Proceedings of the CAS–CERN Accelerator School: Free Electron Lasers and Energy Recovery Linacs, Hamburg, Germany, 31 May–10 June 2016, edited by R. Bailey, CERN Yellow Reports: School Proceedings, Vol. 1/2018, CERN-2018-001-SP (CERN, Geneva, 2018)

Pendulum Equations and Low Gain Regime

S. Reiche
Paul Scherrer Institute, Villigen, Switzerland

Abstract
This paper introduces the theoretical framework for the motion of an electron in the periodic field of an undulator and wiggler. It is a continuation of the previous article on motion in the undulator but includes the interaction with a co-propagating radiation field. The longitudinal motion of each electron is similar to a pendulum and, in resonance with the co-propagating field, energy can be exchanged between the electron beam and the radiation field in the small signal low-gain regime of the free-electron laser.

Keywords
Free-electron laser; theory; low-gain regime; resonance condition.

1 Pendulum equations and low-gain regime

In this section, the interaction of electrons with a radiation field while they move through the undulator is analysed. The approach to this problem is similar to that in the previous paper, except that an additional term in the Hamilton function describes the vector potential of the radiation field [1]. If the emission of radiation is stronger than the absorption, the electrons are losing energy, on average, and the radiation field is amplified. As long as this amplification is small, the radiation field amplitude can be assumed to be constant in the Hamilton function for deriving the equations of motion. The limitations of this model of a 'low-gain' free-electron laser are given at the end of this section. A more self-consistent model of a free-electron laser can be found in the next section, including Maxwell's equation for the radiation field description. Nevertheless, a discussion of the low-gain free-electron laser is fruitful, because it shows the basic principle of how a free-electron laser works, using rather simple equations.

The interaction of charged particles with a radiation field shows two major aspects. The first is the change of the particle momentum and energy. The Hamilton equations of motion are the mathematical representation of this process. The method for solving these equations is very similar to the treatment in the previous paper, but differs in that the electron energy is no longer constant, owing to the electric field components of the radiation field.

The second aspect is the change of the radiation field itself. The fast transverse oscillation of the electrons is a source of radiation. For relativistic particles, this radiation points mainly in the forward direction of the electron beam motion. If the radiation wavelength is shorter than the electron bunch length, the electrons emit at almost all phases and the radiation adds up incoherently. The emission is strongly enhanced if the longitudinal beam profile is modulated on the scale of the radiation wavelength.

Under special conditions, both processes—the change of the particle energy and the emission of radiation—are the source of a collective bunching of the electrons on a resonant frequency, and the radiation field is strongly amplified. The next section analyses this instability—the working principle of the 'high gain' free-electron laser. In contrast with the high gain free-electron laser, the low-gain free-electron laser provides amplification without the necessity of a strong modulation in the electron density.

The discussion begins with the assumption on the radiation field. If a radiation field propagates along the undulator together with the electron bunch, the interaction time is maximized. The electric field components are lying in the transverse xy-plane; thus, only a transverse motion, along or against the field orientation, changes the electron energy. Owing to the symmetry of the magnetic field, the

https://doi.org/10.23730/CYRSP-2018-001.355

radiation emitted in a planar undulator is linearly polarized, while it is circularly polarized for the case of a helical undulator. In this section, the case of a planar undulator is considered. Most of the results are similar or identical for a helical undulator and only important differences are mentioned in the text.

The electric field component of the radiation field

$$\vec{E} = \vec{E}_0 \cos(k(z-ct)+\Psi)\,, \tag{1}$$

is defined by its amplitude $\vec{E}_0$, its wavenumber $k = 2\pi/\lambda$ or wavelength λ, and its initial phase Ψ at the undulator entrance.

The magnetic field component is perpendicular to $\vec{E}$ as well as to the unit vector in the direction of propagation, which mainly coincides with $\vec{e}_z$. Compared with the strong undulator field, the magnetic field of the radiation field is negligible and can be ignored in further discussion. The amplitude $\vec{E}_0$ and the phase Ψ depend on z, because of diffraction. The dependence becomes negligibly small if the transverse extension of the radiation wavefront is much larger than the radiation wavelength.

The change of the electron energy is caused only by the electric field components, which, depending on the radiation phase, accelerate or decelerate the electron with

$$\dot{\gamma} = \frac{e\vec{E}\cdot\vec{\beta}}{mc}\,. \tag{2}$$

Only the parallel components of $\vec{E}$ and $\vec{\beta}$ contribute to Eq. (2). For the planar undulator, they are pointing in the x-direction resulting in a linear polarization of the radiation field.

To obtain the transverse velocities, $\vec{\beta}$ the vector potential $\vec{A}_r$ of the electromagnetic wave must be added to the Hamiltonian,

$$H = \sqrt{(\vec{P}-e\vec{A})^2c^2 + m^2c^4} + e\Phi\,. \tag{3}$$

From the potential,

$$\vec{A}_r = \frac{1}{ck}\sin(k(z-ct)+\Psi)\begin{pmatrix} E_0 \\ 0 \\ 0 \end{pmatrix}\,. \tag{4}$$

the electric field is derived by the time derivative $\vec{E} = -\partial\vec{A}_r/\partial t$. Here, the Lorentz gauge is chosen, which enables the scalar potential to be omitted in the derivation of the electric field.

For an assumed pulse length $L \gg \lambda$, the dependence of the amplitude $\vec{E}_0$, as well as the phase Ψ, on time is negligible and A_r is a valid vector potential for the radiation field of Eq. (1). Inserting the vector potential of the radiation field and the undulator field into the Hamilton function, the transverse velocities are

$$\dot{x} = -\frac{\sqrt{2}cK}{\gamma}\sin(k_{\mathrm{U}}z) - \frac{\sqrt{2}cK_r}{\gamma}\sin(k(z-ct)+\Psi) + \dot{X}\,, \tag{5}$$

$$\dot{y} = \dot{Y}\,. \tag{6}$$

The dimensionless radiation amplitude,

$$K_r = \frac{e\hat{E}}{mc^2k}\,, \tag{7}$$

is defined in an analogous way as the undulator parameter K. The motivation to use the r.m.s. value $\hat{E}$ of the electric field is the same. Most results will be identical for the helical undulator. The velocity terms $\dot{X}$ and $\dot{Y}$ of the betatron oscillation are the same as before.

For the sake of simplicity, any transverse variation of the radiation field is excluded. A radiation field with a finite transverse extension is more difficult to analyse. For small transverse momenta, the longitudinal velocity is approximately

$$\beta_z \approx 1 - \frac{1+K^2+K_r^2}{2\gamma^2} - \frac{\beta_{\mathrm{R}}^2}{2} + \frac{K^2}{2\gamma^2}\cos(2k_{\mathrm{U}}z) + \frac{K_r^2}{2\gamma^2}\cos\left(2k(z-ct)+2\Psi\right) - \frac{2KK_r}{\gamma^2}\sin(k_{\mathrm{U}}z)\sin(k(z-ct)+\Psi)\,. \tag{8}$$

This expression is very similar to that for the electron motion in a pure magnetic field of an undulator, except for three additional terms. The electric field forces an additional transverse oscillation with the frequency of the electromagnetic wave. As for the undulator field, the longitudinal velocity is slowed down and modulated with an oscillation of twice the frequency of the radiation field. It will be shown later that the longitudinal modulation by the radiation field is much smaller than the longitudinal modulation by the undulator field and can be neglected.

The cross term, $\propto KK_r$, can be split into two independent oscillations. If one of them has a small frequency, it can significantly change the longitudinal velocity β_z on a time-scale different to the dominant oscillating term, $\propto K^2$. The explicit calculation of this term is postponed until β_z is further discussed (see Eq. (15)).

Combining all constant or slowly varying terms to β_0, the integration of Eq. (8) up to first order yields

$$z = \beta_0 ct + \frac{K^2}{4\gamma^2 k_{\mathrm{U}}\beta_0}\sin(2k_{\mathrm{U}}\beta_0 ct)\,. \tag{9}$$

With the given expression of the transverse velocities $\dot{x}$ and $\dot{y}$, Eq. (2) can be evaluated. Most of the cross terms between E_x and β_x are quickly oscillating. Over many undulator periods, the net change of the electron energy is negligible. The only possible term that might be constant is the product of $\cos(k(z-ct)+\Psi)$ and $\sin(k_{\mathrm{U}}z)$, similar to the term in Eq. (8). This term is split into two independent oscillations, with the phases $(k \pm k_{\mathrm{U}})z - kct + \Psi$. If one of the phases remains almost constant, the energy change is accumulated over many periods.

With an average longitudinal velocity of $c\beta_0$, the phase relation between the electron and the radiation field remains unchanged if the condition

$$\beta_0 = \frac{k}{k \pm k_{\mathrm{U}}} \tag{10}$$

is fulfilled. As shown later, the interaction between the electron beam and the radiation field needs to add up resonantly over many undulator periods to result in a significant change of the electron energy or the radiation amplitude and phase. This implies that, for a given beam energy and undulator wavelength, the radiation wavelength of the radiation field is well defined according to Eq. (10). The case of the '$-$' sign is excluded because it would demand an electron velocity faster than the speed of light to keep the electrons in phase with the radiation field for any time. The restriction to a well-defined resonant radiation wavelength is called the resonance approximation.

In the limit of a weak electric field ($K_r \to 0$) and a small beam emittance, the resonant radiation wavelength is

$$\lambda_0 = \frac{\lambda_{\mathrm{U}}}{2\gamma^2}(1+K^2)\,. \tag{11}$$

This important equation is also valid for a planar and a helical undulator. A transverse betatron motion and a stronger radiation field shift the resonance condition slightly towards longer wavelengths. If Eq. (11) is exactly fulfilled, the energy change is constant over many undulator periods, pushing the electron off resonance.

So far, the longitudinal oscillation of the electron has not been taken into account. As mentioned previously, it induces higher harmonics in the motion of the electrons.

Inserting Eqs. (1) and (5) into Eq. (2) yields the resonant term

$$\dot{\gamma} = -\frac{2ckKK_r}{\gamma}\cos(k(z-ct)+\Psi)\sin(k_\mathrm{U}z)\,. \tag{12}$$

Note that the choice of the radiation wavenumber k is free and does not need to agree with the resonant wavenumber $k_0 = 2\pi/\lambda_0$, defined by the undulator properties and the particle energy. To evaluate Eq. (12), the sine and cosine function are replaced by complex exponential functions. The oscillating part of the longitudinal motion (Eq. (9)) can be expanded into a series of Bessel functions [2] by the identity

$$\mathrm{e}^{\mathrm{i}a\sin b} = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}mb} J_m(a)\,.$$

The result is a sum of exponential functions with frequencies $[(k+(2m+1)k_\mathrm{U})\beta_0 - k]c$. Beside the ground mode with $m = 0$, some terms are resonant at different wavelengths. The frequencies of these are the odd harmonics of the resonant frequency $\omega_0 = ck_0$.

Collecting all terms belonging to one mode, Eq. (12) becomes

$$\begin{aligned}\dot{\gamma} = -\frac{2ckKK_r}{\gamma}\frac{1}{4i}\Bigg[&\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\Psi}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}2mk_\mathrm{U}\beta_0 ct}(J_m(\chi)-J_{m+1}(\chi)) \\ &- \mathrm{e}^{-\mathrm{i}\theta-\mathrm{i}\Psi}\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}2mk_\mathrm{U}\beta_0 ct}(J_m(\chi)-J_{m+1}(\chi))\Bigg]\,,\end{aligned} \tag{13}$$

with $\chi = kK^2/4\gamma^2 k_\mathrm{U}$ and the so-called ponderomotive phase,

$$\theta = (k+k_\mathrm{U})z - ckt\,. \tag{14}$$

For completeness, it is noted that a transverse non-uniform radiation field also couples the particle motion to the even harmonics of ω_0 [3,4]. If the radiation field is expanded into a Taylor series around the electron position of the betatron oscillation ($x = X + x_0$),

$$\vec{E}(x) = \vec{E}(X) + \left.\frac{\mathrm{d}\vec{E}}{\mathrm{d}x}\right|_X x_0\,,$$

the factor $x_0\dot{x_0}$ is proportional to $\sin(2k_\mathrm{U}z)$ in Eq. (2). Using the same calculation as for Eq. (13), the complex exponential functions have the arguments $[(k+(2m+2)k_\mathrm{U})\beta_0 - k]ct$, being resonant at all even harmonics. The additional pre-exponential factor is $(K/2K_r\gamma k_\mathrm{U}\beta_0)\mathrm{d}K_r/\mathrm{d}x$.

The postponed calculation of the cross term $\sin(k(z-ct)+\Psi)\sin(k_\mathrm{U}z)$ in Eq. (8) is performed in a very similar way. If the phase Ψ is temporarily replaced by $\tilde{\Psi} = \Psi - \pi/2$ to convert the sine function into a cosine function, the expansion into Bessel functions yields

$$\begin{aligned}\beta_z = 1 &- \frac{1+K^2+K_r^2}{2\gamma^2} - \frac{\beta_\mathrm{R}^2}{2} \\ &+ \frac{KK_r}{2\gamma^2}\Bigg[\mathrm{e}^{\mathrm{i}\theta+\mathrm{i}\Psi}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}2mk_\mathrm{U}\beta_0 ct}(J_m(\chi)-J_{m+1}(\chi)) \\ &\qquad + \mathrm{e}^{-\mathrm{i}\theta-\mathrm{i}\Psi}\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}2mk_\mathrm{U}\beta_0 ct}(J_m(\chi)-J_{m+1}(\chi))\Bigg]\,.\end{aligned} \tag{15}$$

The resonant frequencies are well separated, such that only one resonance frequency is of importance for a given radiation field. The coupling factor is smaller for higher modes. Thus, the interaction is the strongest for the fundamental mode [5], which is the only mode considered in the following discussion.

Where the free-electron laser operates at the fundamental frequency, the non-linear terms in the free-electron laser equations will induce an enhanced bunching in the longitudinal position at higher harmonics. This bunching increases more quickly than operating on the higher frequency itself.

For a helical undulator, the amplification of higher modes is much smaller because the dominant longitudinal oscillation, which is why coupling to higher harmonics is strongly suppressed. At the fundamental frequency, the synchronization of the phase front of the ponderomotive wave and the electrons is almost perfect, while it is reduced by a factor $(J_0(\chi) - J_1(\chi))$ for the planar undulator.

Compared with the fast-changing position of the electron, $z \approx \beta_0 ct$, the ponderomotive phase $\theta = (k + k_\mathrm{U})z - ckt$ of the electron is almost constant. It is convenient to change to a moving coordinate system, which is synchronized with the ponderomotive wave. With a simple canonical transformation [6], which keeps the energy unchanged, the equation of motion for the new variable θ becomes $\dot{\theta} = (k + k_\mathrm{U})c\beta_z - kc$. Replacing β_z with Eq. (15), the differential equations for the low-gain free-electron laser are obtained:

$$\dot{\theta} = ck_\mathrm{U} - \omega \frac{1 + K^2 + K_r^2 - 2f_\mathrm{c} K K_r \cos(\theta + \Psi)}{2\gamma^2} - \omega \frac{\beta_\mathrm{R}^2}{2}\,, \tag{16}$$

and

$$\dot{\gamma} = -\omega f_\mathrm{c} \frac{K K_r}{\gamma} \sin(\theta + \Psi)\,. \tag{17}$$

With the definition of the coupling factor

$$f_\mathrm{c} = \begin{cases} J_0(\chi) - J_1(\chi) & \text{planar undulator,} \\ 1 & \text{helical undulator.} \end{cases} \tag{18}$$

and $\chi = kK^2/4\gamma^2 k_\mathrm{U} = K^2/2(1+K^2)$ for the fundamental resonant wavelength, the free-electron laser equations are valid for both types of undulator.

Another way to derive the differential equations is the rigorous canonical and Legendre transformation of the Hamilton function of Eq. (3) [7]. The new Hamilton function, depending on the canonical variable and momentum θ and γ, respectively, is

$$H = ck_\mathrm{U}\gamma + \omega \frac{1 + \gamma^2\beta_\mathrm{R}^2 + K^2 + K_r^2 - 2f_\mathrm{c} K K_r \cos(\theta + \Psi)}{2\gamma}\,. \tag{19}$$

The independent variable is time t. As long as the electric field and the transverse momenta do not change significantly, they can be kept constant in the Hamiltonian. This is the basic assumption of the low-gain free-electron laser. The limitation of this model will be given at the end of this section.

In the limit of a low-gain free-electron laser, the Hamilton function is regarded as independent of t and therefore a constant of motion. Setting the Hamiltonian to $H = 2ck_\mathrm{U}(1 + \alpha)\gamma_\mathrm{R}$ with $\gamma_\mathrm{R}^2 = k(1 + \gamma^2\beta_\mathrm{R}^2 + K^2 + K_r^2)/2k_\mathrm{U}$, the particle energy γ depends on θ as

$$\gamma = \gamma_\mathrm{R}(1 + \alpha) \pm \sqrt{\gamma_\mathrm{R}^2 \alpha(2 + \alpha) + \frac{k f_\mathrm{c} K K_r}{k_\mathrm{U}} \cos(\theta + \Psi)}\,. \tag{20}$$

The lowest boundary of α is $\alpha > -1$, to avoid unphysical negative values of the energy. Other limitations are given by the square root in Eq. (20). Two values of α are of particular interest, for the lowest possible value of the Hamilton function and for an existing solution of γ for all phases θ, respectively.

The smallest value of α is found if the cosine function in the argument of the square root is unity. At $\theta = -\Psi$ the root becomes real for

$$\alpha_0 = -1 + \sqrt{1 - \frac{k f_c K K_r}{k_U \gamma_R^2}} . \tag{21}$$

Inserting α_0 in Eq. (20) yields the corresponding energy

$$\gamma_0 = \sqrt{\gamma_R^2 - \frac{k f_c K K_r}{k_U}} .$$

The position $(-\Psi, \gamma_0)$ in the longitudinal phase space is a stable fixed point, where the electron remains in its position. For any small deviation, the differential equations, Eqs. (16) and (17), can be linearized and combined to produce a second-order differential equation of $\Delta\theta = \theta + \Psi$ with

$$\Delta\theta'' + \Omega^2 \Delta\theta = 0 \tag{22}$$

and $\Omega = \sqrt{2 f_c k k_U K K_r}/\gamma_0$.

This equation is solved by any sine or cosine function with the frequency Ω. The motion in the longitudinal phase space is bound. This is typical for a stable fixed point. For a larger amplitude of $\Delta\theta$, non-linear terms are no longer negligible and the frequency depends on the initial condition of the electron.

Solutions of γ for all phases θ are found for α larger than

$$\alpha_1 = -1 + \sqrt{1 + \frac{k f_c K K_r}{k_U \gamma_R^2}} . \tag{23}$$

The trajectory in phase space is not closed and the electrons have either energy above or below γ_R. A transition is not possible.

The phase space surface for $H = 2ck_U(1 + \alpha_1)\gamma_R$ is called a separatrix. It separates the bound and unbound motion. Any electron within the separatrix is trapped in the ponderomotive wave and oscillates around $-\Psi$. Referring to the acceleration of charged particle in RF cavities, this enclosed area of the separatrix is often called a 'bucket' [8]. The width of the bucket is given by the properties of the undulator and the radiation field and is $\Delta\gamma = \sqrt{8 k f_c K K_r / k_U}$. Electrons outside the separatrix are moving unlimited in θ either faster or slower than the ponderomotive wave.

Figure 1 shows several phase space trajectories for different initial conditions calculated by Eq. (20). Within the bucket, the electrons are moving clockwise; above zero, they move towards larger phases ($\dot{\theta} > 0$), while below zero, they move towards smaller phases. This implies that an electron injected at the ponderomotive phase $0 < \theta + \Psi < \pi$ loses energy. If the undulator length is shorter than the period length of the phase space oscillation $2\pi/\Omega$, the electron will mainly remain in this phase region. Owing to energy conservation, the radiation field has been amplified. This can be generalized for the whole electron bunch. As long as the initial distribution in the longitudinal phase space changes to a final distribution of a mean energy smaller than the initial energy, the gain of the free-electron laser is positive.

Unfortunately, the most obvious way by injecting all electrons at $0 < \theta + \Psi < \pi$ is not realizable. The radiation wavelength depends on the energy as γ^{-2} (Eq. (11)) and is much smaller than a typical bunch length of about 1 mm. The initial ponderomotive phases of the electrons are almost uniformly distributed over 2π. Owing to the finite number of electrons over one radiation wavelength, a small modulation of the electron beam remains. This spontaneous emission provides the initial radiation field for self-amplified spontaneous emission free-electron lasers, discussed in the end of this paper.

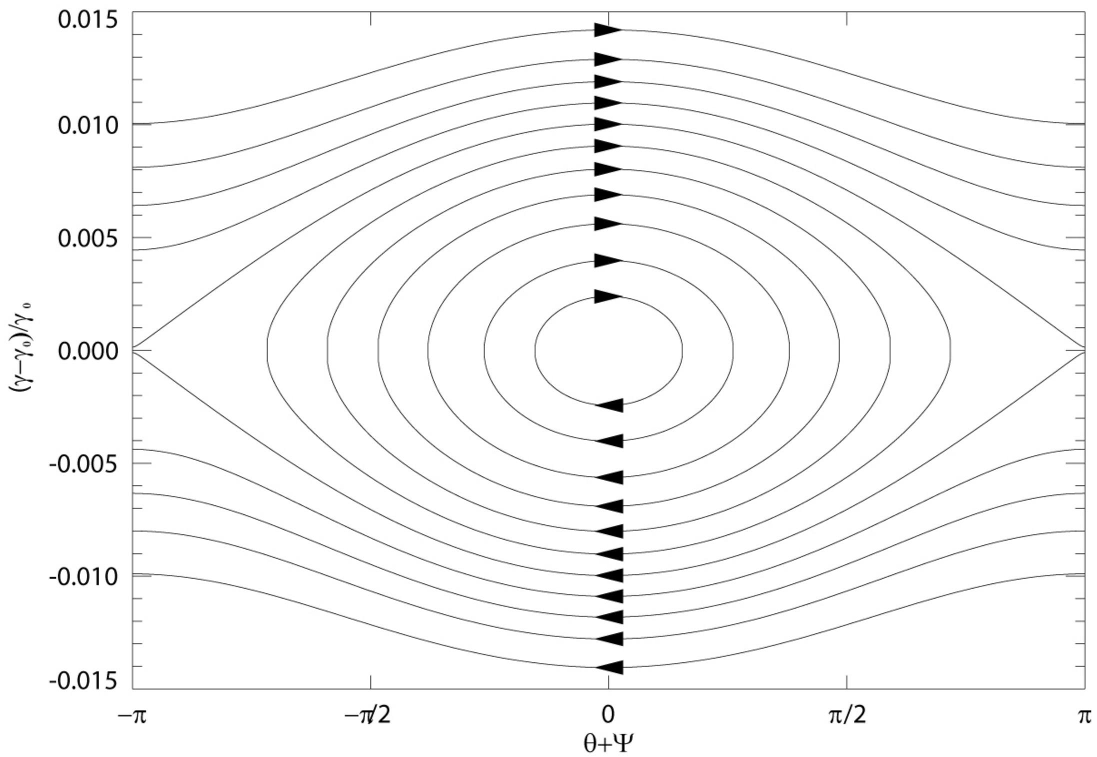

Fig. 1: Electron trajectories in the longitudinal phase space for different initial settings

With an RF photo gun driving the injector for a free-electron laser, relative energy spreads smaller than 1% can be achieved. This width is typically smaller than the width of the bucket and fills it unevenly. For a large energy spread, the bucket is filled almost homogeneously. Any motion of the electrons within the homogeneously filled bucket would not change the mean energy, because the phase space density remains constant, according to Liouville's theorem [9].

Operating as a free-electron laser amplifier, the injection at resonance energy γ_R would not provide any gain at all. For the unmodulated beam, the energy change of one electron is always compensated by a complementary electron, which moves on the same trajectory but which has a phase difference of $2(\theta+\Psi)$. The only visible effect is the increase of the energy spread, because electrons at $-\pi < \theta+\Psi < 0$ gain energy while the complementary electrons at $0 < \theta + \Psi < \pi$ lose energy.

If the injection is off-resonance ($\gamma \neq \gamma_R$) the change of the phase space distribution is no longer symmetrical. For $\gamma > \gamma_R$, electrons at $-\pi < \theta + \Psi < 0$ tend to change the phase rather than the energy, while the opposite is true for the remaining electrons. Averaging over all electrons, the electron beam loses energy and the radiation field is amplified. For injection below the resonant energy, the electron beam will gain energy and the radiation field is weakened.

The gain dependence on the injection energy can be calculated by perturbation theory [10]. The rather long but straightforward calculation is not presented here. The dependence on the injection energy is

$$G \propto -\frac{\mathrm{d}}{\mathrm{d}(\eta/2)} \frac{\sin^2(\eta/2)}{(\eta/2)^2}\,, \tag{24}$$

where $\eta = 4\pi N_U(\gamma - \gamma_R)/\gamma_R$ and N_U is the total number of undulator periods. The gain of the low-gain amplifier is related to the spectrum of the spontaneous undulator radiation [11, 12] by taking the frequency derivative of the intensity spectrum of the spontaneous radiation. This relation is known as Madey's theorem [13].

For the free-electron laser oscillator, as well as for the self-amplified spontaneous emission free-electron laser, the situation is slightly different, because both types of free-electron laser start from spontaneous emission with a broad bandwidth in the frequency domain. As a consequence, the electron beam is always in resonance with the frequency of the largest gain. Using an energy dependence as the argument of Eq. (24) is no longer meaningful and the energy dependence must be replaced with the frequency dependence. The results are made similar by redefining η as $\eta = 2\pi N_{\mathrm{U}}(\omega - \omega_0)/\omega_0$, with ω_0 as the resonant frequency.

In this low-gain approximation, the interaction between the electrons is almost negligible and the gain is proportional to the total number of electrons. In this one-dimensional model of a low-gain free-electron laser, a higher beam current means a larger amplification of the radiation field. Unless the gain does not exceed several per cent of the usage of the free-electron laser, Eqs. (16) and (17) are justified. Otherwise, the assumption of a constant field K_r is no longer valid. The radiation power can increase, which might change the strength of the electron interaction. To cover this aspect, a self-consistent set of free-electron laser equations must be derived.

References

[1] S. Reiche, Motion in the Undulator, these proceedings.

[2] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Function* (Dover Publication, New York, 1972).

[3] D. Noelle, Ph.D. thesis, DELTA Int. Rep. 91–31.

[4] M.J. Schmitt and C.J. Elliott, *Phys. Rev.* **A41** (1990) 3853. https://doi.org/10.1103/PhysRevA.41.3853

[5] J.B. Murphy *et al.*, *Opt. Commun.* **53** (1985) 197. https://doi.org/10.1016/0030-4018(85)90331-1

[6] F. Kuypers, *Klassische Mechanik* (VCH, Weinheim, 1990).

[7] T.M. Tran and J.S. Wurtele, LRP 392/90, CRPP report, Lausanne, Switzerland, 1990.

[8] D.A. Edwards and M.J. Syphers, *An Introduction to the Physics of High Energy Accelerators* (John Wiley and Sons, New York, 1993). https://doi.org/10.1002/9783527617272

[9] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980).

[10] C. Pellegrini, in *Free Electron Lasers*, Eds. S. Martellucci and A.N. Chester (Plenum Press, New York, 1983).

[11] B.M. Kincaid, *J. Appl. Phys.* **48** (1977) 2684. https://doi.org/10.1063/1.324138

[12] W.B. Colson, *IEEE J. Quantum Electron.* **QE17** (1981) 1417. https://doi.org/10.1109/JQE.1981.1071273

[13] J.M.J. Madey, *Il Nuovo Cimen.* **50B** (1979) 64. https://doi.org/10.1007/BF02737622