Chapter III.5

Non-linear beam manipulations in the transverse space

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In recent years, high-energy particle accelerators utilizing superconducting magnets, which inherently possess non-linear field errors, have significantly advanced the field of non-linear beam dynamics. While non-linear dynamics can introduce detrimental effects on the behavior of charged particles, they also offer opportunities for innovation. Non-linear beam dynamics provides the potential for novel approaches to beam manipulation. This contribution explores and examines the central concepts underpinning these advanced manipulations.

There are some well-known paradigms in beam dynamics that are universally accepted, such as invariance of the transverse beam emittances and the Gaussian property of the transverse beam distribution. These statements should be carefully scrutinized, as they are valid under very specific conditions, namely in the absence of linear coupling and non-linear effects. Whenever these conditions are not met, the previous statements no longer hold. The breakdown of these paradigms is perceived negatively, which is the reason why linear coupling and non-linear effects are supposed to be correct at best. On the other hand, it is possible to make use of linear coupling or non-linearities rather than fight them. This is exactly the new paradigm that has been studied intensively for several years, looking at the options offered by the presence of non-linear effects in the transverse beam dynamics. The non-linearities have the property of creating novel structures in the phase space, in addition to those characterizing the linear motion; this fact, combined with the use of slow variation of some parameters characterizing the beam dynamics, typically the transverse tune is essential for the novel beam manipulations. Note that an element of paramount importance is the capability to control accurately the non-linearities and the time-variation of the system parameters. The lack of accurate control generates the risk of spoiling the nice features of the novel beam manipulations, possibly creating beam loss and emittance growth. The combination of non-linearities and slow tune variation opens the door to a new world in which beam splitting, sharing of transverse emittances, and the cooling of annular beam distributions become possible. The four frameworks that are needed to design and understand the novel opportunities are briefly reviewed, and beam manipulations are presented and discussed.

III.5.0.1 Hénon-like maps

These studies rely on models of the non-linear betatron motion. An essential characteristic of a good model is its simplicity, allowing one to perform analytical computations, which, however, should not be

This chapter should be cited as: Non-linear beam manipulations in the transverse space, M. Giovannozzi, DOI: 10.23730/CYRSP-2024-003.2017, in: Proceedings of the Joint Universities Accelerator School (JUAS): Courses and exercises, E. Métral (ed.),
CERN Yellow Reports: School Proceedings, CERN-2024-003, DOI: 10.23730/CYRSP-2024-003, p. 2017.

based on unnecessary simplifications of the dynamics. It turns out that Hénon-like polynomial maps [1,2] are the most suitable models for these studies. The generic map reads

$$\begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = R(\omega_0) \begin{pmatrix} x_n \\ p_n + f(x_n) \end{pmatrix} , \tag{III.5.0.1}$$

where $R(\omega_0)$ is a 2×2 rotation matrix and f stands for a polynomial function. The fundamental model in this family, in which f is a quadratic polynomial, is named after the French astronomer M. Hénon. As shown in Ref. [2] this model represents the transfer map, in normalized Courant–Snyder coordinates [3], of a FODO cell in which a non-linear magnetic element is installed. The effect of the quadrupoles is represented by the rotation matrix, whereas that of the non-linear magnet, represented in the so-called single-kick approximation [2], by the function f. Note that $f(x_n) = x_n^m$ represents a magnetic element with $2(m+1)$ poles. A typical phase-space portrait is shown in Fig. III.5.1, where closed curves are clearly visible together with chains of islands, an inner one with five islands and an outer one with eleven. These chains are the characteristic elements of non-linear motion and represent the key element for the novel beam manipulations. Note that these polynomial maps can be easily generalized to describe the 4D non-linear betatron motion [2].

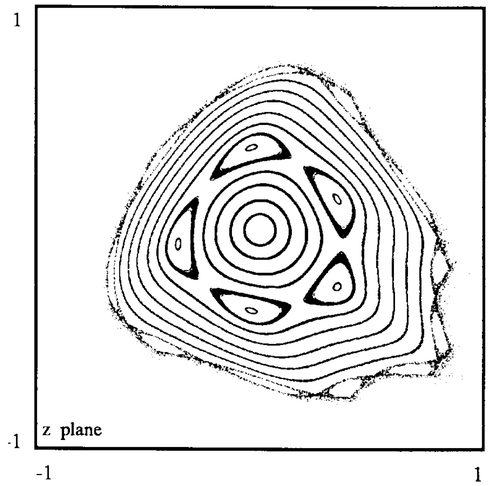

Fig. III.5.1: Phase-space portrait of the Hénon map (III.5.0.1) for $\omega_0 = 0.205/(2\pi)$.

III.5.0.2 Poincaré–Birkhoff theorem

Loosely speaking, the Poincaré–Birkhoff theorem, which has been first stated by Poincaré who provided a proof in some special cases and whose complete proof has been given by Birkhoff, states that for maps of the form (III.5.0.1) the fixed points appear as chains of stable and unstable fixed points, alternating in phase space. A fixed point of period q is the solution of the fixed-point equation $\mathcal{M}^{\circ q}(\mathbf{x}) = \mathbf{x}$, where $\mathcal{M}$ represents the transfer function of the system under consideration. Such a solution is a stable, or elliptic, fixed point whenever the dynamics generates locally closed curves similar to ellipses. The solution is an unstable, or hyperbolic, fixed point whenever the dynamics generates locally open curves similar to hyperbolas. Hyperbolic fixed points in the same chain are connected by a curve called a separatrix, which encloses the stable islands. Figure III.5.1 shows a chain of stable fixed points of period 5 and, according to the statement of the Poincaré–Birkhoff theorem, in between the stable fixed points there is a chain of period 5 unstable fixed points. In fact, a second chain of fixed points is visible as the last curve in the phase-space portrait of Fig. III.5.1. In this case, the period is 11, but once more there is the predicted alternation of stable and unstable fixed points.

III.5.0.3 Normal forms

In general, it is not possible to compute the solution of the equations of motion for a realistic system in closed form. Therefore, perturbative methods have been developed to overcome this difficulty. It is the case of Normal Forms that provide a complete mathematical framework to analyze the dynamics generated by a Hamiltonian system. This tool is available for discrete-time systems (maps) and continuous-time systems (flows), and for typical applications to accelerator physics, the first variant is the one to be considered. The underlying idea is to find a new coordinate system in which the dynamics explicitly shows its symmetries. In mathematical language, this implies solving the following equation:

$$\mathbf{F} \circ \mathbf{\Phi} = \mathbf{\Phi} \circ \mathbf{U}\,, \tag{III.5.0.2}$$

where $\mathbf{F}$ is the original map, $\mathbf{\Phi}$ is the coordinate transformation and $\mathbf{U}$ is the Normal Form. The similarity of Eq. (III.5.0.2) with the similarity relation for the matrices is obvious. In fact, Normal Forms can be seen as an extension of the Courant–Snyder theory that defines new coordinates in which the linear motion generates circles, instead of ellipses as invariant curves. The three functions in Eq. (III.5.0.2) are expressed as homogeneous polynomials, and the solution is obtained by finding the coefficients of the polynomial describing $\mathbf{\Phi}$ when $\mathbf{F}$ is given and $\mathbf{U}$ is predefined based on the symmetries of $\mathbf{F}$ that are considered essential and therefore retained in $\mathbf{U}$ (the details can be found in Ref. [2]).

Once Eq. (III.5.0.2) is solved, the study of $\mathbf{F}$ is replaced by that of $\mathbf{U}$. Note that the latter cannot be used for tracking as, in general, it is non-symplectic, being a truncation of a symplectic function. For these reasons, $\mathbf{U}$ is of limited use, but Normal Forms provide the possibility of constructing a so-called interpolating Hamiltonian, i.e. a Hamiltonian (hence continuous-time) that interpolates at integer times the orbits of $\mathbf{U}$. In this way, it is possible to apply all results from Hamiltonian theory to the interpolating Hamiltonian, and hence indirectly to $\mathbf{U}$ as well.

III.5.0.4 Separatrix-crossing theory

The phenomena that occur when a Hamiltonian system is slowly modulated have been widely studied in the framework of adiabatic theory [4, 5]. As the modulation of the Hamiltonian changes the shape of the separatrices in phase space, the trajectories can cross separatrices and enter into different stable regions that are associated with non-linear resonances. The separatrix crossing is described by a random process in the adiabatic limit, and its probability can be calculated as well as the change in adiabatic invariant due to the crossing [4, 5].

Consider a Hamiltonian $\mathcal{H}(p, q, \lambda = \epsilon\, t)\,,\ \epsilon \ll 1$, where the parameter λ is slowly modulated and whose phase space is sketched in Fig. III.5.2. If we consider an initial condition that lies in Region III, the transition probability into Region I or II of the phase space is given by [4]

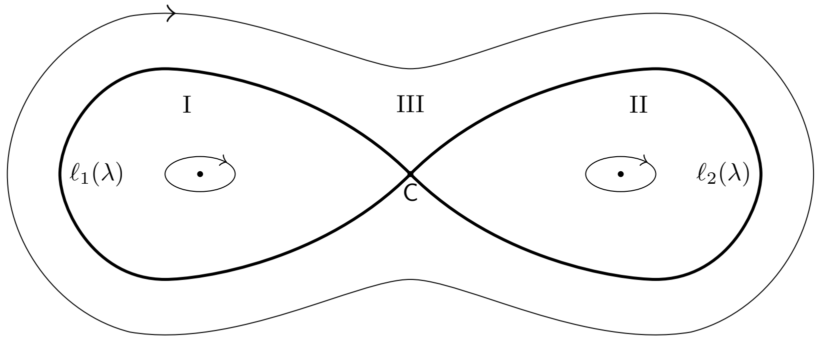

Fig. III.5.2: A generic phase-space portrait divided into three regions (I, II, III) by separatrices $\ell_1(\lambda)$ and $\ell_2(\lambda)$.

$$\mathcal{P}_{\mathrm{III}\to\mathrm{I}} = \frac{\Theta_{\mathrm{I}}}{\Theta_{\mathrm{I}} + \Theta_{\mathrm{II}}} \qquad \mathcal{P}_{\mathrm{III}\to\mathrm{II}} = 1 - \mathcal{P}_{\mathrm{III}\to\mathrm{I}}\,, \tag{III.5.0.3}$$

where

$$\Theta_i = \left.\frac{\mathrm{d}A_i}{\mathrm{d}\lambda}\right|_{\tilde{\lambda}} = \oint_{\partial A_i} \mathrm{d}t \left.\frac{\partial \mathcal{H}}{\partial \lambda}\right|_{\tilde{\lambda}} \qquad i = \mathrm{I},\, \mathrm{II}\,, \tag{III.5.0.4}$$

with A_i the area of the region i, ∂A_i the boundary of region i, and $\tilde{\lambda}$ the value of λ when the separatrix is crossed. Note that in case $\mathcal{P}_{\mathrm{III}\to i} < 0$, then $\mathcal{P}_{\mathrm{III}\to i}$ is set to zero, whereas when $\mathcal{P}_{\mathrm{III}\to i} > 1$ then $\mathcal{P}_{\mathrm{III}\to i}$ is set to unity.

When a separatrix is crossed, the adiabatic invariant J is expected to change according to the area difference between the two regions at the crossing time, so that just after crossing into a region of area A, we have $J = A/(2\pi)$. However, this value is exact only if the modulation is perfectly adiabatic, i.e. it is infinitely slow.

The adiabatic trapping into resonances has been studied in various works [4, 6] to show the possibility of transport in phase space when some system's parameters are slowly modulated. This phenomenon suggests possible applications in different fields and, in particular, in accelerator physics where multi-turn extraction (MTE) has been proposed [7] and successfully implemented as an operational beam manipulation at the CERN Proton Synchrotron (PS) [8, 9]. In this case, an extension of the results of adiabatic theory to quasi-integrable area-preserving maps has been considered, and analogous probabilities (III.5.0.3) to be captured in a resonance can be computed [10] when the Poincaré–Birkhoff theorem [11] can be applied to prove the existence of stable islands in phase space. The properties of such resonance islands for polynomial Hénon-like maps [1] have been studied in [2] and the possibility of performing an adiabatic trapping into a resonance has been analyzed by modulating the linear frequency at the elliptic fixed point [10].

III.5.1 The beginning of it all

The elements discussed above can be used to devise a new beam manipulation. Non-linear magnets, such as sextupoles and octupoles, can be used to generate stable fixed points of period q in the horizontal phase space, and the horizontal tune of the accelerator is varied so as to cross adiabatically the resonance of order q, and the beam can be split in several beamlets. More precisely, if the resonance is stable, $q+1$ beamlets are generated, while if it is unstable, q beamlets are produced. This mechanism has been studied using the simple models outlined before [7], and then made into a real beam manipulation at the CERN Proton Synchrotron ring [12], with the goal of replacing the so-called Continuous Transfer (CT) [13], used to fill the Super Proton Synchrotron (SPS) using two extractions from the PS, each generating a beam spill over five PS turns.

III.5.1.1 Multi-turn extraction

The CERN PS MTE has been designed to replace CT with a scheme characterized by lower beam losses. The beam splitting induced by the resonance-crossing process generates two structures: the beam in the islands creates a structure that winds along the circumference of the PS and closes after four turns, which is visible in Fig. III.5.1.

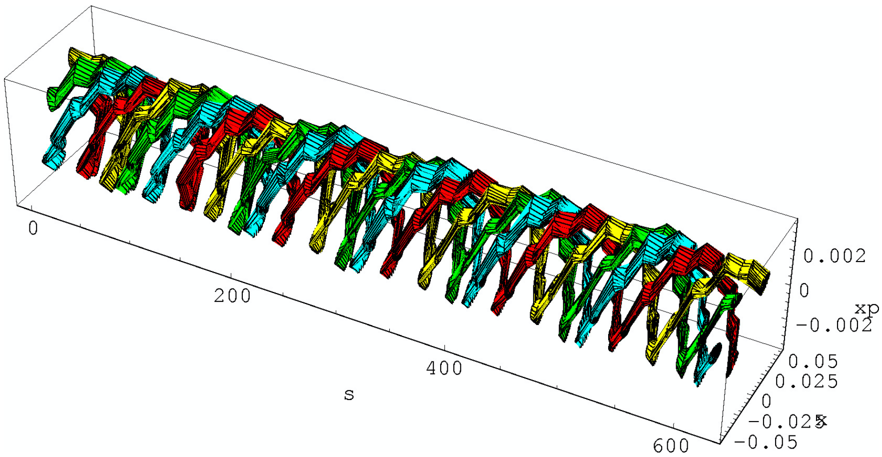

Fig. III.5.1: Extended horizontal phase space, i.e. horizontal phase space along the PS circumference. The four islands are colored differently and their change of phase along the ring is visible. The colors are used for the sake of clarity, but the four structures represent a single structure that extends over four machine turns.

The beam left around the origin of the phase space represents a structure of the same length as the PS ring. In this way, the beam has the structure needed to be extracted over five PS turns.

In reality, several octupole and sextupole magnets are distributed along the ring circumference (see Fig. III.5.2, bottom-left) and are accurately controlled to generate the required phase-space topology, which is needed to perform beam splitting. In Fig. III.5.2 (top-left) the evolution of the strength (in terms of current powering the magnets) during the splitting process is shown, while in the right part of the same figure the evolution of the measure horizontal beam profile is also shown as a waterfall plot.

A measured profile of the split beam is shown in Fig. III.5.2 (bottom) together with the corresponding phase-space portrait. The five Gaussian distributions visible in the measured beam profile are the result of the projection of the beamlets trapped in the stable islands visible in the phase-space portrait on the horizontal dimension.

MTE is the operational method for delivering high-intensity proton beams to the SPS since Oc-

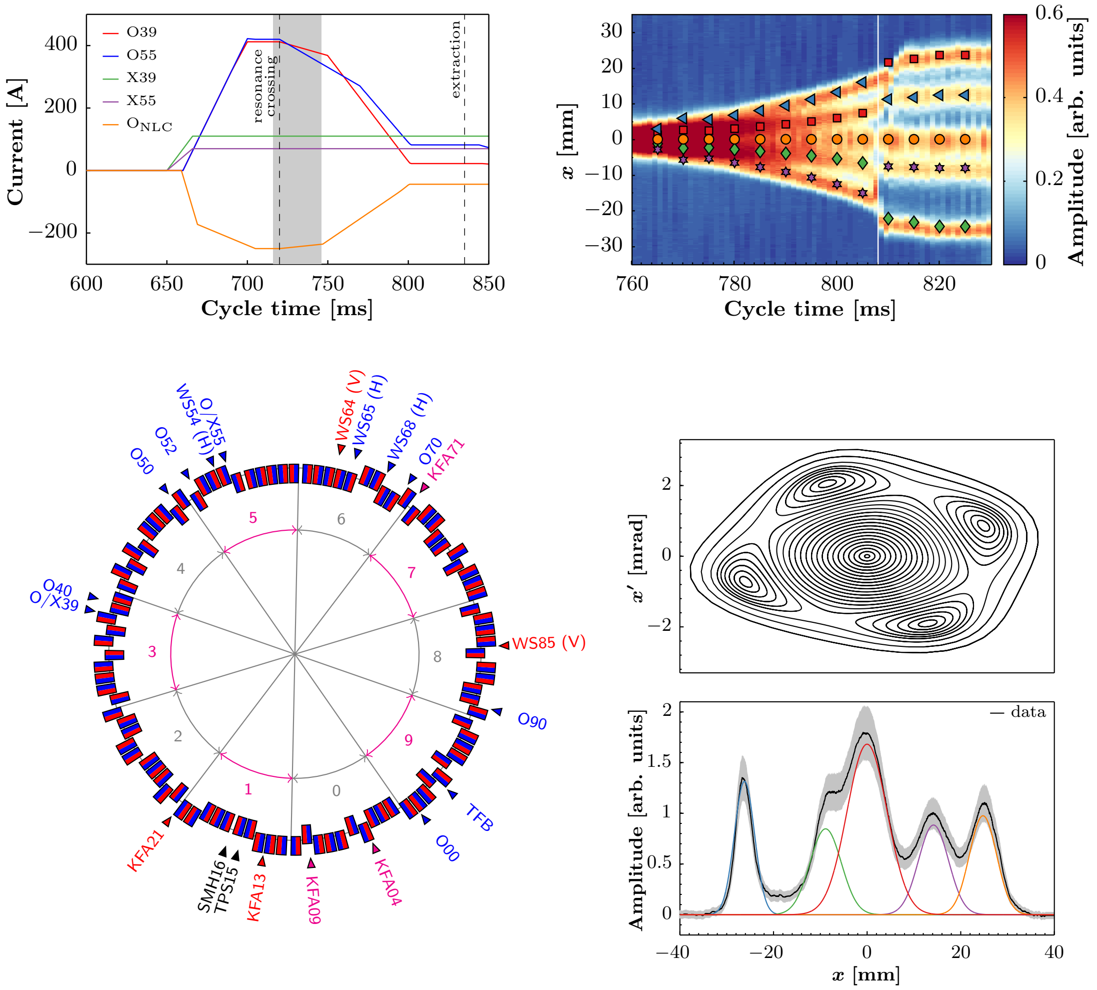

Fig. III.5.2: Top: Evolution of sextupoles and octupoles strength (left); waterfall measurement of the horizontal distribution (right). Bottom: Sketch of the PS ring with the devices used to split the beam (left); a measured profile of a split beam is shown together with the corresponding phase-space portrait (right) (from Ref. [9]).

tober 2015 [8] (a detailed account of the performance can be found in Refs. [9, 14]). Recently, another exotic beam manipulation has been added to beam splitting to further improve overall MTE performance, namely the so-called barrier bucket [15, 16], and combined beam manipulation has been put in operation since October 2022.

III.5.1.2 MTE extensions

Several extensions to MTE have been studied in detail in recent years. The most straightforward is the use of resonances other than the fourth one, and this leads to different configurations of split beams as can be seen in Fig. III.5.3, where the result of crossing the 2nd, 3rd, or 4th order resonance is shown. As mentioned above, one can observe that whenever an unstable resonance is used, such as the 2nd or 3rd order ones, the origin of phase space is essentially depleted of beam. On the other hand, for the stable resonance, the center is still occupied by some left over beam. The dimension of the beamlets decreases with the order of the resonance used to split the beam, which might limit the use of high-order resonances in applications. Note also that the process can be reversed and used to design a Multi-Turn Injection (MTI) [17] that enables shaping the transverse beam profile, which is an interesting aspect to mitigate the effects of space charge.

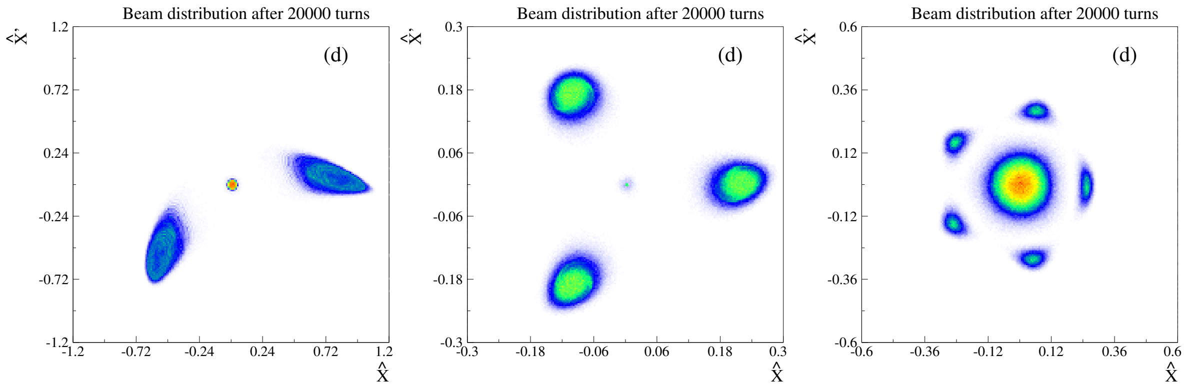

Fig. III.5.3: Examples of MTE extensions: split beam with 2nd-order resonance (left), with 3rd-order resonance (centre), and with 5th-order resonance (from Ref. [18]).

Another extension is the use of AC magnets to perform beam splitting [19], which means considering a system whose transfer map is given by

$$\begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = R(\omega_0) \begin{pmatrix} x_n \\ p_n + x_n^2 + \varepsilon x_n^{\ell-1} \cos \omega n \end{pmatrix} , \tag{III.5.1.1}$$

where in addition to the sextupole non-linearity represented by the quadratic term in Eq. (III.5.1.1), a 2ℓ-pole AC element is considered. The key feature of this system is that the resonance condition is not due to the horizontal tune, i.e. $n\omega_0 \approx q$, but rather between the horizontal tune and the frequency of the AC element ω. Therefore, in this case a nth order resonance condition is given by $\omega \approx n\omega_0$ and the splitting process is performed by varying the frequency ω so as to cross the selected resonance. An example of this process is shown in Fig. III.5.4, where splitting is performed using an AC dipole ($\ell = 1$) crossing the 3rd-order resonance. No difference is observed with respect to the more standard approach based on crossing the resonance with the horizontal tune. Despite similarities in the result of the splitting process, there is one fundamental difference that could be a key to the success of beam splitting in applications. In fact, the use of an AC element allows the machine to remain constant during the manipulation, which might be crucial whenever the transverse tune is constrained in a region that prevents crossing a resonance, e.g. because of space-charge effects. Of course, this advantage should be compared to the potential difficulty in having an AC element installed in the ring against the fact that sextupoles and octupoles are normally already present in modern circular accelerators because of their use for chromaticity correction and for Landau damping.

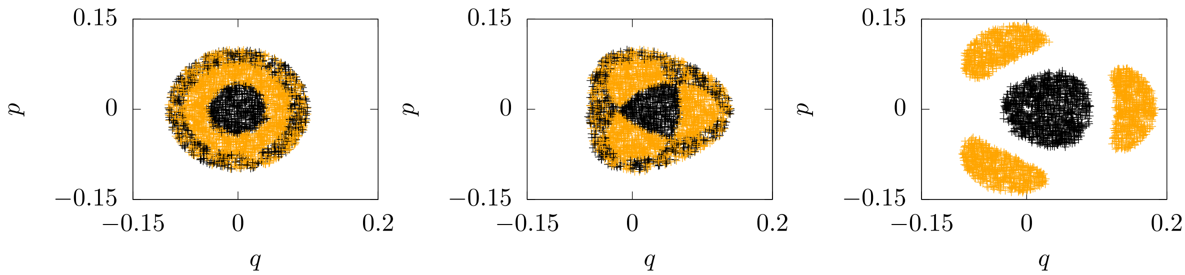

Fig. III.5.4: Evolution of an ensemble of particles in phase space with the colors used to identify in which region each initial condition has been trapped into. The map (III.5.1.1) with $\ell = 1$ has been used and the 3rd order resonance is considered. No difference is seen with respect to a splitting obtained by crossing a resonance with the horizontal tune (from Ref. [19]).

III.5.2 Crossing a 2D resonance

MTE and its extensions are all characterized by the fact that the resonance to be crossed is 1D, i.e. it involves only one tune (customarily the horizontal one). It is rather natural to consider the case of crossing a 2D resonance. In this case, we can aim to share the transverse emittances [20] , thus providing a means of manipulating them. As usual when one has to study a new beam manipulation it is essential to build an appropriate model that is simple to allow for a detailed analysis and realistic to ensure that the manipulation can be implemented in a real accelerator. The model considered is the Hamiltonian

$$\begin{aligned}\mathcal{H}(\phi_x,\, J_x,\, \phi_y,\, J_y) = \omega_x J_x + \omega_y J_y + \alpha_{xx} J_x^2 + 2\alpha_{xy} J_x J_y + \alpha_{yy} J_y^2 + \\ + G J_x^{m/2} J_y^{n/2} \cos(m\phi_x - n\phi_y)\,,\end{aligned} \tag{III.5.2.1}$$

which describes a system with two degrees of freedom in the action-angle coordinates $\phi_x,\, J_x,\, \phi_y,\, J_y$. The coefficients $\alpha_{xx}, \alpha_{xy}, \alpha_{yy}$ represent the amplitude-detuning parameters, and the resonant condition is expressed as $m\,\omega_x - n\,\omega_y \approx 0$. The canonical transformation

$$\begin{aligned} J_x &= mJ_1\,, & \phi_1 &= m\phi_x - n\phi_y\,, \\ J_y &= J_2 - nJ_1\,, & \phi_2 &= \phi_y\,, \end{aligned} \tag{III.5.2.2}$$

casts the Hamiltonian into the form

$$\mathcal{H}(\phi_1, J_1) = (\delta + \alpha_{12} J_2)\, J_1 + \alpha_{11} J_1^2 + G(mJ_1)^{\frac{m}{2}} (J_2 - nJ_1)^{\frac{n}{2}} \cos\phi_1 + \left[\omega_y J_2 + \alpha_{22} J_2^2\right]\,, \tag{III.5.2.3}$$

where $\delta = m\,\omega_x - n\,\omega_y$ is the resonance-distance parameter, and $\alpha_{11}, \alpha_{12}, \alpha_{22}$ are functions of $\alpha_{xx}, \alpha_{xy}, \alpha_{yy}$. As the variable ϕ_2 is missing, the corresponding action is an invariant of motion, and therefore the term in the square brackets can be dropped. The remaining Hamiltonian is a function of J_1 and ϕ_1 and J_2 plays the role of a parameter. Note that even if the original problem is described using two degrees of freedom, it can be reduced to a one-degree-of-freedom problem. Also in this case, one finds separatrices that sweep the phase space and generate trapping phenomena. The key point is that the final outcome of the trapping process is that

$$\epsilon_{x,\mathrm{f}} = \frac{m}{n}\epsilon_{y,\mathrm{i}} \qquad \epsilon_{y,\mathrm{f}} = \frac{n}{m}\epsilon_{x,\mathrm{i}}\,, \tag{III.5.2.4}$$

which corresponds to a redistribution of the values of the transverse emittances ϵ_x, ϵ_y depending on the characteristics of the resonance, i.e. m, n. An example of this process is shown in Fig. III.5.1, where the evolution of the emittances during the crossing of a 2D resonance is shown. The vertical emittance grows by a factor of two and the horizontal emittance shrinks by the same factor according to the expectations. Note that the product of the two emittances remains constant, as it should, with respect to the symplecticity of the dynamics.

III.5.3 Cooling of an annular beam distribution

The last manipulation that will be discussed is that aimed at cooling a beam with a special transverse beam distribution, namely, filling an annular region of the horizontal phase space. The system consists of a simple ring with quadruples and non-linear magnets (no matter whether they are sextupoles or octupoles) and an AC dipole. The Hamiltonian of such a system can be written as [21]

$$\mathcal{H}(\phi, J) = \omega_0 J + \Omega_2 J^2/2 + \varepsilon\sqrt{2J}\cos\phi\cos\omega t\,, \tag{III.5.3.1}$$

where ω_0 is the accelerator tune, Ω_2 is the amplitude detuning, ϵ the strength of the AC dipole and ω the frequency of the AC dipole. The underlying idea is to trap the particles in the annular distribution by means of a stable island and then transport them at lower amplitudes, which has the effect of reducing the initial emittance and hence cooling the distribution. An example of the proposed process is shown in Fig. III.5.1, where the evolution of the initial annular distribution is shown while the parameters ϵ and ω of the AC dipole are changed. The generation of a stable island is clearly seen (center plot) and is used to trap particles inside it. Once all particles are trapped inside, the island is moved towards the center of phase space, and the final distribution looks a standard one, with a much smaller emittance than the initial one. In this way, efficient cooling has been achieved, and with clever engineering of the island size variation, it is possible to optimize the trapping and transport of particles, and hence the cooling performance [21].

III.5.4 Conclusions and outlook

The use of non-linear dynamics can bring accelerator physics to new domains, where a number of novel beam manipulations can be envisaged. The prototype of these manipulations is MTE, which has been proposed to provide a loss-free multi-turn extraction from a circular accelerator. It takes advantage of the full power of resonance crossing. Several MTE extensions have been briefly presented, showing the potential of the original application. Two new applications have been presented, which enable sharing transverse emittances when crossing a 2D non-linear resonance and cooling an annular beam distribution. However, this seems only the beginning of this field, and several novel beam manipulations will be studied in the years to come.

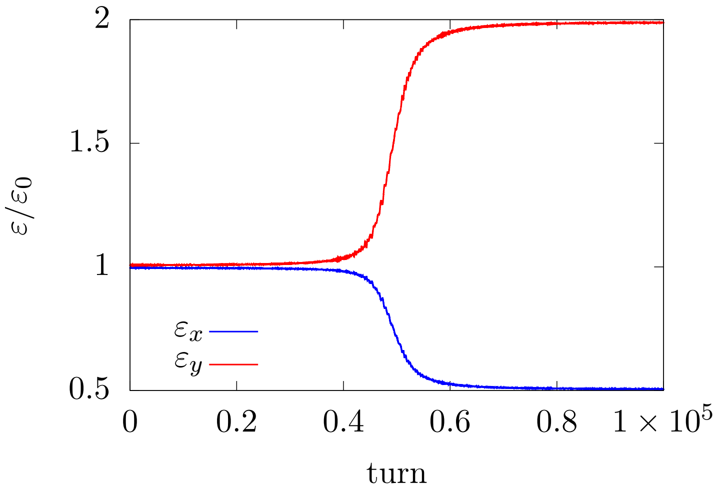

Fig. III.5.1: Evolution of the transverse emittances when crossing the 2D resonance $\omega_x - 2\,\omega_y = 0$ (from Ref. [20]).

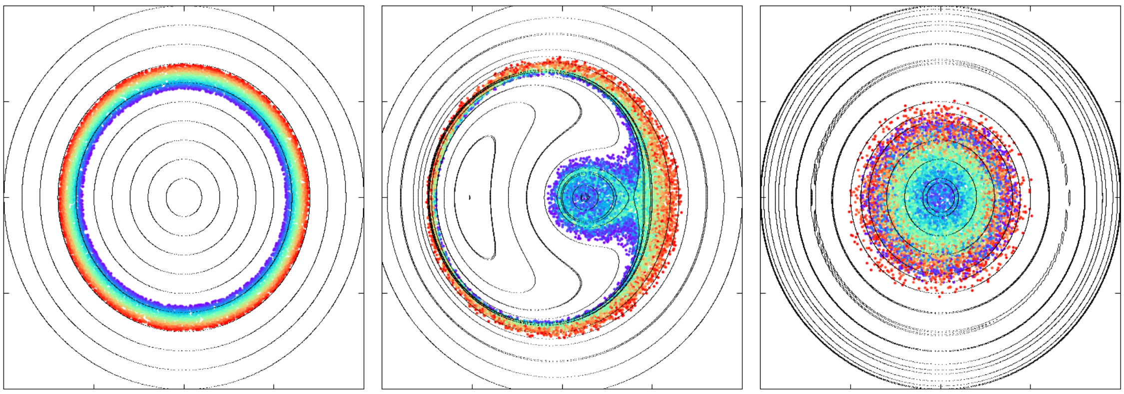

Fig. III.5.1: Evolution of the initial annular distribution (left) when the parameters of the AC dipole are changed (center and right). The island created with the AC dipole is used to trap particles inside it and then to transport them to the center of phase space (from Ref. [21]).

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