Field theory and the electroweak Standard Model

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These lectures offer a concise yet self-contained introduction to the Standard Model. We begin with relativistic quantum field theories for scalars, vectors, and fermions, showing how to compute amplitudes and cross sections. The principle of gauge invariance then guides us through the construction of the Standard Model as a spontaneously broken non-abelian gauge theory, highlighting key phenomenological implications along the way.

1 Introduction

The Standard Model of particle physics, which describes the strong and electroweak interactions, is formulated as a relativistic quantum field theory.

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These lecture notes of the "Field theory & the EW Standard Model" course introduce the foundational elements of relativistic quantum field theory using the Lagrangian formulation. In Section 3 we explore scalar, vector, and fermion quantum fields, demonstrating how to calculate scattering amplitudes and cross sections using perturbation theory and Feynman diagrams.

The principle of local gauge invariance is first illustrated through quantum electrodynamics (QED) with its Abelian gauge symmetry, before in Section 4 generalising to non-Abelian gauge theories. We discuss the structure of quantum chromodynamics (QCD) and the electroweak theory, which combines gauge invariance with spontaneous symmetry breaking through the Higgs mechanism.

Sections 3–4 of these notes are in part based on similar notes of the 2009 incarnation of the CERN school given by W. Hollik [1].

2 Notation and conventions

In these lecture notes we use natural units $c = \hbar = 1$ throughout. A contravariant (upper index) 4-vector x^μ is written as

$$x^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t \\ \vec{x} \end{pmatrix} , \tag{1}$$

where Greek indices $(\mu, \nu, \sigma, \rho, \dots)$ run over $0, 1, 2, 3$, while Latin indices $(i, j, k, l, \dots)$ run over the spatial components $1, 2, 3$.

The Lorentz-invariant spacetime interval can be written as

$$t^2 - \vec{x}^2 = \sum_{\mu,\nu=0}^{3} x^\mu x^\nu \eta_{\mu\nu} \equiv x^\mu \eta_{\mu\nu} x^\nu , \tag{2}$$

where we employ Einstein's summation convention throughout these notes (repeated upper and lower indices are implicitly summed over). The metric tensor $\eta_{\mu\nu}$ is defined as $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. A covariant four-vector (with lower index) x_μ can be obtained from a contravariant four-vector x^ν through contraction with the metric tensor:

$$x_\mu = \eta_{\mu\nu} x^\nu = (t, -\vec{x}) . \tag{3}$$

The spacetime interval can thus be expressed in several equivalent forms:

$$t^2 - \vec{x}^2 = x^\mu \eta_{\mu\nu} x^\nu = x^\mu x_\mu \equiv x^2 \tag{4}$$

For any two four-vectors a^μ and b^μ, we define their Lorentz-invariant scalar product as

$$a \cdot b = ab \equiv a^\mu b_\mu = a^\mu \eta_{\mu\nu} b^\nu . \tag{5}$$

Under a Lorentz transformation $x' = \Lambda x$, the components transform as

$$x'^\mu = \Lambda^\mu_\nu x^\nu , \tag{6}$$

where Λ^μ_ν represents the components of the 4×4 Lorentz transformation matrix Λ.

Important 4-vectors

- 4-momentum: $p^\mu = (E, \vec{p})$, i.e. $p^2 = p^\mu p_\mu = E^2 - \vec{p}^2 \equiv m^2$.
  Note: $p \cdot x = p^\mu x_\mu = Et - \vec{p}\vec{x}$ is invariant (regularly used in QFT).
- partial derivative: $\partial_\mu = (\partial_0, \partial_i)$ is a covariant 4-vector, i.e. $\partial^\mu = (\partial_t, -\partial_i)$,
  $\partial_\mu \partial^\mu = \frac{\partial^2}{\partial_t^2} - \frac{\partial^2}{\partial_x^2} - \frac{\partial^2}{\partial_y^2} - \frac{\partial^2}{\partial_z^2} = \partial_t^2 - \vec{\nabla}^2 = \Box$, and $\partial_\mu p^\mu(x) = \partial_0 p^0 + \partial_i p^i$.

We use the Dirac ket notation $|p\,\sigma\rangle$ to describe the quantum mechanical states of spin-s particles with momentum $p = (p^0, \vec{p})$ and helicity $\sigma = -s, -s+1, \cdots, +s$. Such states are normalised according to

$$\langle p\,\sigma | \,p'\sigma'\rangle = 2p^0\, \delta^3(\vec{p} - \vec{p}')\, \delta_{\sigma\sigma'}\,, \tag{7}$$

which is Lorentz invariant. The vacuum state is denoted by $|0\rangle$ and normalised as

$$\langle 0|0\rangle = 1\,. \tag{8}$$

A concise summary of required basics in quantum mechanics are presented in Appendix 5.2

3 Quantum field theory

3.1 Motivation

Quantum Field Theory (QFT) emerges from the necessity to reconcile quantum mechanics and special relativity. A naive attempt to combine these theories through relativistic quantum mechanics immediately encounters significant challenges, as illustrated in the following. First, we recall the non-relativistic Schrödinger equation,

$$i\partial_t \phi(t, \boldsymbol{x}) = \left(-\frac{1}{2m}\nabla^2 + V(\boldsymbol{x})\right)\phi(t, \boldsymbol{x}) = \hat{H}\,\phi(t, \boldsymbol{x})\,. \tag{9}$$

This equation allows for plane wave solutions of the form:

$$\phi(t, \boldsymbol{x}) \propto e^{-i(Et - \boldsymbol{p}\cdot\boldsymbol{x})} = e^{-ip\cdot x} \tag{10}$$

which satisfy the classical energy-momentum relation:

$$E = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{x})\,. \tag{11}$$

In an attempt to make quantum mechanics relativistic, we might want to consider the Klein–Gordon (KG) equation as "square" of the Schrödinger equation:

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\phi(t, \boldsymbol{x}) = \left(\partial_\mu \partial^\mu + m^2\right)\phi(x) = (\Box + m^2)\phi(x) = 0\,, \tag{12}$$

The KG equation also allows for plane wave solutions as in Eq. (10). These solutions now satisfy the relativistic energy-momentum relation

$$E^2 = m^2 + \boldsymbol{p}^2 \tag{13}$$

However, a fundamental problem emerges: The relativistic energy-momentum relation yields both positive and negative energy solutions

$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2}\,. \tag{14}$$

This presents a serious physical issue: with no lower bound on the energy spectrum, there's nothing to prevent a particle from cascading down to infinitely negative energy states, making the theory unstable. Moreover, the negative energy solutions cannot simply be discarded as they are mathematically necessary for the completeness of the theory. The resolution to this fundamental problem comes through a radical reconceptualisation. Instead of trying to make quantum mechanics relativistic, we quantise classical field theory itself. This approach—treating fields rather than particles as the fundamental objects—leads to quantum field theory, where particles emerge as excitations of these quantum fields. This framework naturally accommodates particle creation and annihilation, and resolves the negative energy problem by reinterpreting negative energy solutions as antiparticles.

3.2 Free scalar field

3.2.1 Lagrange formalism for classical fields

In order to formulate a field quantisation we first have to introduce a Lagrange formalism for fields. In classical field theory a field value is associated to every point in space. For a scalar field $\phi(\vec{x}, t)$ this is a scalar value, while a vector field $A^\mu(\vec{x}, t)$ associates a 4-vector to every point in space. In order to formulate an action-principle for a field theory it is crucial to see the field itself as dynamical variable, while $\vec{x}$ plays the role of a label.

For a scalar field $\phi(\vec{x}, t)$ we can associate $\phi(\vec{x}, t) = \phi_{\vec{x}}(t) \to q_i(t)$ as generalised coordinate in the formulation of Lagrange mechanics (see Appendix 5.1), and $\partial_0\phi(\vec{x}, t) = \dot{\phi}_{\vec{x}}(t) \to \dot{q}_i(t)$ as generalised velocity. The dynamics of the classical field is determined by the Lagrange density $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi, x)$ and the least-action principle reads

$$\delta S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi, x) = 0\,, \tag{15}$$

from which via variation $\phi \to \phi + \delta\phi$ the Euler–Lagrange equation for the field ϕ can be obtained as

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0\,, \tag{16}$$

yielding for an explicit $\mathcal{L}$ the field equation, i.e. the equation of motion, for the field ϕ.

In this field theory for a single scalar field $\phi(\vec{x}, t)$ we can define a conjugate momentum field

$\pi(\vec{x}, t)$ as

$$\pi(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}\,, \tag{17}$$

and the Hamilton density is obtained via a Legendre transformation as in Eq. (192) in Appendix 5.1.3,

$$\mathcal{H}(\phi, \pi, x) = \pi \partial_0 \phi - \mathcal{L}(\phi, \partial_\mu \phi, x)\,. \tag{18}$$

3.2.2 Field quantisation

In the transition from classical to quantum mechanics, classical observables are promoted to operators,

$$q_i, p_i, f(x_i, p_i) \quad \longrightarrow \quad \hat{q}_i, \hat{p}_i, \hat{f}(\hat{x}_i, \hat{p}_i)\,. \tag{19}$$

As part of this transition, the Poisson brackets of classical mechanics are replaced by quantum-mechanical commutators:

$$\begin{array}{lcl} \{q_i, p_j\} = \delta_{ij} & \longrightarrow & [\hat{q}_i, \hat{p}_j] = i\delta_{ij} \\ \{q_i, q_j\} = \{p_i, p_j\} = 0 & & [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 \end{array} \tag{20}$$

In the same way, the identification of classical fields (and conjugate fields) with field operators allows for a field quantisation,

$$\phi, \pi, f(\phi, \pi) \quad \longrightarrow \quad \hat{\phi}(x), \hat{\pi}(x), \hat{f}(\hat{\phi}(x), \hat{\pi}(x))\,, \tag{21}$$

where the equal-time Poisson brackets for the classical fields are replaced with equal-time commutator relations for the field operators,

$$\begin{array}{lcll} \{\phi(t,\vec{x}), \pi(t,\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y}) & \longrightarrow & [\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] & = i\delta^{(3)}(\vec{x}-\vec{y}) \\ \{\phi(t,\vec{x}), \phi(t,\vec{y})\} = \{\pi(t,\vec{x}), \pi(t,\vec{y})\} = 0 & & [\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] & = [\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = 0 \end{array} \tag{22}$$

Here, $\delta^{(3)}(\vec{x}-\vec{y})$ is the three-dimensional Dirac delta function, ensuring that field operators at different spatial points commute. This local commutativity is crucial for maintaining causality in the relativistic theory. These quantisation rules form the foundation for constructing quantum field theories, where the field operators will create and annihilate particles when acting on appropriate quantum states including the vacuum.

3.2.3 Quantisation of the scalar field

We now want to apply the quantisation procedure of Section 3.2.2 to a free scalar field. The Lagrangian for a free real scalar field, describing neutral spinless particles with mass m,

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu \phi)^2 - \frac{m^2}{2}\,\phi^2 \tag{23}$$

yields the *Klein–Gordon equation* as equation of motion according to Eq. (16),

$$(\partial_\mu \partial^\mu + m^2)\,\phi = (\Box + m^2)\,\phi = 0\,. \tag{24}$$

The general solution to the KG equation can be expressed as a Fourier decomposition in terms of plane waves $e^{\pm ikx}$,

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{2k^0}\,[a(k)\,e^{-ikx} \;+\; a^*(k)\,e^{ikx}\,]\,, \tag{25}$$

where $k^0 = \sqrt{\vec{k}^2 + m^2}$ ensures that the relativistic energy-momentum relation is satisfied. The presence of both $a(k)$ and its complex conjugate $a^*(k)$ guarantees that $\phi(x)$ remains real-valued. In order to quantise this field, we promote the Fourier coefficients to operators: $a(k) \to \hat{a}(k)$ and $a^*(k) \to \hat{a}^\dagger(k)$ with the fundamental commutation relations

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = (2\pi)^3\,\delta^{(3)}(\vec{k} - \vec{k}') \tag{26}$$

$$[\hat{a}(\vec{k}), \hat{a}(\vec{k}')] = [\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}')] = 0 \tag{27}$$

This algebra is identical to that of the simple harmonic oscillator (SHO) in quantum mechanics. The operators $\hat{a}(\vec{k})$ and $\hat{a}^\dagger(\vec{k})$ act as ladder operators that annihilate and create single-particle states, see Appendix 5.2.2:

$$a^\dagger(k)\,|0\rangle = |k\rangle \tag{28}$$

$$a(k)\,|k'\rangle = 2E_k\,\delta^{(3)}(\vec{k} - \vec{k}')\,|0\rangle \tag{29}$$

The vacuum state $|0\rangle$ plays a fundamental role. It is defined as the state annihilated by all annihilation operators:

$$\hat{a}(k)|0\rangle = 0 \quad \text{for all } k, \tag{30}$$

with normalisation $\langle 0|0\rangle = 1$. This state represents more than classical "empty space"—it is the state of lowest energy of the quantum field and contains zero-point fluctuations due to the uncertainty principle. The vacuum is Lorentz and translation invariant, meaning all inertial observers agree on this state and it looks the same at all points in space.

In terms of the ladder operators the Hamiltonian of the free scalar field takes the form

$$H = \int d^3k\,\mathcal{H} = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2}\left(\hat{a}^\dagger(k)\hat{a}(k) + \frac{1}{2}\left[\hat{a}(k), \hat{a}^\dagger(k)\right]\right)\,. \tag{31}$$

States created by acting with $a^\dagger(\vec{k})$ on the vacuum state $|0\rangle$ are eigenstates of the Hamiltonian (as for the SHO in non-relativistic Quantum Mechanics),

$$\hat{H}\,\hat{a}^\dagger(\vec{k})\,|0\rangle = E_k\,a^\dagger(\vec{k})\,|0\rangle\,. \tag{32}$$

A general n-particle state can be constructed via subsequent action of creation operators as

$$|\vec{k}_1 \dots \vec{k}_n\rangle = (2E_{k_1})^{1/2} \dots (2E_{k_n})^{1/2}\, \hat{a}^\dagger(\vec{k}_1) \dots a^\dagger(\vec{k}_n)\, |0\rangle \,. \tag{33}$$

Here we have $|\vec{k}_1\vec{k}_2\rangle \sim \hat{a}^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)|\vec{0}\rangle = |\vec{k}_2\vec{k}_1\rangle$, i.e. multi-particle states are symmetric. Thus, the scalar field is a Boson following Bose-Einstein statistics.

The quantum field itself is a superposition of all possible momentum states. When we excite this field, we create what we observe as particles—each particle being a localised excitation with specific momentum. In other words, the field contains within it the complete mathematical freedom to describe any configuration of one or more particles, each with their own momentum state. This is why we say particles are excitations of their corresponding quantum fields. You can think of it like a violin string—the string itself (analogous to the field) can vibrate in many different ways simultaneously (superposition of modes), and each distinct vibration pattern (excitation) represents a "particle" with specific properties. The key difference is that quantum fields exist throughout all of space and can support multiple excitations at once.

The quantum field operator of Eq. (25) acting on the vacuum creates a state localised in position space,

$$|\vec{x}\rangle = \hat{\phi}(0,\vec{x})|0\rangle = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{2k^0} \,[\hat{a}(k)\, e^{+i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger(k)\, e^{-i\vec{k}\cdot\vec{x}}\,]|0\rangle \tag{34}$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{2k^0} e^{-i\vec{k}\cdot\vec{x}}|\vec{k}\rangle \qquad \text{where} \quad |\vec{k}\rangle = \hat{a}^\dagger(\vec{k})|0\rangle \,, \tag{35}$$

i.e. $|\vec{k}\rangle$ represents a single-particle state with definite momentum. This state $|\vec{x}\rangle$ represents a superposition of single-particle states with different momenta—when we "create a particle" at position $\vec{x}$, it doesn't have a unique momentum. The probability amplitude to find it with momentum $\vec{k}$ is given by the wave functions

$$\langle 0|\hat{\phi}(0,\vec{x})|\vec{k}\rangle \sim e^{+i\vec{k}\cdot\vec{x}} \quad \text{for incoming scalar state} \rightarrow \quad \langle 0|\hat{\phi}(t,\vec{x})|k\rangle \sim 1 \cdot e^{-ik\cdot x} \tag{36}$$

$$\langle\vec{k}|\hat{\phi}(0,\vec{x})|0> \sim e^{-i\vec{k}\cdot\vec{x}} \quad \text{for outgoing scalar state} \rightarrow \quad \langle k|\hat{\phi}(t,\vec{x})|0> \sim 1 \cdot e^{+ik\cdot x} \tag{37}$$

where we defined

$$|x\rangle = \hat{\phi}(t,\vec{x})|0\rangle \,. \tag{38}$$

These one-particle state wave functions distinguish between states of incoming and outgoing particles and the factor 1 in Eqs. (36) and (37) will be identified as momentum-space Feynman rule for external scalar particles.

The propagation amplitude between two spacetime points y and x is given

$$\langle x|y\rangle = \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle \equiv D(x,y) \tag{39}$$

$$= D(x-y) = \cdots = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{\vec{k}}} e^{-ik\cdot(x-y)} \tag{40}$$

Here $D(x, y)$ only depends on the distance $x - y$. In order to ensure causality we need to further refine this picture defining the Feynman propagator

$$D_F(x-y) = \begin{cases} D(x-y) & \text{if} \quad x^0 > y^0 \\ D(y-x) & \text{if} \quad y^0 > x^0 \end{cases} = D(x-y)\Theta(x^0-y^0) + D(y-x)\Theta(x^0-y^0)$$
$$= \langle 0|\,\hat{T}\,\hat{\phi}(x)\,\hat{\phi}(y)\,|0\rangle \tag{41}$$

where we make use of the time-ordering operator :

$$\hat{T}\,\hat{\phi}(x)\,\hat{\phi}(y) = \begin{cases} \hat{\phi}(x)\hat{\phi}(y) & \text{if} \quad x^0 > y^0 \\ \hat{\phi}(y)\hat{\phi}(x) & \text{if} \quad y^0 > x^0 \end{cases} \tag{42}$$

The Feynman propagator is a fundamental building block of quantum field theory calculations, and a core ingredient in the Feynman rules used to compute scattering amplitudes via Feynman diagrams.

The Feynman propagator $D_F(x - y)$ is the Green's function of the Klein–Gordon equation,

$$(\partial_\mu \partial^\mu + m^2)\, D_F(x-y) = -\delta^4(x-y)\,. \tag{43}$$

We can solve this equation via Fourier transformation,

$$D_F(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, D(k)\, e^{-ik(x-y)}\,. \tag{44}$$

In momentum-space the inhomogeneous Klein–Gordon equation becomes algebraic

$$(k^2 - m^2)\, D(k) = 1 \tag{45}$$

leading to the momentum-space Feynman propagator

$$i\, D(k) = \frac{i}{k^2 - m^2 + i\epsilon}\,. \tag{46}$$

Here the $i\epsilon$ prescription ensures causality by properly defining the contour integration in Eq. (44). Indeed, performing the energy integral we find

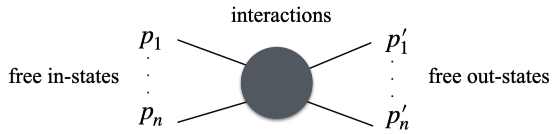
$$iD_F(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \tag{47}$$
$$= \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 k}{2k^0} \left[e^{-ik(x-y)}\, \Theta(x^0 - y^0) \;+\; e^{ik(x-y)}\, \Theta(y^0 - x^0) \right]_{k^0 = \sqrt{\vec{k}^2 + m^2}} \tag{48}$$
$$= \langle 0|\,\hat{T}\,\hat{\phi}(x)\,\hat{\phi}(y)\,|0\rangle \quad . \tag{49}$$

Here, the k^0 integration is performed by closing the contour in the lower half-plane, yielding Eq. (41).

3.3 Interacting fields

3.3.1 The S-matrix and scattering processes

Ultimately our goal is to compute cross sections for scattering processes—probabilities for transitions between initial and final states separated by infinite time



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In this picture a free initial state at $t = -\infty$ evolves subject to an interaction region into a final state at $t = +\infty$,

$$|\mathrm{in}\rangle = |\mathrm{p}_1, \ldots, \mathrm{p}_\mathrm{n}; \mathrm{in}\rangle = |\phi(\mathrm{t} = -\infty)\rangle \quad \longrightarrow \quad |\mathrm{out}\rangle = |\mathrm{p}'_1, \ldots, \mathrm{p}'_\mathrm{n}; \mathrm{out}\rangle = |\phi(\mathrm{t} = +\infty)\rangle \tag{50}$$

In the interaction picture, see Appendix 5.2.3, the free initial state evolves through the interaction region via $|\phi(t)\rangle = U_I(t, -\infty)|\mathrm{in}\rangle$. The S-matrix element is defined as the overlap between this evolved state and the final state

$$S_{\mathrm{fi}} = \langle \mathrm{f}|\hat{\mathrm{S}}|\mathrm{i}\rangle = \lim_{\mathrm{t}\to+\infty} \langle \mathrm{f}|\phi(\mathrm{t})\rangle = \langle \mathrm{out}|\mathrm{U_I}(+\infty, -\infty)|\mathrm{in}\rangle\,. \tag{51}$$

The S-operator has the form

$$\hat{S} = \hat{U}(+\infty, -\infty) = \hat{T}\, e^{-i\int_{-\infty}^{\infty} \hat{H}_I(t')dt'} \tag{52}$$

where $\hat{T}$ is the time-ordering operator. When there are no interactions ($\hat{H}_I = 0$), we have $S = \mathbf{1}$. In general, the S-operator can be expanded perturbatively

$$\hat{S} = \hat{T}\left(1 - i\int_{-\infty}^{\infty} H_I(t')\, dt' + \ldots\right) = \hat{T}\left(1 - i\int_{-\infty}^{\infty} H_I(t')\, dt' + \ldots\right). \tag{53}$$

Considering as a concrete example a ϕ^4 theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\,\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{54}$$

gives an interaction Hamiltonian $\mathcal{H}_I = \frac{\lambda}{4!}\phi^4$. For a $2 \to 2$ scattering process, the scattering amplitude is given by

$$A_{2\to2} = S_{\mathrm{fi}} = \langle \mathrm{f}|\hat{\mathrm{S}}|\mathrm{i}\rangle = \langle 0|\hat{\mathrm{a}}_{\tilde{\mathrm{p}}'_1}\,\hat{\mathrm{a}}_{\tilde{\mathrm{p}}'_2}\,\hat{\mathrm{S}}\,\hat{\mathrm{a}}^\dagger_{\tilde{\mathrm{p}}_1}\hat{\mathrm{a}}^\dagger_{\tilde{\mathrm{p}}_2}|0\rangle\,. \tag{55}$$

At leading-order (LO) in the perturbative expansion in λ we have for the S-matrix:

$$\hat{S} \approx \hat{T}\left(1 - i\int_{-\infty}^{\infty} \mathcal{H}_I(x')\, d^4x'\right) = \hat{T}\left(1 - \frac{i\,\lambda}{4!}\int_{-\infty}^{\infty} \hat{\phi}^4\, d^4x'\right). \tag{56}$$

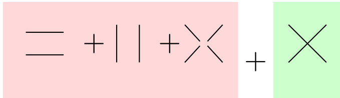
Thus, the $2 \to 2$ scattering amplitude is given by

$$A_{2\to 2} = \langle 0|\hat{a}_{\vec{p}_1'}\,\hat{a}_{\vec{p}_2'}\,\hat{a}^\dagger_{\vec{p}_1}\,\hat{a}^\dagger_{\vec{p}_2}|0\rangle - \frac{i\,\lambda}{4!}\langle 0|\hat{T}(\hat{a}_{\vec{p}_1'}\,\hat{a}_{\vec{p}_2'}\;\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}\,\hat{a}^\dagger_{\vec{p}_1}\,\hat{a}^\dagger_{\vec{p}_2})|0\rangle\,. \tag{57}$$

The first term vanishes for scattering processes where $\langle f|i\rangle = 0$.

The second term can be evaluated using **Wick's theorem**, which decomposes the expectation value into products of two-point functions (propagators),

$$\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = D_{\mathrm{F}}(x-y) = \underset{x}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\underset{y}{\bullet} \tag{58}$$

$\langle 0|\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}|0\rangle =$  $+\cdots\,,$ (59)

where, for the four-field correlator, the first three terms yield disconnected diagrams, which do not contribute to scattering. The external states contribute wave functions,

$$\langle 0|\hat{T}(\hat{\phi}a^\dagger_{\vec{p}})|0\rangle = \langle 0|\hat{\phi}|\vec{p}\rangle = 1\cdot e^{-ip\cdot x} \tag{60}$$
$$\langle 0|\hat{T}(a_{\vec{p}}\hat{\phi})|0> = \langle\vec{p}|\hat{\phi}|0>= 1\cdot e^{ip\cdot x}\,. \tag{61}$$

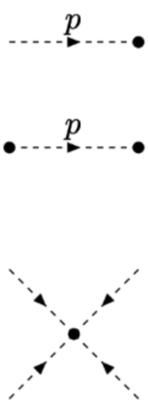
The final result for the $2 \to 2$ amplitude in the ϕ^4 theory at leading order, dropping disconnected diagrams, is

$$A_{2\to 2} = \cdots = -i\lambda(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1' - p_2') \quad , \tag{62}$$

where $\delta^{(4)}(p_1 + p_2 - p_1' - p_2') = \int d^4x e^{-ip_1x}\,e^{-ip_2x}\,e^{+ip_1'x}\,e^{ip_2x}$ originates from the external lines, and the $-i\lambda$ factor can be identified as the momentum-space Feynman rule for the ϕ^4 vertex.

3.3.2 Feynman rules for ϕ^4 theory

In summary, a scattering amplitude within the ϕ^4 theory can be obtained considering all disconnected diagrams at a given multiplicity and perturbative order. Each diagram is evaluated using Feynman rules: external particles contribute plane waves, internal lines give propagators, vertices contribute as factors,



$$1$$

$$\frac{i}{p^2 - m^2 + i\epsilon}$$

$$-i\lambda$$

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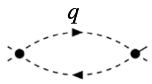
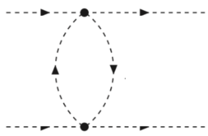
The final amplitude includes appropriate symmetry factors for identical particles and conserves energy-momentum at each vertex. Additionally for higher-order contributions closed loops yield integrals over the undetermined internal loop momenta



$$\int \frac{d^4q}{(4\pi)^4} \quad .$$

As an example consider the one-loop diagram in $2 \to 2$ scattering:



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Applying the ϕ^4 Feynman rules give the amplitude

$$\mathcal{M} = \tfrac{1}{2}(-i\,\lambda)^2 \int \frac{d^4q}{(4\pi)^4} \frac{i}{q^2 - m^2} \frac{i}{(q + p_1 - p_1')^2 - m^2} \quad . \tag{63}$$

Here a factor $\frac{1}{2}$ comes from identical particles in the final state, $(-i\lambda)^2$ from two vertices, two propagators in the loop with momenta q and $(q + p_1 - p_1')$, and we integrate over the undetermined loop momentum q.

3.3.3 Cross section

For a general scattering process

$$a + b \to b_1 + b_2 + \cdots + b_n \tag{64}$$

with momenta $P_i = p_a + p_b = P_f = p_1 + \cdots + p_n$. The S-matrix element connecting initial state $|i\rangle = |a(p_a), b(p_b)\rangle$ to final state $|f\rangle = |b_1(p_1), \cdots b_n(p_n)\rangle$ is

$$S_{fi} = \langle f|\hat{S}|i\rangle = (2\pi)^4\, \delta^{(4)}(P_i - P_f)\, \mathcal{M}_{fi}\, (2\pi)^{-3(n+2)/2} \,. \tag{65}$$

Here $\mathcal{M}_{fi}$ is the matrix element obtained from the Feynman rules, and the delta function ensures momentum conservation: $P_i = p_a + p_b = P_f = p_1 + \cdots + p_n$. The cross section represents the interaction probability per incident flux

$$\sigma = \frac{1}{N} \cdot \text{probability of interactions} \tag{66}$$

$$= \frac{1}{\mathcal{F}} \cdot \prod_f \int d\Phi_f\, (2\pi)^4\, \delta^{(4)}(P_i - P_f)\, |\mathcal{M}_{fi}|^2 \,. \tag{67}$$

where $d\Phi_f$ is the phase space measure for final state particles, and $\mathcal{F}$ is the flux factor ($\mathcal{F} = 2s$ in the massless limit, with $s = (p_a + p_b)^2$). For $2 \to 2$ scattering, this simplifies to:

$$\sigma_{2\to 2} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_1|}{|\vec{p}_a|} \int |\mathcal{M}_{fi}|^2 \mathrm{d}\Omega \tag{68}$$

with only an angular integral in the solid angle element $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$ remaining.

3.4 Standard Model fields

In the SM only the Higgs particle is a fundamental scalar. All other particles are either spin-$\frac{1}{2}$ fermions or spin-1 vector-bosons. In the following we develop their representations in terms of quantum fields.

3.4.1 Vector fields

A vector field $A_\mu(x)$ describes particles with spin 1. Their quantum states $|k\lambda\rangle$ are characterised by two quantum numbers: momentum k that specifies the particle's four-momentum and helicity λ that describes the spin projection, with values $\lambda = \pm 1, 0$ for massive particles (like $W^\pm$, Z bosons), $\lambda = \pm 1$ for massless particles (like photons).

3.4.1.1 Massive vector field

The dynamics of a free massive vector field $Z^\nu = (\phi_Z, \vec{Z})$ is governed by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} Z_\mu Z^\mu \quad \text{with} \quad F_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \,. \tag{69}$$

The Euler–Lagrange equations for Z^ν yield the **Proca equations**,

$$\left[(\Box + m^2)\, g^{\mu\nu} - \partial^\mu \partial^\nu \right] Z_\nu = 0 \,. \tag{70}$$

These equations admit plane wave solutions

$$\sim \epsilon_\mu^{(\lambda)}\, e^{\pm ikx} \tag{71}$$

with polarisation vectors $\epsilon_\mu^{(\lambda)}$ satisfying

$$\epsilon^{(\lambda)} \cdot k = 0 \,, \quad \epsilon^{(\lambda)*} \cdot \epsilon^{(\lambda')} = -\delta_{\lambda\lambda'} \,. \tag{72}$$

These polarisation vectors fulfil the polarisation sum ("completeness"),

$$\sum_{\lambda=1}^{3} \epsilon_\mu^{(\lambda)*} \epsilon_\nu^{(\lambda)} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} \,. \tag{73}$$

The general solutions of the Proca equations can be written as a Fourier expansion,

$$Z_\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_\lambda \int \frac{\mathrm{d}^3 k}{2k^0} \left[a_\lambda(k)\, \epsilon_\mu^{(\lambda)}(k)\, e^{-ikx} + a_\lambda^\dagger(k)\, \epsilon_\mu^{(\lambda)}(k)^*\, e^{ikx} \right]. \tag{74}$$

where $a_\lambda(k)$ and $a_\lambda^\dagger(k)$ are annihilation and creation operators,

$$\begin{aligned} a_\lambda^\dagger(k)\, |0\rangle &= |k\lambda\rangle \\ a_\lambda(k)\, |k'\lambda'\rangle &= 2k^0\, \delta^3(\vec{k} - \vec{k}\,')\, \delta_{\lambda\lambda'}\, |0\rangle \,. \end{aligned} \tag{75}$$

The one-particle state wave functions are

$$\langle 0\,|A_\mu(x)|\,k\lambda\rangle = \frac{1}{(2\pi)^{3/2}}\,\epsilon_\mu^{(\lambda)}(k)\,e^{-ikx} \quad \text{incoming massive vector state}\,, \tag{76}$$

$$\langle k\lambda\,|A_\mu(x)|\,0\rangle = \frac{1}{(2\pi)^{3/2}}\,\epsilon_\mu^{(\lambda)}(k)^*\,e^{ikx} \quad \text{outgoing massive vector state}\,. \tag{77}$$

Thus, in momentum space the wave functions are given by the polarisation vectors.

We can obtain the Feynman propagator $D_{\rho\nu}(x-y)$ of the massive vector field as Green's function of the inhomogenous Proca equations with point-like source,

$$\left[(\Box + m^2)\,g^{\mu\rho} - \partial^\mu\partial^\rho\right] D_{\rho\nu}(x-y) = g^\mu{}_\nu\,\delta^4(x-y)\;. \tag{78}$$

In momentum space this yields an algebraic equation for $D_{\rho\nu}(k)$,

$$\left[(-k^2 + m^2)\,g^{\mu\rho} + k^\mu k^\rho\right] D_{\rho\nu}(k) = g^\mu{}_\nu\,. \tag{79}$$

The solution is the momentum-space Feynman propagator,

$$i\,D_{\rho\nu}(k) = \frac{i}{k^2 - m^2 + i\epsilon}\left(-g_{\nu\rho} + \frac{k_\nu k_\rho}{m^2}\right). \tag{80}$$

As for the scalar propagator in Eq. (46) the $+i\epsilon$ term in the denominator ensures causality, and the factor i is convention.

3.4.1.2 Massless vector field

The dynamics of the 4-potential A_μ of a free massless vector field ("photon") is described by the Maxwell Lagrangian,

$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{81}$$

Applying the Euler–Lagrange equations with respect to the 4-potential A_μ gives the Maxwell equations

$$\left(\Box\,g^{\mu\nu} - \partial^\mu\partial^\nu\right) A_\nu = 0\,. \tag{82}$$

There are two physical polarization vectors $\epsilon_\mu^{(1,2)}$ for the transverse polarization, with $\vec{\epsilon}^{\,(1,2)} \cdot \vec{k} = 0$. The third solution of Eq. (82) with a longitudinal polarization vector $\epsilon_\mu \sim k_\mu$ is unphysical; it can be removed by a gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x) \equiv 0 \quad \text{with} \quad \chi(x) = \pm i e^{\pm ikx}\,. \tag{83}$$

The algebraic equation for the propagator of the massless vector field follows from Eq. (79) setting $m = 0$:

$$\left(-\,k^2\,g^{\mu\rho} + k^\mu k^\rho\right) D_{\rho\nu}(k) \equiv K^{\mu\rho}\,D_{\rho\nu}(k) = g^\mu{}_\nu\,. \tag{84}$$

However, this equation has no solution because the operator $K^{\mu\rho}$ is singular: it has a zero eigenvalue corresponding to eigenvector k_ρ ($K^{\mu\rho}k_\rho = 0$). This mathematical difficulty originates from the gauge invariance of the theory. To resolve this, we must break the gauge symmetry by adding a gauge-fixing term to the Lagrangian. The standard choice is the Lorentz gauge-fixing term

$$\mathcal{L}_{\text{fix}} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 , \tag{85}$$

Here ξ is an arbitrary gauge-fixing parameter, with $\xi = 1$ corresponding to Feynman gauge. The Maxwell Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 . \tag{86}$$

This modifies the operator $K^{\mu\rho}$ to

$$K^{\mu\rho} \to K^{\mu\rho} - \frac{1}{\xi}\, k^\mu k^\rho , \tag{87}$$

and the propagator equation Eq. (84) becomes,

$$\left[-k^2\, g^{\mu\rho} + \left(1 - \frac{1}{\xi}\right)k^\mu k^\rho\right] D_{\rho\nu}(k) = g^\mu{}_\nu . \tag{88}$$

This equation has a solution—the gauge-dependent photon propagator

$$i\, D_{\rho\nu}(k) = \frac{i}{k^2 + i\epsilon}\left[-g_{\nu\rho} + (1-\xi)\,\frac{k_\nu k_\rho}{k^2}\right] . \tag{89}$$

This propagator simplifies considerably in Feynman gauge ($\xi = 1$) where the second term vanishes.

3.4.2 Fermion fields

Spin-$\frac{1}{2}$ particles like electrons and their antiparticles (positrons) are described by 4-component spinor fields,

$$\psi(x) \;=\; \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} . \tag{90}$$

The dynamics of the free spinor field are governed by the Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi , \tag{91}$$

where $\bar{\psi}$ is the adjoint spinor:

$$\overline{\psi} = \psi^\dagger\,\gamma^0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*) . \tag{92}$$

The Dirac matrices γ^μ are 4×4 matrices that satisfy

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \tag{93}$$

and can be written in terms of Pauli matrices as

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}. \tag{94}$$

The Lagrangian Eq. (91) yields the *Dirac equation* as the equation of motion,

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0\,. \tag{95}$$

The Dirac equation has two types of solutions, corresponding to particle and anti-particle wave functions,

$$u(p)\,e^{-ipx} \quad \text{and} \quad v(p)\,e^{ipx} \tag{96}$$

where the spinors u and v satisfy the algebraic equations

$$(\not{p} - m)\,u(p) = 0\,, \qquad (\not{p} + m)\,v(p) = 0\,. \tag{97}$$

Here $\not{a} = \gamma^\mu a_\mu$ is the Feynman slash notation. These solutions in Eq. (97) can be classified by helicity $\sigma = \pm\frac{1}{2}$

$$\frac{1}{2}\,(\vec{\Sigma} \cdot \vec{n})\,u_\sigma(p) = \sigma\,u_\sigma(p)\,, \qquad -\frac{1}{2}\,(\vec{\Sigma} \cdot \vec{n})\,v_\sigma(p) = \sigma\,v_\sigma(p) \tag{98}$$

with $\hat{p} = \vec{p}/|\vec{p}|$ and $\vec{\Sigma}$ being the spin matrices

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad \text{and} \quad \vec{n} = \frac{\vec{p}}{|\vec{p}|}\,. \tag{99}$$

The spinors satisfy important normalisation and completeness relations

$$\overline{u}_\sigma\,u_{\sigma'} = 2m\,\delta_{\sigma\sigma'}\,, \qquad \overline{v}_\sigma\,v_{\sigma'} = -2m\,\delta_{\sigma\sigma'}\,. \tag{100}$$

$$\sum_\sigma u_\sigma\,\overline{u}_\sigma = \not{p} + m\,, \qquad \sum_\sigma v_\sigma\,\overline{v}_\sigma = \not{p} - m\,. \tag{101}$$

As in the case of the scalar and vector fields the general solution of the Dirac equation can be Fourier expanded in terms of creation and annihilation operators,

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int \frac{\mathrm{d}^3k}{2k^0}\,\left[c_\sigma(k)\,u_\sigma(k)\,e^{-ikx} + d_\sigma^\dagger(k)\,v_\sigma(k)\,e^{ikx}\right], \tag{102}$$

where $c_\sigma(k)$ annihilates particles and $d_\sigma(k)$ annihilates antiparticles. The Dirac field has four distinct types of wave functions corresponding to incoming and outgoing particles (here: electrons) and antiparticles (here: positrons),

$$\langle 0|\psi(x)|e^-, k\sigma\rangle = \frac{1}{(2\pi)^{3/2}}\, u_\sigma(k)\, e^{-ikx}\,, \quad \langle e^+, k\sigma|\psi(x)|0\rangle = \frac{1}{(2\pi)^{3/2}}\, v_\sigma(k)\, e^{ikx}\,,$$
$$\langle 0|\overline{\psi}(x)|e^+, k\sigma\rangle = \frac{1}{(2\pi)^{3/2}}\, \overline{v}_\sigma(k)\, e^{-ikx}\,, \quad \langle e^-, k\sigma|\overline{\psi}(x)|0\rangle = \frac{1}{(2\pi)^{3/2}}\, \overline{u}_\sigma(k)\, e^{ikx}\,. \tag{103}$$

In momentum space (omitting the $(2\pi)^{-3/2}$ factors and helicity indices), these wave functions are represented diagrammatically as:

incoming particle	$u(k)$	—>—•
incoming antiparticle	$\overline{v}(k)$	—<—•
outgoing antiparticle	$v(k)$	•—<—
outgoing particle	$\overline{u}(k)$	•—>— ,

where the arrows indicate particle charge flow and k represents the physical momentum flowing toward (for incoming) or away from (for outgoing) the interaction point.

The propagator of the Dirac field, $S(x-y)$, is defined as the solution to the inhomogeneous Dirac equation with a point source,

$$(i\gamma^\mu \partial_\mu - m)\, S(x-y) = \mathbf{1}\, \delta^4(x-y)\,. \tag{104}$$

Working in momentum space via Fourier transformation gives an algebraic equation,

$$(\not{k} - m)\, S(k) = \mathbf{1}\,. \tag{105}$$

The solution is the Dirac propagator, a 4×4 matrix,

$$i\, S(k) = \frac{i}{\not{k} - m + i\epsilon} = \frac{i\,(\not{k} + m)}{k^2 - m^2 + i\epsilon}\,, \tag{106}$$

where the $i\epsilon$ prescription ensures causality, as for the scalar and vector propagators. Diagrammatically, we represent the Dirac propagator as

$i\, S(k)$ •—>—•
k ,

where the arrow indicates particle charge flow direction, and the momentum k flows in the same direction as the arrow. This propagator appears as internal fermion lines in Feynman diagrams.

3.5 QED interactions

The form of QED interaction can be motivated from the inhomogeneous Maxwell equation sourced by the 4-current J^ν

$$\partial_\mu F^{\mu\nu} = J^\nu \,, \tag{107}$$

where current conservation requires $\partial_\nu J^\nu = 0$. The corresponding Lagrange density is given by

$$\mathcal{L}_{\text{MW}} = \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{int}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu \,. \tag{108}$$

A suitable 4-current in terms of a fermion (electron) field can be constructed as: $J^\mu \sim \overline{\psi}\gamma^\mu\psi$, which indeed transforms as a Lorentz vector. This current is conserved when ψ satisfies the Dirac equation,

$$\partial_\mu J^\mu = \bar\psi \overleftarrow{\not\partial} \psi + \bar\psi(\not\partial\psi) = (-m\bar\psi)\psi + \bar\psi(m\psi) = 0 \,. \tag{109}$$

Fixing the proportionality factor in J^μ to $-e$ (charge of electron) yields the QED Lagrangian

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{int}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar\psi\,(i\not\partial - m)\psi + e\bar\psi\,\gamma^\mu\psi A_\mu \tag{110}$$

bar potential gauge-fixing terms. Introducing the covariant derivative

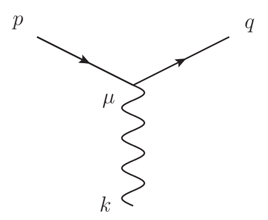
$$\partial_\mu \;\to\; D_\mu = \partial_\mu - ieA_\mu \tag{111}$$

allows to write the QED Lagrangian in the compact form

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar\psi\,(i\not D - m)\psi \,. \tag{112}$$

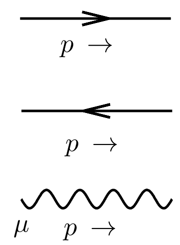
3.5.1 QED Feynman rules

The Feynman rules of QED can be summarised as



$ie\gamma^\mu$

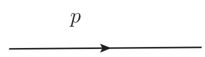
Incoming



$u(p)$

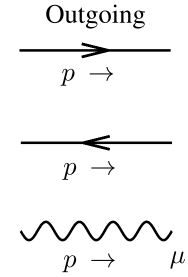
$\overline{v}(p)$

$\varepsilon_\mu(p)$



$$\frac{i(\not p + m)}{p^2 - m^2 + i\varepsilon}$$

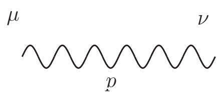
Outgoing



$\overline{u}(p)$

$v(p)$

$\varepsilon^*_\mu(p)$



$$\frac{-i}{p^2 + i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$$

Here the line-type in the diagrams represent different particles: wavy lines represent photons, and straight lines represent charged fermions (like electrons or positrons). The arrows on fermion lines indicate the direction of particle charge flow. When this arrow aligns with the momentum flow, it represents a particle (e.g., electron). When it opposes the momentum flow, it represents an antiparticle (e.g., positron). These rules, combined with diagram symmetry factors and momentum conservation at vertices, allow calculations of any QED process.

3.5.2 Example: Coulomb scattering

As an example, we consider Coulomb scattering:

$$e(p)\,\mu(k) \;\to\; e(p')\,\mu(k')\,. \tag{113}$$

There is only one (t-channel) Feynman diagram contributing to this process and using QED Feynman rules, the amplitude is

$$i\mathcal{M} = ie^2\,[\overline{u}(p')\,\gamma^\mu\,u(p)]\,\frac{g_{\mu\nu}}{q^2}\,[\overline{u}(k')\,\gamma^\nu\,u(k)]\,. \tag{114}$$

where $q = p' - p = k - k'$ is the momentum transfer via the photon propagator. In order to describe an unpolarised physical scattering process, we average over initial-state spins and sum over final-state spins,

$$\overline{|\mathcal{M}|}^2 = \frac{1}{4}\sum_{r,r',s,s'=1}^{2} |\mathcal{M}|^2 \quad . \tag{115}$$

Evaluating the fermion traces and introducing Mandelstam invariants $s = (p+k)^2$, $t = q^2$, and $u = (p-k')^2$, we get

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{t^2}[(s-m^2-M^2)^2 + (u-m^2-M^2)^2 + 2t(m^2+M^2)] \tag{116}$$

with m, M the mass of the electron and muon respectively. In the high-energy limit ($s \gg m^2, M^2$), the differential cross section simplifies to

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} \quad , \tag{117}$$

where $\alpha = e^2/(4\pi)$ is the fine structure constant, θ is the scattering angle in the center-of-mass frame, and there is no dependence on the azimuthal angle ϕ. This result shows the characteristic $1/\sin^4(\theta/2)$ behaviour of Coulomb scattering.

3.5.3 Gauge symmetry

Finally, we arrive at a crucial observation: the QED Lagrangian $\mathcal{L}_{\text{QED}}$ in eq. (110) is invariant under spacetime-dependent (= x dependent) transformations of both the matter field $\psi(x)$ and the gauge field $A_\mu(x)$,

$$\psi(x) \to \psi'(x) = e^{-ie\alpha(x)}\psi(x)\,, \qquad A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x)\,. \tag{118}$$

Here, $\alpha(x)$ is an arbitrary function of spacetime, making this a local symmetry. This locality has deep physical implications. The electromagnetic field strength tensor $F_{\mu\nu}$ remains invariant under these transformations by construction. When we transform the fields as in Eq. (118), the change in the interaction term precisely cancels the additional terms arising from the transformation of the Dirac kinetic term. We can actually see this $U(1)$ gauge symmetry as a guiding principle to construct the QED interaction term. The requirement of gauge invariance naturally leads us to introduce the covariant derivative via the replacement Eq. (111). This "minimal coupling" prescription automatically ensures gauge invariance of the theory. The resulting structure forbids certain terms in the Lagrangian. For instance, a mass term for the photon of the form $A^\mu A_\mu$ (which appears in the Proca Lagrangian for massive vector fields) would violate gauge invariance. This mathematical constraint explains the physical observation that photons must be massless. The gauge principle thus serves as both a powerful constraint on structure of the theory structure, and a guide to constructing physically meaningful interactions.

4 The Standard Model

The Standard Model of particle physics is built on several fundamental principles that constrain and guide its construction: causality, unitarity, symmetry, renormalisability, minimality / Occam's razor. In the following we construct the Lagrangian of the Standard Model based on the requirement for invariance under Lorentz transformations (space-time symmetry) and under gauge transformations (internal symmetry).

4.1 Symmetries in quantum field theory

Symmetries in quantum field theory can be classified into discrete and continuous transformations. The discrete transformations include for example parity P, time-reversal T, and charge conjugation C. Parity transformation is defined as

$$\phi'(t,\vec{x}) = P\phi(t,\vec{x}) = \phi(t,-\vec{x})\,,$$

representing spatial reflection. Time-reversal acts as

$$\phi'(t,\vec{x}) = T\phi(t,\vec{x}) = \phi(-t,\vec{x})\,,$$

reversing the direction of time evolution through an anti-unitary operator. Charge conjugation transforms as

$$\phi'(t,\vec{x}) = C\phi(t,\vec{x}) = \phi^\dagger(t,\vec{x})\,,$$

interchanging particles with their antiparticles while preserving mass and spin but reversing charge. Continuous transformations encompass space-time symmetry and internal symmetry that lead to conservation laws via Noether's theorem. Space-time translations transform scalar fields as

$$\phi'(x) = \phi(x - a)\,.$$

Internal symmetry manifests through gauge transformations

$$\phi'(x) = e^{i\alpha(x)}\phi(x)\,,$$

representing local phase transformations.
For a general quantum state, symmetry transformations act as

$$|\phi'\rangle = U|\phi\rangle\,.$$

The requirement of probability conservation demands

$$\langle\phi'|\phi'\rangle = \langle\phi|U^\dagger U|\phi\rangle = \langle\phi|\phi\rangle\,, \tag{119}$$

implying the unitarity condition $U^\dagger U = 1$. The mathematical framework for understanding such symmetry transformations is provided by group theory (see Appendix 5.3), which offers the precise language and tools needed to classify and analyse the various symmetries present in the Standard Model. The combination of C, P, and T transformations (CPT) represents a fundamental symmetry of any local, Lorentz-invariant quantum field theory, even though individual symmetries or pairs may be violated in certain interactions.

4.2 Symmetries and field content of the Standard Model

The Standard Model is a highly predictive quantum field theory describing all known elementary particles and their strong, weak and electromagnetic (EM) interactions. It is based on the principles of **global Poincaré space-time symmetry**

$$\mathbf{R}^{1,3} \times \mathrm{SL}(2, \mathbf{C}), \tag{120}$$

and **local gauge invariance** under the direct product

$$SU(3)_C \times SU(2)_L \times U(1)_Y \tag{121}$$

of compact Lie groups.

Matter fields

In the SM, matter particles—leptons and quarks—appear in three families. They are chiral spin-1/2 fermions with different charges under the gauge groups. Here, these fermions are described by left- and right-handed Weyl spinors (in the SM literature often instead an equivalent Dirac spinor nota-

Table 1: Field content of the Standard Model together with corresponding spin, representation under $SU(3)_C$, $SU(2)_L$ and hypercharge Y. Matter fields are shown in their $SU(2)$ representations.

	field			spin	$SU(3)_C$	$SU(2)_L$	Y
quarks	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\begin{pmatrix} c \\ s \end{pmatrix}_L$	$\begin{pmatrix} t \\ b \end{pmatrix}_L$	1/2	**3**	**2**	1/3
	u_R	c_R	t_R	1/2	**3**	**1**	4/3
	d_R	s_R	b_R	1/2	**3**	**1**	-2/3
leptons	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$	1/2	**1**	**2**	-1
	e_R	μ_R	τ_R	1/2	**1**	**1**	-2
Higgs-doublet		$\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}_L$		0	**1**	**2**	1
gauge bosons		G_μ^a		1	**8**	**1**	0
		W_μ^i		1	**1**	**3**	0
		B_μ		1	**1**	**1**	0

tion is used). If charged, they transform in the fundamental representation of the corresponding group. All left(right)-handed fermions are doublets (singlets) under $SU(2)_L$; the charge Y (called *hypercharge*) of $U(1)_Y$ for all fermions is determined from validity of the Gell-Mann–Nishijima relation,

$$Q = I_3 + \frac{Y}{2} \tag{122}$$

Here, Q is the electric charge and I_3 the third component of the *isospin* **I**, the generator of $SU(2)_L$. In this way $SU(2)_L \times U(1)_Y$ unifies *quantum electrodynamics* (QED) with a weak theory into the *electroweak* Standard Model. All quarks are triplets under $SU(3)_C$, the gauge group of *quantum chromodynamics* (QCD).

All matter fields are summarised together with their spin and group representations in Table 1. Here, the subscript L/R denotes left-/right-handed spinors, where $f_R = \bar{f}_L^c$ in terms of left-handed Weyl spinors. The left-handed lepton doublets are built out of left-handed electrons e_L, muons μ_L, taus τ_L and corresponding neutrinos ν_{iL}. There are no right-handed neutrinos in the SM. As already mentioned, also the quarks appear in three families: the up- u and down-type d quarks of the first generation, charm c and strange s quarks of the second generation, and in the third generation top t and bottom b quarks.

Gauge fields

As all the gauge symmetries are local symmetries, corresponding spin-1 bosonic vector fields have to be introduced. They transform in the adjoint representation of the respective gauge group. Thus, there is the octet G_μ^a of QCD, the isotriplet W_μ^i belonging to $SU(2)_L$ and the isosinglet B_μ of $U(1)_Y$. For these fields no gauge invariant mass terms can be formulated. Thus, only the G_μ^a can directly be identified with the physical gluons. The EW subgroup $SU(2)_L \times U(1)_Y$ has to be *spontaneously* broken to allow mass terms, as observed for the $W^\pm$ and Z^0 particles, in a gauge invariant way which thus does not spoil renormalizability.

Higgs fields

In the SM the spontaneous symmetry breaking (SSB) is achieved in a minimal way via the Higgs mechanism. Here, an additional spin-0 complex scalar $SU(2)_L$-doublet field Φ with hypercharge $Y = 1$ is introduced with a potential that spontaneously breaks

$$SU(2)_L \times U(1)_Y \to U(1)_{\text{EM}}\,, \tag{123}$$

and in this way leaves the electromagnetic $U(1)_{\text{EM}}$ with the photon A_μ as a gauge field as a symmetry of nature. In the following section we construct the dynamics of the Standard Model explicitly.

4.3 Construction of the Standard Model

4.3.1 Non-abelian gauge interactions

For the discussion of non-abelian gauge interactions extending the $U(1)$ symmetry of QED with the SM gauge groups the Dirac Lagrangian provides our starting point:

$$\mathcal{L}_{\text{Dirac}} = \overline{\psi}\,(i\partial\!\!\!/ - m)\,\psi\,. \tag{124}$$

We demand ψ to transform in the fundamental representation $\mathbf{N}$ and $\overline{\psi}$ in the anti-fundamental representation $\overline{\mathbf{N}}$. With explicit indices in $SU(N)$ space the Lagrangian reads

$$\mathcal{L}_{\text{Dirac}} = \overline{\psi}_i\,(i\partial\!\!\!/\,\delta_{ij} - m\delta_{ij})\,\psi_j\,. \tag{125}$$

This Lagrangian is invariant under global $SU(N)$ transformations $\psi \to U\,\psi$, but it breaks local gauge invariance when $U = U(x)$. To restore local gauge invariance, we introduce the covariant derivative through minimal coupling,

$$\partial^\mu \to D^\mu_{ij} \;=\; \partial^\mu \delta_{ij} - ig\,\mathbf{V}^\mu_{ij}\,, \tag{126}$$

where

$$\mathbf{V}^\mu_{ij}(x) = \sum_{a=1}^{N^2-1} T^a_{ij}\,V^{\mu,a}(x)\,. \tag{127}$$

The gauge field $\mathbf{V}^\mu_{ij}(x)$ decomposes into generators and vector fields

$$\mathbf{V}^\mu_{ij}(x) = \sum_{a=1}^{N^2-1} T^a_{ij} V^{\mu,a}(x)\,. \tag{128}$$

This introduces a coupling between the fermion and the vector field:

$$\mathcal{L}_{\text{Dirac}} \to \mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{int}} \quad \text{with} \quad \mathcal{L}_{\text{int}} = g\,\overline{\psi}\gamma^{\mu}\mathbf{V}_{\mu}\psi \;= g\,\overline{\psi}\gamma^{\mu}T_a\psi V^a_{\mu}\,. \tag{129}$$

The complete Lagrangian transforms covariantly under local gauge transformations:

$$\psi \to \psi' = U\,\psi \tag{130}$$

$$\mathbf{V}_{\mu} \to \mathbf{V}'_{\mu} = U\,\mathbf{V}_{\mu}\,U^{\dagger} - \frac{i}{g}(\partial_{\mu}U)U^{\dagger}\,, \tag{131}$$

and we only need to add a kinetic term for the gauge field to allow it to propagate. For this we can generalises the electromagnetic field strength tensor to the non-abelian case:

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} = \frac{i}{e}[D_{\mu}, D_{\nu}] \;\to\; \mathbf{F}_{\mu\nu} = \frac{i}{g}[\mathbf{D}_{\mu}, \mathbf{D}_{\nu}] \tag{132}$$

$$\to\; \mathbf{F}_{\mu\nu} = \partial_{\mu}\mathbf{V}_{\nu} - \partial_{\nu}\mathbf{V}_{\mu} - i\,g\,[\mathbf{V}_{\mu}, \mathbf{V}_{\nu}] = T^a F^a_{\mu\nu}\,. \tag{133}$$

With Eq. (127) we can identify

$$F^a_{\mu\nu} = \partial_{\mu}V^a_{\nu} - \partial_{\nu}V^a_{\mu} + g\,f^{abc}\,V^b_{\mu}\,V^c_{\nu}\;. \tag{134}$$

Under the gauge transformation Eq. (131) we have

$$\mathbf{F}_{\mu\nu} \to \mathbf{F}'_{\mu\nu} = U\mathbf{F}_{\mu\nu}U^{\dagger}\,. \tag{135}$$

Thus, a Lagrangian term

$$\text{Tr}(\mathbf{F}'_{\mu\nu}\mathbf{F}'^{\mu\nu}) = \text{Tr}(U\mathbf{F}_{\mu\nu}U^{\dagger}\,U\mathbf{F}^{\mu\nu}U^{\dagger}) = \text{Tr}(U^{\dagger}U\mathbf{F}_{\mu\nu}U^{\dagger}\,U\mathbf{F}^{\mu\nu}) = \text{Tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) \tag{136}$$

is gauge-invariant and yields a kinetic term for the non-abelian vector-field

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2}\,\text{Tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) = -\frac{1}{2}\,\text{Tr}(T^aT^b)F^a_{\mu\nu}\,F^{b,\mu\nu} = -\frac{1}{4}\,F^a_{\mu\nu}\,F^{a,\mu\nu} \tag{137}$$

Expanding this Lagrangian in terms of Eq. (134) yields the Yang–Mills Lagrangian with kinetic terms and self-interactions

$$\begin{aligned}\mathcal{L}_{\text{YM}} = &-\frac{1}{4}\,(\partial_{\mu}V^a_{\nu} - \partial_{\nu}V^a_{\mu})\,(\partial^{\mu}V^{a,\nu} - \partial^{\nu}V^{a,\mu})\\ &-\frac{g}{2}\,f_{abc}\,(\partial_{\mu}V^a_{\nu} - \partial_{\nu}V^a_{\mu})\,V^{b,\mu}\,V^{c,\nu}\\ &-\frac{g^2}{4}\,f_{abc}f_{ade}\,V^b_{\mu}\,V^c_{\nu}\,V^{d,\mu}\,V^{e,\nu}\,.\end{aligned} \tag{138}$$

Thus, we can construct $SU(N)$ gauge-invariant fermionic theories via

$$\mathcal{L} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{Dirac}} = -\frac{1}{2}\,\text{Tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) + \Psi\,(i\not{\mathbf{D}} - m\delta_{ij})\,\Psi \tag{139}$$

and corresponding gauge-invariant complex scalar theories via

$$\mathcal{L} = \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm cKG} = -\frac{1}{2}\,{\rm Tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) + (\mathbf{D}_\mu\Phi)^\dagger(\mathbf{D}^\mu\Phi) - m^2\,\Phi^\dagger\Phi\,. \tag{140}$$

Here we already want to note that the gauge field self-interactions arise naturally from the non-abelian structure, the relation between trilinear and quartic interactions is determined by the gauge structure, and this theory framework unifies matter-gauge couplings with gauge self-interactions through a single coupling constant g.

4.3.2 The unbroken Standard Model

Demanding local $SU(3)_C \times SU(2)_L \times U(1)_Y$ invariance introduces the gauge fields

$$\mathcal{L}_{\rm YM} = -\tfrac{1}{4}G^{a\,\mu\nu}G^a_{\mu\nu} - \tfrac{1}{4}W^{i\,\mu\nu}W^i_{\mu\nu} - \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu}\,, \tag{141}$$

where

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc}G^b_\mu G^c_\nu\,, \tag{142}$$

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^a_\mu + g_2\epsilon^{ijk}W^j_\mu W^k_\nu\,, \tag{143}$$

$$B_{\mu\nu} = \partial_\mu B^i_\nu - \partial_\nu B^a_\mu\,. \tag{144}$$

The $SU(3)_C$ fields G^a_μ entail 8 vector-bosons, the gluons, the $SU(2)_L$ fields W^i_μ entail 3 vector-bosons, W^0, W^1, W^2, and the $U(1)_Y$ field entails the B_μ vector-boson.

QCD

Demanding invariance under local $SU(3)_C$ yields QCD. The quark matter field transform in the fundamental representation of $SU(3)$, i.e. as triplets: they carry an additional colour-charge index. The corresponding gauge field (=gluons) transforms in the adjoint rep. of SU(3), i.e. as **8**.

The QCD Lagrangian for one quark-type of mass m follows from Eq. (139) and is given by:

$$\mathcal{L}_{\rm QCD} = -\tfrac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} + \overline{\psi}_i\,(i\not{D}_{ij} - m\,\delta_{ij})\,\psi_j\,, \tag{145}$$

with gluon-colour index, $a = 1\ldots 8$, quark-colour indices $i, j = 1, 2, 3$, and

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc}G^b_\mu G^c_\nu\,, \qquad D^\mu_{ij} = \partial^\mu\delta_{ij} + ig_s t^a_{ij}\,, G^{a\mu} \tag{146}$$

with f^{abc} the structure constants of SU(3). The generators of $SU(3)$ in the fundamental representation are given by 3×3 matrices with $[t^a, t^b] = if^{abc}t^c$.

We can introduce 6 identical copies of the QCD Lagrangian Eq. (145) for the different quark flavours $f = \{u, d, c, s, t, b\}$

$$\mathcal{L}_{\rm QCD} = -\tfrac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} + \sum_f \overline{\psi}^f_i\,(i\not{D}_{ij} - m^f\,\delta_{ij})\,\psi^f_j\,. \tag{147}$$

EW sector

In 1957, Wu's experiment demonstrated that weak interactions violate parity conservation, revealing a fundamental asymmetry in nature. This violation stems from the fact that weak charged currents couple exclusively to left-handed particles (and right-handed antiparticles), distinguishing the weak force from other fundamental interactions.

Within the $SU(2)_L$ gauge symmetry of the electroweak sector of the Standard Model, this chiral nature is encoded in the fermion representations: left-handed fermions transform as doublets under $SU(2)_L$, while right-handed fermions are singlets that do not participate in weak isospin transformations. This fundamental difference in how the gauge group acts on fermions of different chiralities necessitates treating left- and right-handed fields as distinct entities in the theory, with only left-handed fields carrying weak isospin charge.

Starting from a Dirac fermion ψ we define

$$\psi_L = \frac{1-\gamma_5}{2}\,\psi\,, \quad \psi_R = \frac{1+\gamma_5}{2}\,\psi\,, \tag{148}$$

with $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Left-handed components of the fermion fields are grouped into doublets

$$\psi_L^j = \begin{pmatrix} \psi_{L+}^j \\ \psi_{L-}^j \end{pmatrix} \tag{149}$$

while the right-handed fields are singlets

$$\psi_{R\pm}^j\,. \tag{150}$$

Each left- and right-handed multiplet is an eigenstate of the weak hypercharge Y such that the relation Eq. (122) is fulfilled.

We can now define the $SU(2)_L \times U(1)_Y$-invariant covariant derivatives:

$$\mathbf{D}_\mu^L = \partial_\mu + i\,g_2\,\mathbf{I}^i\,W_\mu^i + i\,g_1\,\frac{Y}{2}\,\mathbf{1}\,B_\mu \tag{151}$$

$$\mathbf{D}_\mu^R = \partial_\mu + i\,g_1\,\frac{Y}{2}\,1\,B_\mu \tag{152}$$

with the $SU(2)_L$ generators given by the Pauli matrices $\mathbf{I}^i = \frac{1}{2}\sigma^i$, and a trivial $U(1)_Y$ generator.

Fermion-gauge field interactions follow via minimal coupling as

$$\begin{aligned}\mathcal{L}_{\text{Dirac}} = \sum_{i=1}^{3}[&q_L^{i\,\dagger}\bar{\sigma}^\mu D_\mu q_L^i \;+ u_R^{i\,\dagger}\sigma^\mu D_\mu u_R^i \;+ d_R^{i\,\dagger}\sigma^\mu D_\mu d_R^i \\ &+ l_L^{i\,\dagger}\bar{\sigma}^\mu D_\mu l_L^i \;+ e_R^{i\,\dagger}\sigma^\mu D_\mu e_R^i\;]\end{aligned} \tag{153}$$

The resulting fermion-femion-vector (F-F-V) couplings are directly related to the trilinear vector-vector-vector (V-V-V), and quartic vector-vector-vector-vector (V-V-V-V) couplings which follow from the kinetic terms of the EW gauge fields. This relationship is not coincidental but rather a consequence of demanding local gauge invariance. The precise form and strength of these self-interaction vertices are thus fixed by the same structure constants of the gauge group that determine how the gauge bosons couple to fermions. Still, at the level of this theory vector-boson mass terms are not allowed by gauge invariance. Also: no fermion mass terms are allowed as $m\overline{\psi}\psi = m\left(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\right)$ would mix left- and right-handed fields. The solution for the generation of both gauge boson and fermion mass terms is given by spontaneous symmetry breaking (SSB), as explained in the following.

4.3.3 The broken Standard Model

The Standard Model acquires mass through spontaneous symmetry breaking, where the Lagrangian respects the full $SU(2)_L \times U(1)_Y$ gauge symmetry but the vacuum state does not. Indeed, the vacuum configuration spontaneously breaks the EW $SU(2)_L \times U(1)_Y$ symmetry. According to Goldstone's theorem, each broken generator of a continuous symmetry produces a massless scalar mode (Goldstone boson) in the spectrum. However, when spontaneous symmetry breaking occurs in a gauge theory, the Higgs mechanism intervenes: the would-be massless Goldstone bosons are "eaten" by the gauge fields and reappear as the longitudinal polarisation states of the gauge bosons associated with the broken generators. This process converts massless gauge bosons into massive ones while preserving the renormalisability of the theory, simultaneously explaining how the W and Z bosons acquire mass while the photon—corresponding to the unbroken electromagnetic symmetry—remains massless.

To construct the SSB in the Standard Model the unbroken Lagrangian is extended by the Higgs and Yukawa terms,

$$\mathcal{L}_{\text{SM}}^{\text{classical}} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}\,, \tag{154}$$

where

$$\mathcal{L}_{\text{Higgs}} = (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi)\,, \tag{155}$$

with a complex scalar $SU(2)$-doublet

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix}\,. \tag{156}$$

The Standard Model Higgs potential is given by ($\mu^2, \lambda > 0$)

$$V(\Phi) = -\mu^2\,\Phi^\dagger\Phi + \frac{\lambda}{4}\,(\Phi^\dagger\Phi)^2 \tag{157}$$

and has a minimum at $\Phi^\dagger\Phi = \frac{2\mu^2}{\lambda}$. We can choose the minimum to be at

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \quad \text{with} \quad v = \frac{2\mu}{\sqrt{\lambda}} \tag{158}$$

with the vacuum expectation value (vev) v. For this minimum

$$Q\langle\Phi\rangle = \left(I_3 + \frac{Y}{2}\right)\langle\Phi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} = 0\,. \tag{159}$$

Therefore this choice ensures the vacuum to be electrically neutral, i.e. it remains invariant under $U(1)_{\mathrm{EM}}$. However, this vacuum is not invariant under $SU(2)_L \times U(1)_Y$ transformations.

In order to investigate the implications of this symmetry breaking we can expand the Φ-field around the minimum:

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}\left(v + H(x) + i\chi(x)\right) \end{pmatrix}, \tag{160}$$

with the would-be Goldstone bosons $\phi^\pm, \chi$. For the fields $h, \phi^\pm, \chi$ we have $\langle h^0\rangle = \langle \chi^0\rangle = \langle\phi^\pm\rangle = 0$. Exploiting $SU(2)_L$ invariance we can eliminate $\phi^\pm, \chi$ via a suitable Gauge transformation. This gauge is called unitary gauge, with

$$\Phi(x) = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h^0(x) \end{pmatrix}. \tag{161}$$

4.3.3.1 Higgs potential

Expanding the Higgs potential Eq. (157) in unitary gauge yields

$$V = \mu^2 (h^0)^2 + \frac{\mu^2}{v}(h^0)^3 + \frac{\mu^2}{4v^2}(h^0)^4 = \frac{m_h}{2}(h^0)^2 + \ldots \tag{162}$$

The first (quadratic) term in this potential can be identified with the squared mass of the h^0 state—the infamous Higgs boson,

$$m_{h^0} = \sqrt{2}\mu = \tfrac{v\mu}{2}\,. \tag{163}$$

The remaining terms yield trilinear and quartic Higgs self-interactions.

4.3.3.2 Kinetic term

We can also expand the kinetic term $(D^\mu\Phi)^\dagger(D_\mu\Phi)$ for the Φ field around the vev,

$$\begin{aligned}(D^\mu\Phi)^\dagger(D_\mu\Phi) &= \frac{1}{2}\left(\frac{g_2}{2}v\right)^2 (W_1^2 + W_2^2) + \frac{1}{2}\left(\frac{v}{2}\right)^2 \left(W^3_\mu, B_\mu\right)\begin{pmatrix} g_2^2 & g_1 g_2 \\ g_1 g_2 & g_1^2 \end{pmatrix}\begin{pmatrix} W^{3,\mu} \\ B^\mu \end{pmatrix} + \ldots \\ &= M_W^2\, W^+_\mu W^{-\mu} + \frac{1}{2}(A_\mu, Z_\mu)\begin{pmatrix} 0 & 0 \\ 0 & M_Z^2 \end{pmatrix}\begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix} + \cdots \end{aligned} \tag{164}$$

Thus, quadratic terms in the $SU(2)_L \times U(1)_Y$ vector bosons are generated. In the second step of Eq. (164) these quadratic forms are diagonalised in order to interpret them as canonical mass terms via

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(W^1_{\mu} \mp iW^2_{\mu}), \qquad \begin{pmatrix} Z_{\mu} \\ A_{\mu} \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W^3_{\mu} \\ B_{\mu} \end{pmatrix}, \tag{165}$$

which allows us to identify

$$M_W = \frac{1}{2} g_2 v, \quad M_Z = \frac{1}{2}\sqrt{g_1^2 + g_2^2}\, v, \quad M_A = 0. \tag{166}$$

Thus, the $W^{\pm}, Z$-boson masses are determined by the gauge couplings and the vev, and are not independent parameters, while the photon A remains massless. In Eq. (165) we have introduced the electroweak mixing angle θ_W, which is determined by the gauge couplings or alternatively by the ratio of the weak boson masses,

$$\cos\theta_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} = \frac{M_W}{M_Z}. \tag{167}$$

Considering the remaining terms in the expansion of $(D^{\mu}\Phi)^{\dagger}(D_{\mu}\Phi)$ we find

$$\begin{aligned} (D^{\mu}\Phi)^{\dagger}(D_{\mu}\Phi) &= M_W^2\, W^+_{\mu} W^{-\mu} + \frac{1}{2}(A_{\mu}, Z_{\mu}) \begin{pmatrix} 0 & 0 \\ 0 & M_Z^2 \end{pmatrix} \begin{pmatrix} A^{\mu} \\ Z^{\mu} \end{pmatrix} \\ &+ \tfrac{g_2^2 v}{2} h^0 W^+ W^- + \frac{g_1^2 + g_2^2}{4} v\, h^0 ZZ \\ &+ \tfrac{g_2^2 v^2}{4} h^0 h^0 W^+ W^- + \frac{g_1^2 + g_2^2}{8} v^2\, h^0 h^0 ZZ, \end{aligned} \tag{168}$$

i.e. trilinear $h^0W^+W^-$ and h^0ZZ, and quartic $h^0h^0W^+W^-$ and h^0h^0ZZ interactions, whose strength are all fixed by the gauge couplings and the vev.

4.3.3.3 Yukawa terms

In order to generate masses for the fermions, we introduce Yukawa interactions of the form $y\overline{\psi}\Phi\psi$ between the Higgs field Φ and the charged fermion fields ψ,

$$\mathcal{L}_{\text{Yukawa}} = -\sum_{i,j=1}^{3} \left[y^d_{ij}(q^i_L)^{\dagger}\Phi d^j_R + y^u_{ij}(q^i_L)^{\dagger}\Phi^c u^j_R + y^l_{ij}(q^i_L)^{\dagger}\Phi e^j_R + \text{h.c.} \right] \tag{169}$$

where $\Phi^c \equiv i\sigma^2\Phi^*$, and the indices i, j run over the three fermion generations introducing a mixing between the different generations via the Yukawa matrices $y^d_{ij}, y^u_{ij}, y^l_{ij}$. Expanding the Higgs field Φ in Eq. (169) around the vev yields terms of the form

$$\sim -\sum_f m_f\, \overline{\psi}_f \psi_f - \sum_f \frac{m_f}{v}\, \overline{\psi}_f \psi_f\, h^0 \tag{170}$$

which corresponds to fermion mass terms and fermion-fermion-Higgs couplings. In fact, we obtain mass matrices

$$m_{ij}^{f} = \frac{v}{\sqrt{2}} y_{ij}^{f} \,, \tag{171}$$

which can be diagonalised via a bi-unitary transformation to yield

$$m_{f,i} = \frac{v}{\sqrt{2}} \sum_{k,m}^{3} U_{ik}^{f,L} y_{km}^{f} \left(U_{mi}^{f,R} \right)^{\dagger} \equiv \frac{v}{\sqrt{2}} \lambda_i^f \,, \tag{172}$$

where λ_i^f are the Yukawa coupling of a fermion f.

A crucial consequence of this diagonalisation procedure is that the unitary transformation matrices cancel out in neutral current (NC) interactions due to their unitarity properties, eliminating flavor-changing neutral currents (FCNCs) at tree level in the Standard Model—a feature consistent with experimental observations. However, in charged current (CC) interactions, the mismatch between the diagonalisation matrices for up-type and down-type quarks leaves a non-trivial unitary matrix: the Cabibbo–Kobayashi–Maskawa (CKM) matrix $\mathbf{V}_{\mathrm{CKM}}$, which parameterises quark mixing and CP violation in the weak sector (for further details, see Timothy Gershon's flavour physics course[1]).

4.3.3.4 Gauge interactions

Starting from the Fermion kinetic term Eq. (153) we can identify the interactions of the physical gauge boson fields (after the redefinitions in Eq. (165)) with fermions as

$$\begin{aligned} \mathcal{L}_{\mathrm{Dirac}} &= \cdots + J_{\mathrm{em}}^{\mu} A_{\mu} + J_{\mathrm{NC}}^{\mu} Z_{\mu} + J_{\mathrm{CC}}^{\mu} W_{\mu}^{+} + J_{\mathrm{CC}}^{\mu\,\dagger} W_{\mu}^{-} \\ &= \cdots - \frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}} \bar{e} \gamma_{\mu} e A^{\mu} + \ldots \end{aligned} \tag{173}$$

with

$$J_{\mathrm{EM}}^{\mu} = -e \sum_{f=l,q} Q_f \, \overline{\psi}_f \gamma^{\mu} \psi_f \,, \tag{174}$$

$$J_{\mathrm{NC}}^{\mu} = \frac{g_2}{2\cos\theta_W} \sum_{f=l,q} \overline{\psi}_f (v_f \gamma^{\mu} - a_f \gamma^{\mu} \gamma_5) \psi_f \,, \tag{175}$$

$$J_{\mathrm{CC}}^{\mu} = \frac{g_2}{\sqrt{2}} \left(\sum_{i=1,2,3} \overline{\nu}^i \gamma^{\mu} \frac{1-\gamma_5}{2} e^i + \sum_{i,j=1,2,3} \overline{u}^i \gamma^{\mu} \frac{1-\gamma_5}{2} V_{ij} d^j \right) \,, \tag{176}$$

where

$$v_f = I_3^f - 2Q_f \sin^2\theta_W \,, \tag{177}$$

$$a_f = I_3^f \tag{178}$$

[1] https://indico.cern.ch/event/1378334/timetable/?view=standard

and in the second line of Eq. (173) we identify the gauge coupling of the remaining unbroken $U(1)_{\mathrm{EM}}$ photon field as

$$e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} \tag{179}$$

with the gauge couplings g_1, g_2 of the $SU(2)_L \times U(1)_Y$ theory.

Finally, as in Eq. (138) the Yang-Mills term for the vector bosons $\mathcal{L}_{\mathrm{YM}}$ in Eq. (141) introduces trilinear and quartic gauge-boson self-interactions which for the physical fields read

$$\begin{aligned}
\mathcal{L}_{\mathrm{YM}} = \cdots + e &\left[(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)\, W^{-\mu} A^\nu \,+\, W_\mu^+ W_\nu^- \, F^{\mu\nu} \,+\, h.c.\right] \\
+ e \cot\theta_W &\left[(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)\, W^{-\mu} Z^\nu \,+\, W_\mu^+ W_\nu^- \, Z^{\mu\nu} \,+\, h.c.\right] \\
&- e^2/(4\sin\theta_W)\, [(W_\mu^- W_\nu^+ - W_\nu^- W_\mu^+) W_\mu^+ W_\nu^- + h.c.] \\
&- e^2/4\, (W_\mu^+ A_\nu - W_\nu^+ A_\mu)(W^{-\mu} A^\nu - W^{-\nu} A^\mu) \\
&- e^2/4\, \cot^2\theta_W\, (W_\mu^+ Z_\nu - W_\nu^+ Z_\mu)(W^{-\mu} Z^\nu - W^{-\nu} Z^\mu) \\
&+ e^2/2\, \cot\theta_W\, (W_\mu^+ A_\nu - W_\nu^+ A_\mu)(W^{-\mu} Z^\nu - W^{-\nu} Z^\mu) + h.c. \quad ,
\end{aligned} \tag{180}$$

which are all determined by the gauge couplings and the weak mixing angle.

4.4 SM input parameters

The Standard Model can be parameterised in terms of different sets of input parameters depending on whether we work in the unbroken or broken phase of the electroweak symmetry.

In the unbroken theory, the fundamental parameters are the gauge couplings g_1, g_2, g_S for the $U(1)_Y$, $SU(2)_L$, and $SU(3)_C$ groups respectively, the parameters of the Higgs potential μ and λ, and the Yukawa coupling matrices y_{ij}^f that determine fermion masses and mixing.

After electroweak symmetry breaking, it becomes more convenient to trade the original parameters for physical observables. The gauge couplings can be replaced by α_{EM}, $\sin\theta_W$, and α_S, while the Higgs sector parameters are traded for the physical masses of the electroweak bosons m_{h^0}, m_W, m_Z and fermions m_f, along with the CKM matrix elements $\mathbf{V}_{\mathrm{CKM}}$ that encode quark mixing.

Importantly, these parameters are not all independent due to tree-level relations imposed by the gauge structure, such as $\cos\theta_W = \frac{m_W}{m_Z}$. The electroweak couplings and boson masses are constrained by the gauge symmetry, and similarly the Yukawa couplings are related to the physical fermion masses. These tree-level relations receive higher-order quantum corrections that depend on all input parameters.

4.4.0.1 EW input schemes

Different electroweak input schemes are commonly employed in precision calculations, where the gauge couplings are expressed through $e = \sqrt{4\pi\alpha}$, $g_1 = e/\cos\theta_W$, and $g_2 = e/\sin\theta_W$. The three most common schemes are:

– The $\{\alpha(0), m_W, m_Z\}$-scheme, where $\alpha(0) \approx 1/137 = 0.0073\ldots$ corresponds to the Thomson limit ($Q \to 0$).

– The $\{G_\mu, m_W, m_Z\}$-scheme, which uses the precisely measured Fermi constant $G_\mu = 1.166371 \times 10^{-5}\,\text{GeV}^{-2}$ and yields

$$\alpha|_{G_\mu} = \sqrt{2/\pi}\, G_\mu m_W^2 \sin^2\theta_W \approx 1/132 = 0.0076\ldots \tag{181}$$

– The $\{\alpha(m_Z), m_W, m_Z\}$-scheme, with $\alpha(m_Z) \approx 1/128 = 0.0078\ldots$.

These EW schemes are supplemented by additional inputs such as m_{h^0} and m_f.

The G_μ-scheme is defined through the relation $\left|\frac{8}{\sqrt{2}}G_\mu\right|^2 = \left|\frac{g_2^2}{2m_W^2}\right|^2 = |\mathcal{M}|^2$, which connects the squared matrix elements for muon decay in the Fermi theory to the corresponding W-exchange matrix elements in the low-energy limit. At next-to-leading order, this scheme incorporates the radiative correction Δr (which depends on all Standard Model parameters) through

$$\alpha|_{G_\mu}/(s_W^2 m_W^2) = \sqrt{2}G_\mu/\pi = \alpha(0)(1+\Delta r)/(s_W^2 m_W^2)\,. \tag{182}$$

The quantity Δr is given by Ref. [2]

$$\begin{aligned}\Delta r = \Pi^{AA}(0) - \frac{c_W^2}{s_W^2}\left(\frac{\Sigma_T^{ZZ}(M_Z^2)}{M_Z^2} - \frac{\Sigma_T^{W}(M_W^2)}{M_W^2}\right) + \frac{\Sigma_T^{W}(0) - \Sigma_T^{W}(M_W^2)}{M_W^2} \\ + 2\frac{c_W}{s_W}\frac{\Sigma_T^{AZ}(0)}{M_Z^2} + \frac{\alpha}{4\pi s_W^2}\left(6 + \frac{7-4s_W^2}{2s_W^2}\log c_W^2\right)\,.\end{aligned} \tag{183}$$

where $\Pi^{AA}(0)$ is the photon vacuum polarisation, Σ_T^{VV} denote the transverse gauge-boson self-energies, and Σ_T^{AZ} is the photon–Z mixing self-energy. By incorporating these universal corrections into the leading-order couplings, the G_μ-scheme provides improved perturbative convergence for processes dominated by $SU(2)$ interactions at or above the electroweak scale.

5 Appendix

5.1 Classical mechanics

5.1.1 Least-action principle

Classical mechanics can be formulated as a *least-action principle*. Consider a particle moving in one dimension between time $t = t_A$ and time $t = t_B$, with its position as a function of time denoted as $x(t)$. The classical path is such that

$$\delta S[x(t)] = S[x(t) + \delta x(t)] - S[x(t)] = 0\,, \tag{184}$$

where $\delta S[x(t)]$ is the variation of the classical action S with respect to any variations in the path $x(t) \to x(t) + \delta x(t)$ with $\delta x(t_A) = \delta x(t_B) = 0$. The action is given by

$$S[x(t)] = \int_{t_A}^{t_B} L(x(t), \dot{x}(t), t)\, dt\,, \tag{185}$$

with the Lagrange function $L = L(x(t), \dot{x}(t), t)$, which in turn is given by $L = T - V$ with $T = T(x, \dot{x}, t)$ the kinetic energy, and $V = V(x, t)$ the potential energy of the particle. For a free particle in one-dimension $T = \frac{1}{2}m^2\dot{x}^2$ and $V = 0$.

The least-action principle, Eqs. (184) and (185), can be generalised to three dimensions and N particles, i.e. the Lagrange function L becomes a function of $3N$ coordinates and $3N$ velocities. More generally, the action principle holds for a system depending on M generalised coordinates $q_i(t)$ and M generalised velocities $\dot{q}_i(t)$ with $i = 1 \dots M$, where $L = L(q_i(t), \dot{q}_i(t), t)$.

5.1.2 Lagrange equations of motion

The least-action principle Eq. (184) is equivalent to the Euler–Lagrange (EL) equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0\,, \tag{186}$$

which schematically can be shown via (IBP = Integration-by-parts)

$$\delta S[x(t)] = \int_{t_A}^{t_B} \delta L(x(t), \dot{x}(t), t) dt = \int_{t_A}^{t_B} \left(\frac{\partial L}{\partial x}\delta x(t) + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}(t) \right) dt \tag{187}$$

$$\overset{\text{IBP}}{=} \int_{t_A}^{t_B} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) \delta x(t) dt + \frac{\partial L}{\partial \dot{x}}\delta x(t)\Big|_{t_A}^{t_B} \overset{!}{=} 0\,. \tag{188}$$

The least-action principle needs to hold for any variation $\delta x(t)$ from which it follows that the integrand in Eq. (188) needs to be zero, which yields Eq. (186).

For a single particle in one-dimension subject to a potential $V(x)$, i.e. $L = \frac{m^2}{2}\dot{x}^2 - V(x)$, the Euler–Lagrange equation yields

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}m\dot{x} + \frac{\partial V(x)}{\partial x} = m\ddot{x} + \frac{\partial V(x)}{\partial x} = 0\,. \tag{189}$$

This is Newton's second law: $m\ddot{x} = -\frac{\partial V(x)}{\partial x} = F(x)$.

Generalising the least-action principle to M generalised coordinates and velocities yields M Euler–Lagrange equations—one for each coordinate:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,. \tag{190}$$

5.1.3 Hamilton formalism in classical mechanics

For a system of M generalised coordinates and velocities whose dynamics is determined by the least-action principle with the Lagrange function $L = L(q_i(t), \dot{q}_i(t), t)$ we can define generalised momenta p_i via

$$p_i = \frac{\partial L}{\partial \dot{q}_i}\,. \tag{191}$$

A Legendre transformation of the Lagrange function L defines the Hamilton function $H = H(q_i, p_i, t)$,

$$H = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t)\,, \tag{192}$$

where the velocities $\dot{q}_i = \dot{q}_i(q_i, p_i)$ are found using Eq. (191). The Hamilton function is a function of the M generalised coordinates and M generalised momenta (instead of the generalised velocities). For a single particle in one dimension subject to a potential $V(x)$ we obtain $H = \frac{p^2}{2m} + V(x)$. In general, for systems where the kinetic energy T is a bilinear function of the generalised velocities we have $H = T + V$.

The Euler–Lagrange equations Eq. (190) are equivalent to the Hamilton equations of motion

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial q_i}\,, \tag{193}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\,. \tag{194}$$

5.2 Quantum mechanics

5.2.1 Quantum mechanics basics

Quantum mechanics describes the state of a physical system using vectors in a Hilbert space $\mathcal{H}$.

States and Operators

- Quantum state: The state of a system is represented by a normalised vector $|\psi\rangle \in \mathcal{H}$, called a ket. The corresponding dual vector is a bra $\langle\psi|$. The normalisation condition ensures probability conservation: $\langle\psi|\psi\rangle = 1$.
- Inner product: The probability amplitude of a state $|\psi\rangle$ being found in state $|\phi\rangle$ is given by the inner product $\langle\phi|\psi\rangle$. The probability of measuring state $|\phi\rangle$ is then $|\langle\phi|\psi\rangle|^2$.
- Observables: Physical observables (like position, momentum, energy) are represented by Hermitian operators $\hat{A}$, which satisfy $\hat{A} = \hat{A}^\dagger$. This Hermiticity guarantees that their eigenvalues (the possible measurement results) are real numbers.

Measurement and eigenvalues

A measurement of an observable $\hat{A}$ yields one of its real eigenvalues a_n.

- Eigenvalue equation: The possible measured values are determined by the equation:

  $$\hat{A}|a_n\rangle = a_n|a_n\rangle\,,$$

  where $|a_n\rangle$ are the eigenstates of $\hat{A}$. After a measurement, the state of the system collapses to the corresponding eigenstate $|a_n\rangle$.
- Expectation value: The average value of repeated measurements of $\hat{A}$ on a system in state $|\psi\rangle$ is the expectation value:

  $$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle\,.$$

Commutation relations and uncertainty

The order in which two operators are applied matters. Their relationship is quantified by the commutator:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}\,.$$

– Canonical Commutation Relation: For position ($\hat{x}$) and momentum ($\hat{p}$), the fundamental relation is:

$$[\hat{x}, \hat{p}] = i\hbar\hat{I}\,.$$

Since this commutator is non-zero, these observables are incompatible, meaning they cannot be simultaneously measured with arbitrary precision.

5.2.2 *Simple harmonic oscillator in quantum mechanics*

The one-dimensional simple harmonic oscillator (SHO) is a crucial system in quantum mechanics, serving as a basis for understanding quantum field theory.

Hamiltonian

The Hamiltonian operator $\hat{H}$ for the one-dimensional SHO is defined as:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\,,$$

where $\hat{x}$ is the position operator, $\hat{p}$ is the momentum operator, m is the mass, and ω is the classical angular frequency of the oscillator.

Ladder operators

The algebraic solution is conveniently achieved by introducing ladder (or creation and annihilation) operators, $\hat{a}^\dagger$ and $\hat{a}$, respectively. We first introduce a dimensionless coordinate X and momentum $\hat{P}$:

$$\hat{X} = \sqrt{\frac{m\omega}{\hbar}}\hat{x}, \quad \hat{P} = \frac{1}{\sqrt{m\omega\hbar}}\hat{p}\,.$$

The Hamiltonian can then be rewritten in terms of $\hat{P}$ and $\hat{X}$:

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{P}^2 + \hat{X}^2)\,.$$

The annihilation and creation operators are defined as:

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P})\,.$$

Their fundamental commutation relation is derived from the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$:

$$[\hat{a}, \hat{a}^\dagger] = 1\,.$$

Energy eigenvalues

The Hamiltonian can be expressed in terms of the ladder operators:

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\,.$$

The operator $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator, which counts the number of energy quanta. The eigenstates $|n\rangle$ of the Hamiltonian satisfy $\hat{H}|n\rangle = E_n|n\rangle$. The energy eigenvalues are found to be quantized:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad \text{for } n = 0, 1, 2, \ldots \quad .$$

The ground state energy $E_0 = \frac{1}{2}\hbar\omega$ is the non-zero zero-point energy of the system.

The action of the ladder operators on the eigenstates is:

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle\,.$$

5.2.3 Quantum pictures

The time evolution of a quantum system can be described using three main pictures: the Schrödinger picture, the Heisenberg picture, and the Interaction picture. These pictures are physically equivalent but assign time dependence differently between the state vectors and the operators.

Schrödinger picture

In the Schrödinger picture, the time evolution is carried by the states, while the operators are generally time-independent.

- States $|\phi_S(t)\rangle$ are time-dependent, evolving according to the Schrödinger equation:

$$i\frac{\partial}{\partial t}|\phi_S(t)\rangle = \hat{H}_S|\phi_S(t)\rangle\,. \tag{195}$$

  The formal solution for the state evolution is:

$$|\phi_S(t)\rangle = e^{-i\hat{H}_S(t-t_0)}\,|\phi_S(t_0)\rangle = U(t, t_0)|\phi_S(t_0)\rangle\,, \tag{196}$$

  where $\hat{H}_S$ is the time-independent Hamiltonian operator and $U(t, t_0) = e^{-i\hat{H}_S(t-t_0)}$ is the time-evolution operator (assuming a time-independent Hamiltonian $\hat{H}_S$).
- Operators $\hat{A}_S$ are time-independent (unless they have an explicit time dependence, i.e., $\frac{\partial \hat{A}_S}{\partial t} \neq 0$).

Heisenberg picture

In the Heisenberg picture, the time evolution is transferred entirely to the operators, making the states time-independent.

– States $|\phi_H\rangle$ are time-independent, fixed at a reference time t_0:

$$|\phi_H\rangle = |\phi_S(t_0)\rangle \,. \tag{197}$$

– Operators $\hat{A}_H(t)$ are time-dependent, related to the Schrödinger operators $\hat{A}_S$ by:

$$\hat{A}_H(t) = U^\dagger(t,t_0)\,\hat{A}_S\,U(t,t_0)\,, \tag{198}$$

where $U(t,t_0)$ is the time-evolution operator from the Schrödinger picture. Their time evolution is governed by the Heisenberg equation of motion:

$$i\frac{d\hat{A}_H(t)}{dt} = [\hat{A}_H(t), \hat{H}_H] + i\frac{\partial \hat{A}_H}{\partial t}\,, \tag{199}$$

where $\hat{H}_H$ is the Hamiltonian in the Heisenberg picture ($\hat{H}_H = \hat{H}_S$).

Interaction picture (Dirac picture)

The Interaction picture is useful when the Hamiltonian can be separated into a free (exactly solvable) part $\hat{H}_0$ and an interacting part $\hat{H}_I$:

$$\hat{H} = \hat{H}_0 \,+\, \hat{H}_I \,. \tag{200}$$

In this picture, the evolution due to $\hat{H}_0$ is assigned to the operators, and the evolution due to $\hat{H}_I$ is assigned to the state vectors.

– States $|\phi_I(t)\rangle$ are time-dependent, absorbing the evolution due to $\hat{H}_I$:

$$|\phi_I(t)\rangle = e^{i\hat{H}_0(t-t_0)}\,|\phi_S(t)\rangle = \hat{U}_0^\dagger(t,t_0)|\phi_S(t)\rangle \tag{201}$$

$$= \hat{U}_I(t,t_0)|\phi_I(t_0)\rangle\,, \tag{202}$$

where $|\phi_I(t_0)\rangle = |\phi_S(t_0)\rangle$ and $\hat{U}_0(t,t_0) = e^{-i\hat{H}_0(t-t_0)}$. The states evolve according to the interaction Schrödinger equation:

$$i\frac{\partial}{\partial t}|\phi_I(t)\rangle = \hat{H}_I(t)|\phi_I(t)\rangle\,, \tag{203}$$

where $\hat{H}_I(t)$ is the interaction Hamiltonian in the Interaction picture

$$\hat{H}_I(t) = \hat{U}_0^\dagger(t,t_0)\hat{H}_I\hat{U}_0(t,t_0)\,. \tag{204}$$

The formal solution for the state evolution is given by the time-ordered exponential:

$$\hat{U}_I(t,t_0) = \hat{T}\,e^{-i\int_{t_0}^{t}\hat{H}_I(t')dt'}\,. \tag{205}$$

– Operators $\hat{A}_I(t)$ are time-dependent, evolving with the free Hamiltonian $\hat{H}_0$:

$$\hat{A}_I(t) = \hat{U}_0^\dagger(t, t_0)\, \hat{A}_S\, \hat{U}_0(t, t_0)\,. \tag{206}$$

Their time evolution is governed by an equation similar to the Heisenberg equation, but with $\hat{H}_0$:

$$i\frac{d\hat{A}_I(t)}{dt} = [\hat{A}_I(t), \hat{H}_0(t)] + i\frac{\partial \hat{A}_I}{\partial t}\,. \tag{207}$$

5.3 Group theory basics

5.3.1 General group theory

The mathematical language used to describe symmetry transformations is group theory.

5.3.1.1 Definition of a group

A Group G is a set of elements $\{g_1, g_2, g_3, \dots\}$ with a binary operation "$\cdot$" (often called the group product), such that if $g_1, g_2 \in G$, then $g_3 = g_1 \cdot g_2$ is also an element of the group (G is closed under the operation). The operation must satisfy the following three axioms:

– Associativity: $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ for all $g_1, g_2, g_3 \in G$.
– Identity (Unity) Element: A unique element $e \in G$ exists such that $e \cdot g = g \cdot e = g$ for all $g \in G$.
– Inverse Element: For every element $g \in G$, there exists a unique inverse element $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

Examples of groups include: integers with addition (identity $e = 0$), rotations in 2D about a fixed axis, modular arithmetic, and the set of all invertible $N \times N$ matrices under matrix multiplication.

5.3.1.2 Types of groups

– Abelian group: A group where the group operation is commutative:

$$g_i \cdot g_j = g_j \cdot g_i \quad \text{for all } g_i, g_j \in G$$

Example: The $U(1)$ gauge group of quantum electrodynamics (QED).

– Non-abelian group (non-commutative): A group where the group operation is not commutative for all elements:

$$g_i \cdot g_j \neq g_j \cdot g_i \quad \text{for at least two } g_i, g_j \in G$$

Example: The $SU(3)$ gauge group of quantum chromodynamics (QCD).

– Lie group: A group that is also a smooth manifold, allowing for the use of differential calculus (e.g., continuous groups like rotations). Any element $U(x)$ close to the identity can be obtained via exponentiation of its generators:

$$U(\alpha) = 1 + i\alpha^a T^a + \cdots = e^{i\alpha^a T^a}\,. \tag{208}$$

Here T^a are the *generators* of the group, and α^a are continuous real parameters. The $U(1)$ group is a special case where there is only one generator, which can be seen as $T^a = 1$.

5.3.2 *SU(N) group theory*

Here we summarise several relevant concepts in $SU(N)$ group theory. For further details see e.g. Ref. [3].

5.3.2.1 *Definition of $SU(N)$*

The special unitary group of degree N, denoted $SU(N)$, is the Lie group of all $N \times N$ unitary matrices with a determinant of 1. For every group element $U \in SU(N)$, the defining properties are:

$$UU^\dagger = U^\dagger U = \mathbf{1}_N, \quad \det(U) = 1\,, \tag{209}$$

where $\mathbf{1}_N$ is the $N \times N$ identity matrix.

5.3.2.2 *Generators and Lie algebra*

Every group element U in $SU(N)$ can be obtained from the identity via exponentiation:

$$U = e^{i\alpha^a \frac{1}{2}\lambda^a} \quad , \tag{210}$$

where λ^a are the generators of the group (often normalized to $T^a = \frac{1}{2}\lambda^a$), and α^a are the real parameters. The generators form the Lie algebra $SU(N)$, which is defined by the commutation relation:

$$[\lambda^a, \lambda^b] = i f_{abc} \lambda^c\,, \tag{211}$$

where f_{abc} are the real and completely antisymmetric structure constants of the group.

The generators λ^a have the following properties, derived from the $SU(N)$ conditions:

- Hermiticity (from $U^{-1} = U^\dagger$): $(\lambda^a)^\dagger = \lambda^a$.
- Tracelessness (from $\det(U) = 1$, since $\det(e^A) = e^{\text{Tr}(A)}$):

$$\det(U) = 1 = \det(e^{i\alpha^a \frac{1}{2}\lambda^a}) = e^{i\alpha^a \frac{1}{2}\text{Tr}(\lambda^a)} \quad \Longrightarrow \quad \text{Tr}(\lambda^a) = 0\,.$$

We fix the normalisation of the generators by the condition:

$$\text{Tr}(\lambda^a \lambda^b) = T_R\, \delta^{ab}\,, \tag{212}$$

where T_R is the index of the representation, which determines the overall normalisation. For the fundamental representation, $T_R = 1/2$ is often chosen for $SU(N)$.

The special unitary group $SU(N)$ has $\mathbf{N^2 - 1}$ independent generators. Thus, for the Standard Model gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, there are $(3^2-1)+(2^2-1)+(1^2-1+1) = 8+3+1 = 12$ generators (and thus 12 corresponding gauge bosons).

5.3.2.3 Representations

Definition An N-dimensional matrix representation $D(G)$ of a group G is a map $D : G \to GL(N)$, where $GL(N)$ is the general linear group of degree N, i.e. the set of all $N \times N$ invertible matrices.

This map must preserve the group structure:

1. $D(e) = \mathbf{1}_N$, where e is the identify element of G.
2. $D(g_1)D(g_2) = D(g_1 g_2)$ for all $g_1, g_2 \in G$.

Fundamental (N) and anti-fundamental ($\bar{\mathbf{N}}$) representations The map $D(U) = U$ for all $U \in SU(N)$ defines the most obvious representation, called the fundamental representation.

– Objects ψ that transform under this representation are N-component column vectors and are denoted as $\mathbf{N}$ (e.g., $SU(3)$ quarks are $\mathbf{3}$, $SU(2)$ doublets are $\mathbf{2}$). They transform as:

$$\psi \to U\psi \,. \tag{213}$$

– The anti-fundamental representation ($\bar{\mathbf{N}}$) corresponds to the transformation of the conjugate state $\psi^\dagger$ (an N-component row vector):

$$\psi^\dagger \to \psi^\dagger U^\dagger \,. \tag{214}$$

Singlet (1) representation The trivial singlet representation is defined by $D(U) = \mathbf{1}$ (the scalar number 1, or the 1×1 identity matrix), which corresponds to the transformation:

$$\phi \to \phi \,. \tag{215}$$

Objects that transform as a singlet are denoted as $\mathbf{1}$ (they are invariant under the transformation). Given ψ transforms in the $\mathbf{N}$ and $\psi^\dagger$ in the $\bar{\mathbf{N}}$ representation, the combination $\psi^\dagger\psi$ transforms as a $\mathbf{1}$.

Adjoint ($\mathbf{N^2 - 1}$) representation Another natural representation is the adjoint representation. A complex tensor Ψ_{ij} is said to transform under the adjoint representation when:

$$\Psi \to U\Psi U^\dagger \,. \tag{216}$$

Since an $N \times N$ matrix Ψ can be expanded in terms of the $N^2 - 1$ generators, these objects are denoted as $\mathbf{N^2} - \mathbf{1}$ (e.g., the gauge bosons of $SU(3)$ transform as an $\mathbf{8}$). The gauge bosons of $SU(N)$ always transform in this representation.

In the following we list the explicit generators λ_a of $SU(3)$, where $a = 1 \ldots 8$. These can be seen as a generalisation of the three Pauli matrices $\sigma_{1\ldots3}$ of $SU(2)$. The λ_a matrices, known as the Gell-Mann matrices, are given by:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad .$$

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