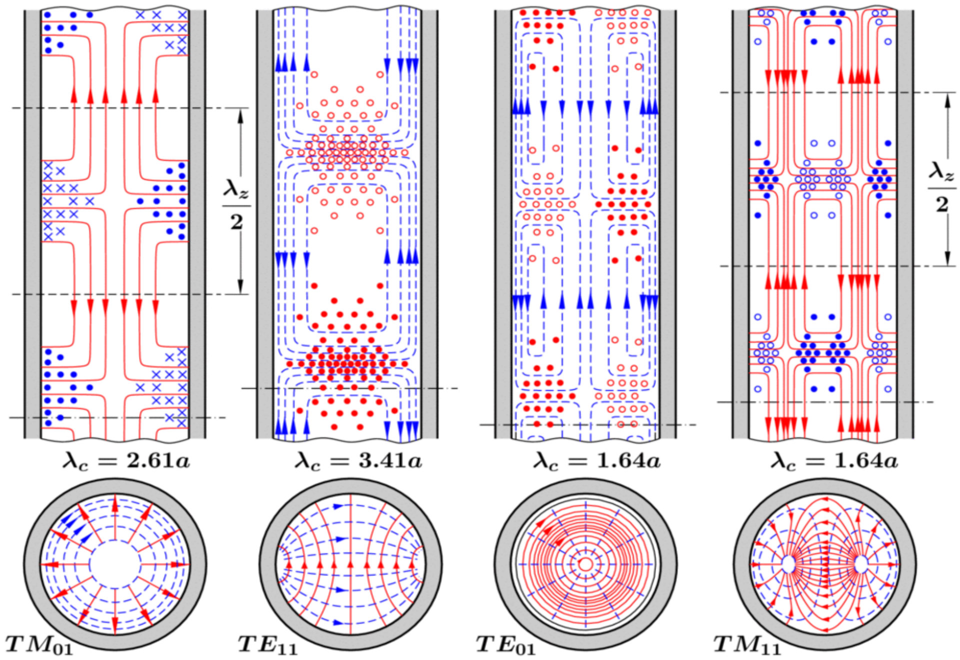


Fig. I.1.15: Field plots of four low-order modes.

I.1.8.2 Impedance boundary condition on good conductors

In materials with high conductivity κ (metals) one can neglect the displacement current compared to the conduction current, see Sec. I.1.5. On the surface of the material the electric field is essentially perpendicular and the magnetic field is parallel. Tangential to the surface the typical length in which the fields change is λ. In the material and in normal direction to the surface the length in which changes take place is $\delta_s \ll \lambda$. Under these assumptions a very good approximation for the ratio of the tangential fields can be derived.

One decomposes the fields and the nabla operator into tangential and normal components, see Fig. I.1.16,

$$\boldsymbol{E} = \boldsymbol{E}_t + E_z\boldsymbol{e}_z, \quad \boldsymbol{H} = \boldsymbol{H}_t + H_z\boldsymbol{e}_z, \quad \boldsymbol{\nabla} = \boldsymbol{\nabla}_t + \boldsymbol{e}_z\frac{\partial}{\partial z}.$$

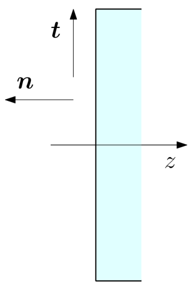


Fig. I.1.16: Geometry on the surface of a medium with high conductivity.

Maxwell's equations become

$$
\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} = \kappa \boldsymbol{E}: \quad & \boldsymbol{E}_t = -\frac{1}{\kappa}\boldsymbol{e}_z \times \boldsymbol{\nabla}_t H_z + \frac{1}{\kappa}\boldsymbol{e}_z \times \frac{\partial \boldsymbol{H}_t}{\partial z} \\
& E_z \boldsymbol{e}_z = \frac{1}{\kappa}\boldsymbol{\nabla}_t \times \boldsymbol{H}_t \\
\boldsymbol{\nabla} \times \boldsymbol{E} = -\mathrm{i}\omega\mu \boldsymbol{H}: \quad & \boldsymbol{H}_t = -\frac{\mathrm{i}}{\omega\mu}\boldsymbol{e}_z \times \boldsymbol{\nabla}_t E_z + \frac{\mathrm{i}}{\omega\mu}\boldsymbol{e}_z \times \frac{\partial \boldsymbol{E}_t}{\partial z} \\
& H_z \boldsymbol{e}_z = \frac{\mathrm{i}}{\omega\mu}\boldsymbol{\nabla}_t \times \boldsymbol{E}_t.
\end{aligned}
\tag{I.1.69}
$$

Since tangential to the surface the typical length of change is λ, an order of magnitude approximation is

$$
|\boldsymbol{\nabla}_t| \approx \frac{1}{\lambda}.
$$

Then, $|E_z|$, $|H_z|$ in Eq. (I.1.69) become

$$
\begin{aligned}
|E_z| &= \frac{1}{\kappa}|\boldsymbol{\nabla}_t \times \boldsymbol{H}_t| \approx \frac{1}{\kappa\lambda}|\boldsymbol{H}_t| = \pi\left(\frac{\delta_s}{\lambda}\right)^2 Z|\boldsymbol{H}_t| \\
Z|H_z| &= \frac{1}{\omega\mu}|\mathrm{i}\boldsymbol{\nabla}_t \times \boldsymbol{E}_t| = \frac{1}{\omega\mu}\frac{Z}{\lambda}|\boldsymbol{E}_t| = \frac{1}{2\pi}|\boldsymbol{E}_t|,
\end{aligned}
$$

and the first terms on the right side in Eq. (I.1.69)

$$
\begin{aligned}
\frac{1}{\kappa}|\boldsymbol{e}_z \times \boldsymbol{\nabla}_t H_z| &\approx \frac{1}{\kappa\lambda}|H_z| = \pi\left(\frac{\delta_s}{\lambda}\right)^2 Z|H_z| \approx \frac{1}{2}\left(\frac{\delta_s}{\lambda}\right)^2|\boldsymbol{E}_t| \\
\frac{1}{\omega\mu}|\mathrm{i}\boldsymbol{e}_z \times \boldsymbol{\nabla}_t E_z| &\approx \frac{1}{\omega\mu\lambda}|E_z| = \frac{1}{2\pi Z}|E_z| \approx \frac{1}{2}\left(\frac{\delta_s}{\lambda}\right)^2|H_t|.
\end{aligned}
$$

Due to $\delta_s \ll \lambda$, these terms can be neglected compared to the left sides $|\boldsymbol{E}_t|$, $|\boldsymbol{H}_t|$ in Eq. (I.1.69) and the equations simplify to

$$
\begin{aligned}
\kappa \boldsymbol{E}_t &\approx \boldsymbol{e}_z \times \frac{\partial \boldsymbol{H}_t}{\partial z} \\
\mathrm{i}\omega\mu \boldsymbol{H}_t &\approx -\boldsymbol{e}_z \times \frac{\partial \boldsymbol{E}_t}{\partial z}.
\end{aligned}
\tag{I.1.70}
$$

Eliminating $\boldsymbol{E}_t$ one gets a differential equation for $\boldsymbol{H}_t$

$$
\frac{\partial^2 \boldsymbol{H}_t}{\partial z^2} - \mathrm{i}\omega\mu\kappa \boldsymbol{H}_t = 0
$$

with solution

$$
\boldsymbol{H}_t = \boldsymbol{H}_{t0}\mathrm{e}^{-(1+\mathrm{i})z/\delta_s}. \tag{I.1.71}
$$

After substituting Eq. (I.1.71) into Eq. (I.1.70) one gets a relation between the tangential field components on the surface

$$
\boldsymbol{E}_{t0} \approx Z_w\left(\boldsymbol{n} \times \boldsymbol{H}_{t0}\right) \tag{I.1.72}
$$

where Z_w is the wall impedance

$$Z_w = \frac{1+\mathrm{i}}{\kappa\delta_s}, \quad \delta_s = \sqrt{\frac{2}{\omega\mu\kappa}}. \tag{I.1.73}$$

I.1.8.3 Attenuation in waveguides (power-loss method)

One starts with a small piece Δz of waveguide, Fig. I.1.17.

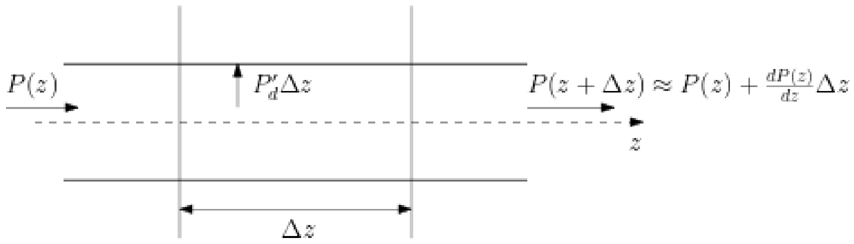


Fig. I.1.17: Input power $P(z)$, power loss $P'_d\Delta z$ and output power $P(z+\Delta z)$ for a piece Δz of waveguide.

Conservation of power requires

$$\frac{dP(z)}{dz} = -P'_d \tag{I.1.74}$$

and since $\boldsymbol{E}$, $\boldsymbol{H}$ are proportional to $\mathrm{e}^{-\alpha z}$ the transported power is

$$P(z) \sim \mathrm{e}^{-2\alpha z},$$

and Eq. (I.1.74) becomes

$$-2\alpha P(z) = -P'_d \rightarrow \alpha = \frac{1}{2}\frac{P'_d}{P(z)}. \tag{I.1.75}$$

In order to calculate the dissipation per unit length P'_d one takes the power radiated into a small area ΔA of the waveguide wall and uses Eq. (I.1.72) and Eq. (I.1.73)

$$\begin{aligned}\frac{\Delta P_d}{\Delta A} &= -\boldsymbol{n}\cdot\mathrm{Re}\{\boldsymbol{S}_c\} = -\frac{1}{2}\mathrm{Re}\{\boldsymbol{n}\cdot(\boldsymbol{E}_{t0}\times\boldsymbol{H}^*_{t0})\}\\ &= \frac{1}{2}\mathrm{Re}\{Z_w\}|\boldsymbol{H}_{t0}|^2 = \frac{1}{2\kappa\delta_s}|H_{t0}|^2.\end{aligned} \tag{I.1.76}$$

P'_d is obtained by integrating Eq. (I.1.76) over the boundary of the waveguide cross-section

$$P'_d = \frac{1}{2\kappa\delta_s}\oint|\boldsymbol{H}_{t0}|^2 ds.$$

The transported power in the waveguide gives the Poynting vector integrated over the cross-section

$$\begin{aligned}P(z) &= \iint_A \mathrm{Re}\{\boldsymbol{S}_c\}\cdot d\boldsymbol{A} = \frac{1}{2}\iint_A \mathrm{Re}\{\boldsymbol{E}\times\boldsymbol{H}^*\}\cdot\boldsymbol{e}_z dA\\ &= \frac{1}{2}\iint_A \mathrm{Re}\{\boldsymbol{E}_{\mathrm{transv}}\times\boldsymbol{H}^*_{\mathrm{transv}}\}dA = \frac{1}{2}Z_F\iint_A|\boldsymbol{H}_{\mathrm{transv}}|^2 dA.\end{aligned}$$

Then, the attenuation Eq. (I.1.75) is

$$\alpha = \frac{1}{2\kappa\delta_s} \frac{\oint |\boldsymbol{H}_{t0}|^2 ds}{Z_F \iint |\boldsymbol{H}_{\text{transv}}|^2 dA}. \tag{I.1.77}$$

It is worth noting, that the tangential field $\boldsymbol{H}_{t0}$ and transverse field $\boldsymbol{H}_{\text{transv}}$ are the fields of the ideal conducting waveguide.

Although the power-loss method was used here to calculate the waveguide attenuation, Eq. (I.1.76) can also be used to calculate the losses in any metallic structure.

I.1.9 Resonant cavities

Resonant cavities are used to accelerate charged particles. The simplest arrangement is a cylindrical cavity with radius a and length g. In order to have a longitudinal electric field it is operated in a TM-mode.

We take the E_φ component in Eq. (I.1.64) and superimpose a forward and backward travelling wave

$$E_\varphi = \frac{k_{zmn}}{\omega\varepsilon}\frac{m}{\rho} D_{mn} \sin m\varphi J_m\left(j_{mn}\frac{\rho}{a}\right)\left(\mathrm{e}^{-\mathrm{i}k_{zmn}z} - r_m \mathrm{e}^{\mathrm{i}k_{zmn}z}\right).$$

The boundary conditions are

$$\begin{aligned} E_\varphi(z=0) &= 0 \quad \to \quad & r_m &= 1, \quad & E_\varphi &\sim \sin k_{zmn}z \\ E_\varphi(z=g) &= 0 \quad \to \quad & k_{zmnp}g &= p\pi, \quad & p &= 0,1,2,\ldots \end{aligned}$$

therefore

$$\begin{aligned} E_\varphi &= -\mathrm{i}2\frac{p\pi/g}{\omega\varepsilon}\frac{m}{\rho} D_{mn} \sin m\varphi J_m\left(j_{mn}\frac{\rho}{a}\right)\sin\left(p\pi\frac{z}{g}\right) \\ E_z &= -\mathrm{i}\frac{2}{\omega\varepsilon}\left(\frac{j_{mn}}{a}\right)^2 D_{mn}\cos m\varphi J_m\left(j_{mn}\frac{\rho}{a}\right)\cos\left(p\pi\frac{z}{g}\right) \\ E_\rho &= \mathrm{i}2\frac{p\pi/g}{\omega\varepsilon}\frac{j_{mn}}{a} D_{mn}\cos m\varphi J'_m\left(j_{mn}\frac{\rho}{a}\right)\sin\left(p\pi\frac{z}{g}\right) \\ H_\rho &= -2\frac{m}{\rho} D_{mn}\sin m\varphi J_m\left(j_{mn}\frac{\rho}{a}\right)\cos\left(p\pi\frac{z}{g}\right) \\ H_\varphi &= -2\frac{j_{mn}}{a} D_{mn}\cos m\varphi J'_m\left(j_{mn}\frac{\rho}{a}\right)\cos\left(p\pi\frac{z}{g}\right), \end{aligned} \tag{I.1.78}$$

where

$$\frac{j_{mn}}{a} = \sqrt{k^2 - \left(p\frac{\pi}{g}\right)^2}.$$

For acceleration the resonator is operated in the TM_{010}-mode, i.e. no azimuthal dependence ($m = 0$), no longitudinal dependence ($p = 0$) and the first non-vanishing zero j_{01} of J_m. The mode has field components, see Eq. (I.1.78)

$$\begin{aligned} E_z &= -\mathrm{i}\frac{2}{\omega\varepsilon}\left(\frac{j_{01}}{a}\right)^2 D_{01}J_0\left(j_{01}\frac{\rho}{a}\right), \quad & E_\rho &= E_\varphi = 0 \\ H_\varphi &= -2\frac{j_{01}}{a} D_{01}J'_0\left(j_{01}\frac{\rho}{a}\right), & H_\rho &= 0, \end{aligned} \tag{I.1.79}$$

and a resonant frequency

$$k_r = \frac{\omega_r}{c} = \frac{j_{01}}{a} \to f_r = \frac{\omega_r}{2\pi} = \frac{j_{01}c}{2\pi a}.$$

A figure of merit is the Q-value. It is determined by the stored energy

$$\begin{aligned}\overline{W} = \overline{W}_e + \overline{W}_m = 2\overline{W}_e &= \frac{1}{2}\iiint_V \boldsymbol{E}\cdot\boldsymbol{D}^* dV = \frac{\varepsilon}{2}\iiint_V |E_z|^2 dV \\ &= \frac{4\pi g}{\omega_r^2\varepsilon}\frac{j_{01}^4}{a^2} D_{01}^2 \int_0^1 J_0^2\left(j_{01}\frac{\rho}{a}\right)\frac{\rho}{a}d\frac{\rho}{a} \\ &= \frac{2\pi g}{\omega_r^2\varepsilon}\frac{j_{01}^4}{a^2} D_{01}^2 J_1^2(j_{01})\end{aligned}$$

and the dissipated power, see Eq. (I.1.76),

$$\begin{aligned}P_d &= \oint\!\!\oint P_d'' dA = \frac{1}{2\kappa\delta_s}\oint\!\!\oint |H_{t0}|^2 dA \\ &= \frac{1}{2\kappa\delta_s}\left[\int_0^g\int_0^{2\pi}|H_\varphi(\rho=a)|^2 a d\varphi dz + 2\int_0^a\int_0^{2\pi}|H_\varphi(z=0)\rho d\varphi d\rho\right] \\ &= \frac{4\pi}{\kappa\delta_s} j_{01}^2\left(1+\frac{g}{a}\right) D_{01}^2 J_1^2(j_{01}).\end{aligned}$$

Then,

$$Q = \frac{\omega_r \overline{W}}{P_d} = \frac{1}{\delta_s}\frac{g}{1+g/a} \to \delta_s Q = 2\frac{\text{volume}}{\text{surface}}. \tag{I.1.80}$$

Q is proportional to the ratio volume over surface. Obviously, a spherical resonator will have the highest Q-value. The importance of Q stems from the fact that it defines the decay rate of the stored energy or the filling time T_f. From power conservation

$$-\frac{d\overline{W}}{dt} = P_d = \frac{\omega_r}{Q}\overline{W}$$

follows

$$\overline{W} = \overline{W}_0 \mathrm{e}^{-2t/T_f} \text{ with } T_f = 2\frac{Q}{\omega_r}. \tag{I.1.81}$$

The example of a copper cavity at 3 GHz with

$$\begin{aligned} g &= \lambda_r/2 = 5\,\text{cm}, & \kappa &= 58\cdot 10^6\,\Omega^{-1}\,\text{m}^{-1}, \\ j_{01} &= 2.405, & J_1(j_{01}) &= 0.5191 \end{aligned}$$

yields

$$a = 3.83\,\text{cm},\quad \delta_s = 1.21\,\mu\text{m},\quad Q = 17\,963,\quad T_f = 1.9\,\mu\text{s}.$$

I.1.9.1 Resonance behavior of a cavity mode

To make the problem directly accessible for an analytical treatment, it is convenient to take ideal conducting walls but a lossy dielectric filling. In that way ideal modes with losses exist.

The cavity is driven by a current J passing through it. J splits into a conduction current $J_c = \kappa E$, responsible for the losses in the dielectric, and in an enforced current J_0 as driving term. From the first two Maxwell's equations

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\mu \frac{\partial}{\partial t} \nabla \times \boldsymbol{H} = -\mu \frac{\partial}{\partial t} \left(\boldsymbol{J}_0 + \kappa \boldsymbol{E} + \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \right)$$

follows together with $\nabla \cdot \boldsymbol{E} = 0$

$$\nabla^2 \boldsymbol{E} - \mu\kappa \frac{\partial \boldsymbol{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \mu \frac{\partial \boldsymbol{J}_0}{\partial t}. \tag{I.1.82}$$

One expands $\boldsymbol{E}$ in modes

$$\boldsymbol{E} = \sum_r a_r(t) \boldsymbol{e}_r(x, y, z) \tag{I.1.83}$$

where r runs over all m, n, p and where it is assumed that

$$\begin{aligned} &\nabla^2 \boldsymbol{e}_r + k_r^2 \boldsymbol{e}_r = 0 \\ &\iiint \boldsymbol{e}_r \cdot \boldsymbol{e}_s dV = \delta_r^s \\ &\boldsymbol{n} \times \boldsymbol{e}_r = 0 \text{ on walls} \\ &\nabla \cdot \boldsymbol{e}_r = 0 \text{ in volume.} \end{aligned}$$

One substitutes Eq. (I.1.83) into Eq. (I.1.82)

$$\sum_r \left[\frac{d^2 a_r}{dt^2} + \frac{\kappa}{\varepsilon} \frac{da_r}{dt} + \frac{k_r^2}{\mu\varepsilon} a_r \right] \boldsymbol{e}_r = -\frac{1}{\varepsilon} \frac{d\boldsymbol{J}_0}{dt},$$

multiplies with $\boldsymbol{e}_s$ and integrates over V, then

$$\frac{d^2 a_s}{dt^2} + \frac{\kappa}{\varepsilon} \frac{da_s}{dt} + \frac{k_s^2}{\mu\varepsilon} a_s = -\frac{1}{\varepsilon} \iiint \frac{d\boldsymbol{J}_0}{dt} \cdot \boldsymbol{e}_s dV = \frac{df_s}{dt}$$

and in case of time-harmonic excitation

$$\begin{gathered} \left[-\omega^2 + \mathrm{i} \frac{\kappa}{\varepsilon} \omega + \frac{k_s^2}{\mu\varepsilon} \right] a_s = \mathrm{i}\omega f_s, \\ a_s = \frac{Q_s}{\omega_s} \frac{f_s}{1 + \mathrm{i} Q_s \left(\frac{\omega}{\omega_s} - \frac{\omega_s}{\omega} \right)}, \quad \omega_s = c k_s, \quad Q_s = \frac{\varepsilon \omega_s}{\kappa}. \end{gathered} \tag{I.1.84}$$

For a realistic cavity with lossy walls one replaces Q_s by Q and ω_s by ω_{mnp}. Figure I.1.18 shows the magnitude and angle of a mode amplitude $a(\omega)$.

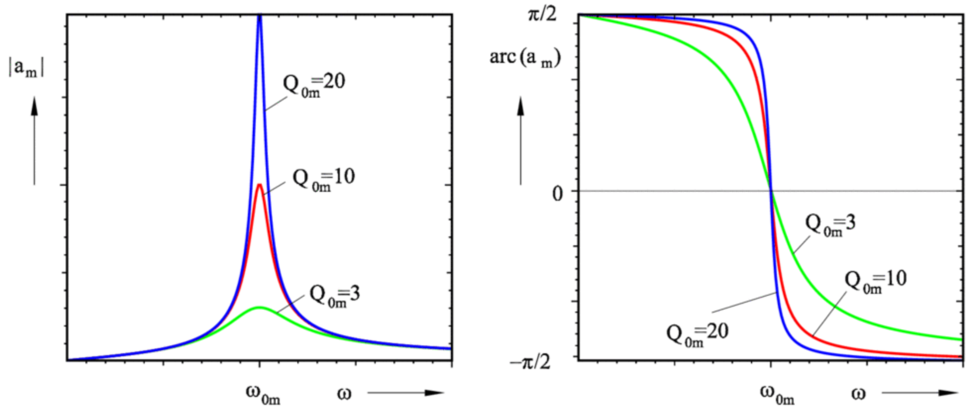


Fig. I.1.18: Magnitude and phase of a resonant mode amplitude.

If the modes in a cavity are well separated they can be represented by lumped element resonant circuits, Fig. I.1.19.

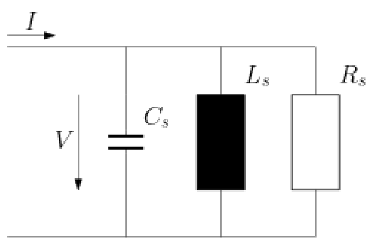


Fig. I.1.19: Lumped element resonant circuit.

The corresponding parameters are

$$\begin{aligned}
&\omega_s = \frac{1}{\sqrt{L_s C_s}}, && Q_s = \frac{\omega_s W_s}{P_{ds}} = \omega_s R_s C_s \\
&\text{bandwidth} && B_s = \frac{(\omega_s + \delta\omega) - (\omega_s - \delta\omega)}{\omega_s} = 2\frac{\delta\omega}{\omega_s} = \frac{1}{Q_s} \\
&\text{filling time} && T_{fs} = 2\frac{Q_s}{\omega_s} = \frac{1}{\delta\omega}.
\end{aligned} \tag{I.1.85}$$

While ω_s and Q_s are directly measurable, one needs one more quantity to determine the lumped elements. This is the R-upon-Q

$$\frac{R_{sh}}{Q_s} = \frac{V_s^2}{\omega_s W_s} = \frac{2}{\omega_s C_s}, \tag{I.1.86}$$

with

$$R_{sh} = \frac{V_s^2}{P_{ds}} = 2R_s$$

and the accelerating voltage for a particle passing the cavity on-axis with velocity v

$$V_s = \left| \int_0^g a_s \boldsymbol{e}_s \cdot \boldsymbol{e}_z \mathrm{e}^{\mathrm{i}\omega t} dz \right|, \quad z = vt.$$

The shunt impedance R_{sh} is a measure of the available voltage for a given dissipation. Whereas the R-upon-Q determines the available voltage for a given stored energy. It is independent of the cavity losses and therefore measurable.

I.1.9.2 Influence of beam pipe

It is important to estimate the influence of the beam pipe on the accelerating voltage, Fig. I.1.20.

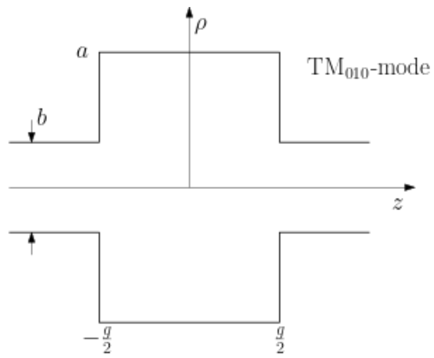


Fig. I.1.20: Cross-section of a pill-box cavity with beam pipes.

Let us assume that the cavity is excited in the TM$_{010}$-mode. In the pipe region, $0 \le \rho \le b$, a spectral expansion of the longitudinal field is used

$$E_z(\rho, z) = \int_{-\infty}^{\infty} A(k_z) I_0(K\rho) \mathrm{e}^{-\mathrm{i}k_z z} dk_z, \quad K = \sqrt{k_z^2 - k^2} \tag{I.1.87}$$

with

$$A(k_z) I_0(K\rho) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_z(\rho, z) \mathrm{e}^{\mathrm{i}k_z z} dz.$$

On the interface between pipe and cavity, $\rho = b$, the field is approximated by

$$E_z(\rho = b, z) = \begin{cases} E_0 & -g/2 \le z \le g/2 \\ 0 & |z| > g/2 \end{cases}$$

and one obtains

$$A(k_z) I_0(Kb) = \frac{E_0}{2\pi} \int_{-g/2}^{g/2} \mathrm{e}^{\mathrm{i}k_z z} dz = \frac{E_0 g}{2\pi} \frac{\sin(k_z g/2)}{k_z g/2}. \tag{I.1.88}$$

Using Eq. (I.1.87) together with Eq. (I.1.88) the accelerating voltage becomes

$$\begin{aligned} V(\rho) &= \int_{-\infty}^{\infty} E_z(\rho, z) \mathrm{e}^{\mathrm{i}\omega t} dz = \int_{-\infty}^{\infty} E_z(\rho, z) \mathrm{e}^{\mathrm{i}kz/\beta} dz \\ &= \frac{E_0 g}{2\pi} \int_{-\infty}^{\infty} dk_z \frac{\sin(k_z g/2)}{k_z g/2} \frac{I_0(K\rho)}{I_0(Kb)} \int_{-\infty}^{\infty} dz \mathrm{e}^{\mathrm{i}(k/\beta - k_z)z} = E_0 g T F(\rho, b), \quad k = \frac{\omega}{c}, \quad \beta = \frac{v}{c}, \end{aligned} \tag{I.1.89}$$

with the gap factor (transit time factor)

$$T = \frac{\sin(k_z g/2)}{k_z g/2}$$

and the reduction factor due to the beam pipe

$$F(\rho, b) = \frac{I_0(k\rho/\beta\gamma)}{I_0(kb/\beta\gamma)}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

For a particle on-axis the reduction of the accelerating voltage is

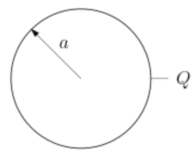
$$\frac{V(\rho=0)}{V(\rho=b)} = \frac{1}{I_0(kb/\beta\gamma)}.$$

For $kb/\beta\gamma \geq 2$ it is approximately exponential

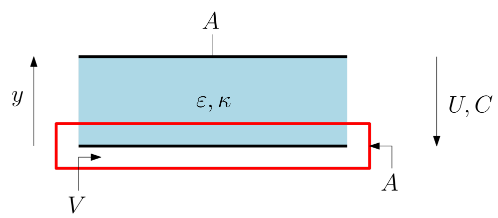
$$\frac{V(\rho=0)}{V(\rho=b)} \approx \sqrt{2\pi kb/\beta\gamma}\,\mathrm{e}^{-kb/\beta\gamma}.$$

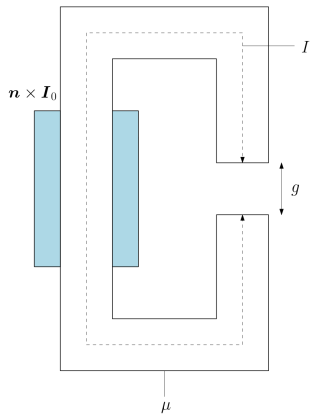
I.1.10 Exercises

I.1.10.1 Exercise 1: Given is a conducting hollow sphere carrying a charge Q. What is the field in- and outside and what is the stored energy? If the sphere were a model for an electron ($E_{0e} = 511$ keV) what is then the classical electron radius $r_e = a$?



I.1.10.2 Exercise 2: A parallel plate capacitor is filled with a lossy dielectric and charged to a voltage V. What is the time constant for discharge?





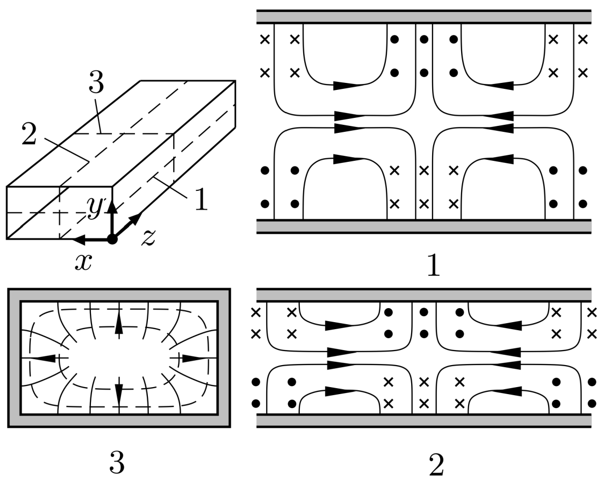
I.1.10.3 Exercise 3: A long dipole magnet is excited by a coil with n windings and current I_0. Calculate the magnetic field in the gap.

I.1.10.4 Exercise 4: Derive the multi-poles for a static 2-dimensional magnetic field. Remark: Solve the equation for the magnetic potential in circular cylindrical coordinates.

I.1.10.5 Exercise 5: Give the E- and H-field of a z-polarized plane wave which propagates in x-direction. What is the time-averaged radiation power density?

I.1.10.6 Exercise 6: Derive the longitudinal vector potential for TM-modes in a rectangular waveguide. What is the equation for the separation constants?

I.1.10.7 Exercise 7: Give the guide wavelength, phase, and group velocity of a TM_{11}-mode in a rectangular waveguide.



I.1.10.8 Exercise 8: Calculate the accelerating voltage, shunt impedance and R-upon-Q of a TM_{110}-mode in a rectangular cavity resonator of quadratic cross-section, $a = b$, and length g.

I.1.11 Solutions to the exercises

I.1.11.1 Solution to exercise 1

From Eq. (I.1.1) and due to spherical symmetry

$$\oiint_A \boldsymbol{D} \cdot d\boldsymbol{A} = \iiint_V \rho dV = 4\pi\varepsilon_0 r^2 E_r \begin{cases} 0 & r < a \\ Q & r \geq a \end{cases}.$$

Inside the sphere there is no field and no stored energy. Outside the sphere the energy stored in the field is

$$W_e = \frac{1}{2} \iiint_V \boldsymbol{E} \cdot \boldsymbol{D} dV = \frac{1}{2} \left(\frac{Q}{4\pi\varepsilon_0} \right)^2 4\pi\varepsilon_0 \int_a^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0 a}.$$

To find the classical electron radius the stored energy must be equal the electron rest energy

$$\frac{e^2}{8\pi\varepsilon_0 a} = m_{0e} c^2 \to a = \frac{e^2}{8\pi\varepsilon_0 m_{0e} c^2}.$$

Since there exist several models for an electron with slightly different factors the radius is defined as

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_{0e} c^2} = 2.81 \times 10^{-15}\,\mathrm{m}.$$

I.1.11.2 Solution to exercise 2

From Eq. (I.1.2)

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) = 0 = \boldsymbol{\nabla} \cdot \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right).$$

After integration over V and application of Gauss' law

$$\oiint_A \left(\kappa \boldsymbol{E} + \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \right) \cdot d\boldsymbol{A} = 0 \to \frac{dE}{dt} = -\frac{\kappa}{\varepsilon} E.$$

Where we have used the fact that E has only a y-component in the capacitor and vanishes outside. The solution of the differential equation is

$$E = E_0 \mathrm{e}^{-t/T_\varepsilon}$$

with the **relaxation time** $T_\varepsilon = \varepsilon/\kappa$.

I.1.11.3 Solution to exercise 3

From Eq. (I.1.1) and Eq. (I.1.13)

$$\oint \boldsymbol{H} \cdot d\boldsymbol{s} = \iint_A \boldsymbol{J} \cdot d\boldsymbol{A}$$
$$H_{\text{iron}} l + H_{\text{gap}} g = n I_0$$
$$B_{\text{iron}} = B_{\text{gap}} \to \mu_{\text{iron}} H_{\text{iron}} = \mu_0 H_{\text{gap}}.$$

Then

$$H_{\text{gap}} = \frac{n I_0}{g + \mu_0 l / \mu_{\text{iron}}} \approx n \frac{I_0}{g}.$$

I.1.11.4 Solution to exercise 4

From Eq. (I.1.30)

$$\boldsymbol{\nabla}^2 \phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2} = 0.$$

Substituting a Bernoulli ansatz

$$\phi(\rho, \varphi) = R(\rho) F(\varphi)$$

yields

$$\frac{\rho}{R} \frac{d}{d\rho} \left(\rho \frac{dR}{d\rho} \right) + \frac{1}{F} \frac{d^2 F}{d\varphi^2} = 0.$$

The second term is independent of ρ and must be a constant, put to $-\nu^2$. 2π-periodic fields require $\nu = n$

$$F_n = a_n \cos n\varphi, \quad n = 0, 1, 2, \ldots$$

Substituting F_n into the differential equation gives

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} - \frac{n^2}{\rho^2} R = 0$$

with solutions

$$R_n = b_n \rho^n + c_n \rho^{-n}.$$

A finite potential for $\rho \to 0$ requires $c_n = 0$ and the general solution is

$$\phi(\rho, \varphi) = \sum_{n=0}^{\infty} a_n \rho^n \cos n\varphi$$
$$B = \boldsymbol{\nabla} \phi = \sum_{n=1}^{\infty} a_n n \rho^{n-1} \left[\cos n\varphi \boldsymbol{e}_\rho - \sin n\varphi \boldsymbol{e}_\varphi \right].$$

$n = 1, 2, 3, \ldots$ are dipole, quadrupole, sextupole,... fields.

I.1.11.5 Solution to exercise 5

It is

$$\boldsymbol{E} = E_0 \mathrm{e}^{\mathrm{i}(\omega t - kx)} \boldsymbol{e}_z, \qquad k = \frac{\omega}{c},$$
$$Z\boldsymbol{H} = \boldsymbol{e}_x \times \boldsymbol{E} = -E_0 \mathrm{e}^{\mathrm{i}(\omega t - kx)} \boldsymbol{e}_y, \quad Z = \sqrt{\frac{\mu}{\varepsilon}}, \quad \boldsymbol{S}_c = \frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^* = \frac{1}{2Z}|E_0|^2 \boldsymbol{e}_x.$$

I.1.11.6 Solution to exercise 6

The vector potential satisfies the homogeneous Helmholtz equation

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} + k^2 A_z = 0, \quad k = \frac{\omega}{c}.$$

Using a Bernoulli ansatz

$$A_z(x, y, z) = X(x)Y(y)Z(z)$$

the equation becomes

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0.$$

Each term depends only on its own variable and must therefore be constant. The constants we call $-k_x^2$, $-k_y^2$, $-k_z^2$. They form the equation of the separation constants (dispersion relation)

$$k^2 = k_x^2 + k_y^2 + k_z^2.$$

With the separation constants each function X, Y, Z satisfies the standard equation for harmonic oscillation, e.g.

$$\frac{d^2X}{dx^2} + k_x^2 X = 0$$

with linear combinations of solutions

$$X(x) = \left\{ \begin{array}{c} \cos k_x x \\ \sin k_y y \end{array} \right\} \text{ or } \left\{ \begin{array}{c} \mathrm{e}^{-\mathrm{i}k_x x} \\ \mathrm{e}^{\mathrm{i}k_x x} \end{array} \right\}.$$

Then, the vector potential is

$$A_z(x, y, z) = \left\{ \begin{array}{c} \cos k_x x \\ \sin k_x x \end{array} \right\} \left\{ \begin{array}{c} \cos k_y y \\ \sin k_y y \end{array} \right\} \left\{ \begin{array}{c} \mathrm{e}^{-\mathrm{i}k_z z} \\ \mathrm{e}^{\mathrm{i}k_z z} \end{array} \right\}.$$

Here, standing waves were used for the transverse directions x, y and traveling waves for the longitudinal direction z.

It is understood that all quantities are proportional to $\mathrm{e}^{\mathrm{i}\omega t}$, which was dropped for simplicity.

The constants k_x, k_y follow from the boundary conditions.

$$\begin{aligned}
&\text{TM-waves:} && \boldsymbol{H} = \boldsymbol{\nabla} \times (A_z \boldsymbol{e}_z), \quad \mathrm{i}\omega\varepsilon \boldsymbol{E} = \boldsymbol{\nabla} \times \boldsymbol{H} \\
& && \mathrm{i}\omega\varepsilon E_z = -\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2}\right) = (k_x^2 + k_y^2) A_z \\
&\text{Boundary condition:} && E_z(x = 0, a) = E_z(y = 0, b) = 0 \\
& && A_z(x, y, z) = C_{mn} \sin k_{xm} x \sin k_{yn} y \mathrm{e}^{-\mathrm{i}k_{zmn} z} \\
& && k_{xm} = m\frac{\pi}{a}, \quad k_{yn} = n\frac{\pi}{b}, \quad m, n = 1, 2, 3, \ldots
\end{aligned}$$

The propagation constant k_{zmn} is fixed by the equations of separation constants and the frequency k

$$k^2 = \left(\frac{\omega}{c}\right)^2 = \left(m\frac{\pi}{a}\right)^2 + \left(n\frac{\pi}{b}\right)^2 + k_{zmn}^2.$$

I.1.11.7 Solution to exercise 7

It is $m = n = 1$, $k = \omega/c = 2\pi/\lambda$

$$\begin{aligned}
k_{z11} &= \frac{2\pi}{\lambda_{z11}} = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2 - \left(\frac{\pi}{b}\right)^2} \\
\lambda_{z11} &= \frac{\lambda}{\sqrt{1 - (\lambda/2a)^2 - (\lambda/2b)^2}} \\
v_{\text{ph}} &= \frac{\omega}{k_{z11}} = \frac{c}{\sqrt{1 - (\lambda/2a)^2 - (\lambda/2b)^2}} \\
v_g &= \frac{\partial\omega}{\partial k_{z11}} = c\sqrt{1 - (\lambda/2a)^2 - (\lambda/2b)^2} \\
v_{\text{ph}} v_g &= c^2.
\end{aligned}$$

I.1.11.8 Solution to exercise 8

It is $m = n = 1$, $p = 0$, $E_z \sim A_z$

$$E_z = C \sin\pi\frac{x}{a} \sin\pi\frac{y}{a}, \quad E_x = E_y = 0, \quad k_r = \frac{\omega_r}{c_0} = \sqrt{2}\frac{\pi}{a}.$$

The accelerating voltage is with $z = vt$

$$V = \left|\int_0^g E_z\left(x = \frac{a}{2}, y = \frac{a}{2}\right) \mathrm{e}^{\mathrm{i}\omega_r t} dz\right| = |C| \int_0^g \mathrm{e}^{\mathrm{i}\omega_r z/v} dz = |C| g T, \quad T = \frac{\sin \omega_r g/2v}{\omega_r g/2v}.$$

The magnetic fields are

$$\begin{aligned}
-\mathrm{i}\omega_r \mu_0 \boldsymbol{H} &= \boldsymbol{\nabla} \times \boldsymbol{E} \\
-\mathrm{i}\omega_r \mu_0 H_x &= \frac{\partial E_z}{\partial y} = C\frac{\pi}{a} \sin\pi\frac{x}{a} \cos\pi\frac{y}{a} \\
-\mathrm{i}\omega_r \mu_0 H_y &= -\frac{\partial E_z}{\partial x} = -C\frac{\pi}{a} \cos\pi\frac{x}{a} \sin\pi\frac{y}{a} \\
H_z &= 0.
\end{aligned}$$

Stored energy

$$\overline{W} = 2\overline{W}_e = \frac{\varepsilon_0}{2} \iiint_V |E_z|^2 dx dy dz = \frac{\varepsilon_0}{8} a^2 g C^2.$$

Dissipated power

$$\begin{aligned} P_d = \oiint P_d'' dA &= \frac{2}{2\kappa\delta_s}\left[\int_0^g \int_0^a |H_x(y=0)|^2 dxdz + \int_0^g \int_0^a |H_y(x=0)|^2 dydz \right. \\ &\left. + \int_0^a \int_0^a |H_x(z=0)|^2 + |H_y(z=0)|^2 dxdy \right] \\ &= \frac{C^2}{\kappa\delta_s\omega_r^2\mu_0^2}\frac{\pi^2}{2}\left(1+2\frac{g}{a}\right). \end{aligned}$$

Shunt impedance

$$R_{\text{sh}} = \frac{V^2}{P_d} = \frac{4\sqrt{2}}{\pi}\frac{g^2 Z_0}{\delta_s a(1+2g/a)}T^2.$$

Q-value

$$Q = \frac{\omega_r \overline{W}}{P_d} = \frac{g}{\delta_s 2(1+2g/a)} = \frac{1}{\delta_s}\frac{\text{volume}}{\text{surface}}.$$

R-upon-Q

$$\frac{R_{\text{sh}}}{Q} = \frac{V^2}{\omega_r \overline{W}} = \frac{8\sqrt{2}}{\pi}\frac{g}{a}Z_0 T^2, \quad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}.$$

$\omega_r, Q, R_{\text{sh}}/Q$ can be measured. They determine the lumped elements R, L, C in an equivalent circuit.

Literature

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