Proceedings of the CAS–CERN Accelerator School: Free Electron Lasers and Energy Recovery Linacs, Hamburg, Germany, 31 May–10 June 2016, edited by R. Bailey, CERN Yellow Reports: School Proceedings, Vol. 1/2018, CERN-2018-001-SP (CERN, Geneva, 2018)

Classical Electrodynamics and Applications to Particle Accelerators

W. Herr
CERN, Geneva, Switzerland

Abstract
Classical electrodynamic theory is required theoretical training for physicists and engineers working with particle accelerators. Basic and empirical phenomena are reviewed and lead to Maxwell's equations, which form the framework for any calculations involving electromagnetic fields. Some necessary mathematical background is included in the appendices so the reader can follow the work and the conventions used in this text. Plane waves in vacuum and in different media, radio frequency cavities, and propagation in a waveguide are presented.

Keywords
Electrodynamics; Maxwell's equations; electromagnetic waves; cavities; polarization.

1 Introduction and motivation

Together with classical mechanics, quantum theory, and thermodynamics, the theory of classical electrodynamics forms the framework for the introduction to theoretical physics. Classical electrodynamics can be applied where the length scale does not required a treatment on the quantum level. Although both electricity and magnetism can exert forces on other objects, they were for a long time treated as distinct effects. The empirical laws were unified in a single theory by Maxwell and culminated in the prediction of electromagnetic waves. Although very successful in describing most phenomena, it is not possible to reconcile the theory with the concepts of classical mechanics. This was solved by the introduction of special relativity by Einstein, in studying the effects of moving charges. This reformulation not only explained the origin of such effects as the Lorentz force, but also showed that electricity and magnetism are two different aspects of the same underlying physics. Since in accelerator physics we are mainly concerned about moving charges, the topic of special relativity is treated in a separate lecture at this school [1].

This paper touches on many different areas of electromagnetic theory, with a strong focus on applications to accelerator physics [2]. It covers the field of electrostatics and the equations of Gauss and Poisson, magnetic fields generated by linear and circular currents, and electromagnetic effects in vacuum and different media, and leads to Maxwell's equations [3–5].

Electromagnetic waves and their behaviour at boundaries and in waveguides and cavity resonators are treated in some detail. Because of their importance, such phenomena as polarization and propagation in perfect and resistive conductors are presented.

The paper is intended as a recapitulation for physicists and engineers and mathematical subtleties are avoided where it is acceptable.

This paper cannot replace a full course on electromagnetic theory. This is, in particular, true for students less familiar with this subject. Although they will not be able to understand everything in this lecture, it is attempted to provide access to the core material and the direct features relevant for accelerator physics.

The background required is a knowledge of calculus and differential equations; some more advanced concepts, such as vector calculus are summarized in the appendices.

https://doi.org/10.23730/CYRSP-2018-001.1

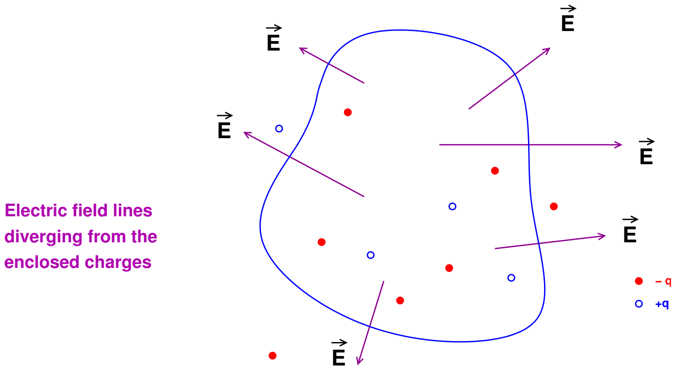

Fig. 1: Charges enclosed within a closed surface

2 Electrostatics

Electrostatics deals with phenomena related to time-independent charges. It was found empirically that charged bodies exert a force on each other, attracting in the case of unlike charges or repelling for charges of equal sign. This is described by the introduction of electric fields and the Coulomb force acting on the particles. Charges are the origin of electric fields, which form a vector field.

2.1 Gauss's theorem

The fields of a distribution of charges add to form the overall field and the latter can be computed when the distribution of charge is known. This treatment is based on the mathematical framework worked out by Gauss and others and is summarized in Gauss's theorem. Gauss's theorem in its simplest form is illustrated in Fig. 1.

We assume a surface S enclosing a volume V, within which are charges: $q_1, q \ldots$, producing electromagnetic fields $\vec{E}$ originating from the charges and passing through the surface (Fig. 1).

Summing the normal component of the fields passing through the surface, we obtain the flux ϕ:

$$\phi = \int_S \vec{E} \cdot \vec{n} \mathrm{d}A = \sum_i \frac{q_i}{\epsilon_0} = \frac{Q}{\epsilon_0}\,, \tag{1}$$

where $\vec{n}$ is the normal unit vector and $\vec{E}$ the electric field at an area element $\mathrm{d}A$ of the surface. The surface integral of $\vec{E}$ equals the total charges Q inside the enclosed volume.

This holds for *any* arbitrary (closed) surface S and is:

- independent of *how* the particles are distributed inside the volume;
- independent of *whether* the particles are moving or at rest;
- independent of *whether* the particles are in vacuum or material.

Using Gauss's formula (see Appendix B), we can formulate the theorem as:

$$\begin{aligned} &\textstyle\int_S \vec{E} \cdot \mathrm{d}\vec{A} = \int_V \frac{\rho}{\epsilon_0} \cdot \mathrm{d}V = \frac{Q}{\epsilon_0} = \phi_E \\ &\underbrace{\int_S \vec{E} \cdot \mathrm{d}\vec{A} = \int_V \nabla\vec{E} \cdot \mathrm{d}V = \int_V \mathsf{div}\vec{E} \cdot \mathrm{d}V}_{\text{Gauss's formula}} \qquad \text{(relates surface and volume integrals)} \end{aligned} \tag{2}$$

It follows that from Eq. (2):

$$\mathsf{div}\ \vec{E} = \nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}\,, \tag{3}$$

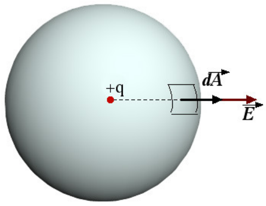

Fig. 2: Flux through a surface element created by a single point charge

which is Maxwell's *first* equation.

As a physical picture, the divergence 'measures' outward flux ϕ_E of the field. The simplest possible example is the flux from a single charge q, shown in Fig. 2. A charge q generates a field $\vec{E}$ according to (Coulombs law):

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}\,. \tag{4}$$

It is enclosed by a sphere and obviously $\vec{E} = \mathrm{const.}$ on a sphere (area, $4\pi \cdot r^2$):

$$\int\int_{\text{sphere}} \vec{E}\cdot \mathrm{d}\vec{A} = \frac{q}{4\pi\epsilon_0}\int\int_{\text{sphere}}\frac{\mathrm{d}A}{r^2} = \frac{q}{\epsilon_0}\,. \tag{5}$$

The surface integral through the sphere A equals the charge inside the sphere (for any radius of the sphere), consistent with Eq. (1).

2.2 Electrostatic potential and Poisson's equation

We can derive the field $\vec{E}$ from a scalar electrostatic potential $\phi(x, y, z)$, i.e.,

$$\vec{E} = -\mathsf{grad}\ \phi = -\nabla\phi = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)\,, \tag{6}$$

then we have

$$\nabla\vec{E} = -\nabla^2\phi = -\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = \frac{\rho(x, y, z)}{\epsilon_0}$$

This is *Poisson's equation.*

Once we can compute ϕ for a given distribution of the charge density ρ, we can derive the fields. As an example, the simplest possible charge distribution is an isolated point charge with the potential:

$$\phi(r) = \frac{q}{4\pi\epsilon_0 r}$$

$$\vec{E} = -\nabla\phi(r) = \frac{q}{4\pi\epsilon_0}\cdot\frac{\vec{r}}{r^3}$$

As a realistic case, we assume a distribution $\rho(x, y, z)$ that is Gaussian in all three dimensions:

$$\rho(x, y, z) = \frac{Q}{\sigma_x\sigma_y\sigma_z\sqrt{2\pi}^3}\exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\right)$$

($\sigma_x, \sigma_y, \sigma_z$ are the r.m.s. sizes).

The potential $\phi(x, y, z, \sigma_x, \sigma_y, \sigma_z)$ becomes (see e.g., Ref. [6]):

$$\phi(x, y, z, \sigma_x, \sigma_y, \sigma_z) = \frac{Q}{4\pi\epsilon_0}\int_0^\infty \frac{\exp\left(-\frac{x^2}{2\sigma_x^2+t} - \frac{y^2}{2\sigma_y^2+t} - \frac{z^2}{2\sigma_z^2+t}\right)}{\sqrt{(2\sigma_x^2+t)(2\sigma_y^2+t)(2\sigma_z^2+t)}}\mathrm{d}t\,. \tag{7}$$

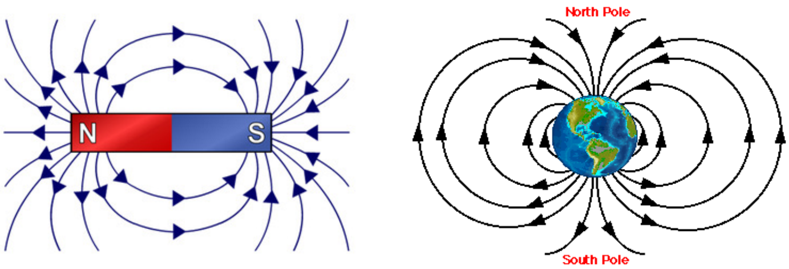

Fig. 3: Field lines between magnetic dipoles

In many realistic cases, the charge distribution shows a strong symmetry, Then we can rewrite the Poisson equation and obtain some very important formulae in practice.

Poisson's equation in polar co-ordinates (r, φ):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} = -\frac{\rho}{\epsilon_0}\,; \tag{8}$$

Poisson's equation in cylindrical co-ordinates (r, φ, z):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{\rho}{\epsilon_0}\,; \tag{9}$$

Poisson's equation in spherical co-ordinates (r, θ, φ):

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = -\frac{\rho}{\epsilon_0}\,. \tag{10}$$

Examples for solutions of these equations are found in Ref. [3].

3 Magnetostatics

In the treatment of magnetostatic phenomena, we follow the strategy developed for electrostatics. The striking difference is the absence of magnetic charges, i.e., magnetic 'charges' occur only in combination with opposite 'charges', i.e., in the form of a magnetic dipole.

The field lines between magnetic poles for a magnet and the Earth's magnetic field are shown in Fig. 3.

We start with some basic definitions and properties.

- Magnetic field lines always run from north to south.
- They are described as vector fields by the magnetic flux density $\vec{B}$.
- All field lines are closed lines from the north to the south pole.

3.1 Gauss's theorem

We follow the same procedure as for electrostatic charges and enclose a magnetic dipole within a closed surface Fig. 4.

From very simple considerations, it is rather obvious that field lines passing outwards through the surface also return through this surface, i.e., the overall flux is zero. This is formally described by Gauss's second theorem, for magnetic fields:

$$\int_S \vec{B}\mathrm{d}\vec{A} = \int_V \nabla\vec{B}\mathrm{d}V = 0\,. \tag{11}$$

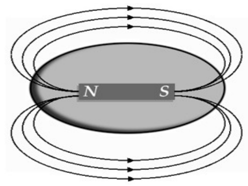

Fig. 4: Closed surface around magnetic dipole

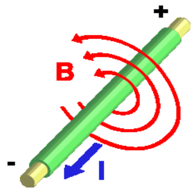

Fig. 5: Static electric current inducing an encircling (curling) magnetic field

This leads to Maxwell's second equation:

$$\nabla \vec{B} = 0\,. \tag{12}$$

The physical significance of this equation is that magnetic charges (monopoles) do not exist (although Maxwell's equations could easily be modified if necessary).

3.2 Ampère's law

Static currents produce a magnetic field described by Ampère's law (Fig. 5).

Assuming a current density $\vec{j}$, we can compute the magnetic field:

$$\mathrm{curl}\vec{B} = \nabla \times \vec{B} = \mu_0 \vec{j}\,, \tag{13}$$

or in integral form, where the current density becomes the current I,

$$\int\int_A \nabla \times \vec{B}\mathrm{d}\vec{A} = \int\int_A \mu_0 \vec{j}\mathrm{d}\vec{A} = \mu_0 \vec{I}\,. \tag{14}$$

For a *static electric current* I in a *single wire* (Fig. 6), we get the Biot–Savart law (we have used Stoke's theorem and the area of a circle, $A = r^2 \cdot \pi$):

$$\begin{aligned} \vec{B} &= \frac{\mu_0}{4\pi} \oint \vec{I} \cdot \frac{\vec{r} \times \mathrm{d}\vec{s}}{r^3} \\ \vec{B} &= \frac{\mu_0}{2\pi} \frac{\vec{I}}{r} \end{aligned} \tag{15}$$

4 Time-varying electromagnetic fields

Extending the subject of static electric and magnetic fields opens a large range of new phenomena. Furthermore it shows a close connection between electricity and magnetism.

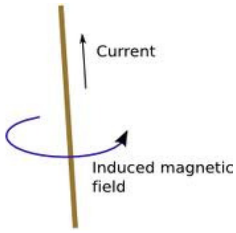

Fig. 6: Induced magnetic fields by static current

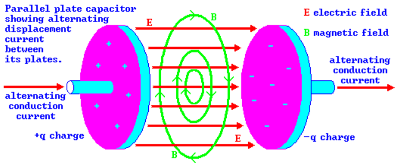

Fig. 7: Maxwell's displacement current, e.g., charging capacitor

4.1 Maxwell and time-varying electric fields

We need to address the question of whether we need an electric current to produce magnetic fields. This was addressed by Maxwell, which led him to the introduction of the displacement current $\vec{j}_{\mathrm{d}}$.

We define this displacement current by:

$$\vec{I}_d = \frac{\mathrm{d}q}{\mathrm{d}t} = \epsilon_0 \cdot \frac{\mathrm{d}\phi}{\mathrm{d}t} = \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int\int_{\mathrm{area}} \vec{E} \cdot \mathrm{d}\vec{A}\,. \tag{16}$$

It must be understood that this is not a current from *moving charges* but from *time-varying electric fields*.

The displacement current I_{d} produces magnetic fields, just like 'actual currents' do. An example for a displacement current is a charging capacitor (Fig. 7).

Time-varying electric fields induce magnetic fields (using the current density $\vec{j}_{\mathrm{d}}$). We can formulate this as:

$$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\,. \tag{17}$$

The bottom line of this result is that magnetic fields $\vec{B}$ can be generated in two ways:

$$\nabla \times \vec{B} = \mu_0 \vec{j} \tag{18}$$

are the magnetic fields produced by an electric current (Ampère), while

$$\nabla \times \vec{B} = \mu_0 \vec{j}_{\mathrm{d}} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{19}$$

are the magnetic fields produced by a changing electric field (Maxwell).

Putting them together we obtain Maxwell's third law:

$$\nabla \times \vec{B} = \mu_0 (\vec{j} + \vec{j}_{\mathrm{d}}) = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\,. \tag{20}$$

Using Stoke's formula, this can be rewritten in integral form:

$$\underbrace{\oint_C \vec{B} \cdot \mathrm{d}\vec{s} = \int_A \nabla \times \vec{B} \cdot \mathrm{d}\vec{A}}_{\text{Stoke's formula}} = \int_A \left(\mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot \mathrm{d}\vec{A}\,. \tag{21}$$

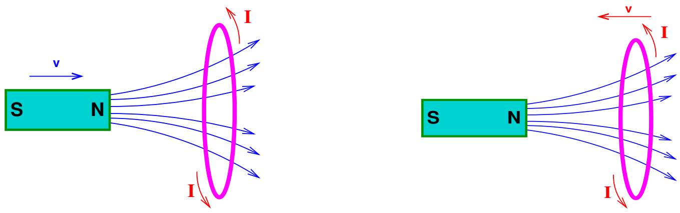

Fig. 8: Electromotive force (EMF) produced by changing magnetic flux Ω

4.2 Faraday's law and varying magnetic fields

Assuming a conducting coil in a static magnetic field $\vec{B}$ (Fig. 8). The area enclosed by the coil should be A. Changing the magnetic flux Ω through the area A produces an electromotive force (EMF) in the coil resulting in a current I:

$$\mathsf{flux} = \Omega = \int_A \vec{B}\mathrm{d}\vec{A}, \qquad \mathsf{EMF} = \oint_C \vec{E}\cdot\mathrm{d}\vec{s}, \tag{22}$$

$$-\frac{\partial\Omega}{\partial t} = -\frac{\partial}{\partial t}\underbrace{\int_A \vec{B}\mathrm{d}\vec{A}}_{\mathrm{flux}\Omega} = \oint_C \vec{E}\cdot\mathrm{d}\vec{s}, \tag{23}$$

$$-\frac{\partial\Omega}{\partial t} = -\int_A \frac{\partial}{\partial t}\vec{B}\mathrm{d}\vec{A} = \oint_C \vec{E}\cdot\mathrm{d}\vec{s}. \tag{24}$$

The magnetic flux can be changed by:

– moving the magnet relative to the conducting coil;
– moving the coil relative to the magnet.

4.3 Ampère and Maxwell's law

In a more general form, this can be written using Stoke's formula, which relates line integrals and surface integrals. It is then rewritten as:

$$-\int_A \frac{\partial\vec{B}}{\partial t}\mathrm{d}\vec{A} = \underbrace{\int_A \nabla\times\vec{E}\mathrm{d}\vec{A} = \oint_C \vec{E}\cdot\mathrm{d}\vec{s},}_{\text{Stoke's formula}} \tag{25}$$

and we arrive at the well-known formulation:

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}. \tag{26}$$

A changing magnetic field through *any* closed area induces electric fields in the (arbitrary) boundary. A sketch demonstrating Stoke's formula is shown in Fig. 9. This formulation is known as the Maxwell–Faraday law.

5 Maxwell's equations

The empirical concepts and experimental findings can be put together in a set of differential equations, usually referred to as Maxwell's equations.

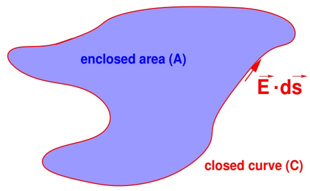

Fig. 9: Stoke's formula

5.1 Maxwell's equations in vacuum

Putting together Eqs. (3), (12), (20), and (26), Maxwell's equations in vacuum (so-called microscopic equations) read:

$$\begin{aligned}
\nabla \vec{E} &= \frac{\rho}{\epsilon_0} = -\Delta\phi\,, && \text{(I)}\\
\nabla \vec{B} &= 0\,, && \text{(II)}\\
\nabla \times \vec{E} &= -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}\,, && \text{(III)}\\
\nabla \times \vec{B} &= \mu_0 \left(\vec{j} + \epsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t}\right)\,, && \text{(IV)}
\end{aligned} \tag{27}$$

or, written in integral form (using Gauss's and Stoke's theorems):

$$\begin{aligned}
&\int_A \vec{E}\cdot \mathrm{d}\vec{A} = \frac{Q}{\epsilon_0}\,,\\
&\int_A \vec{B}\cdot \mathrm{d}\vec{A} = 0\,,\\
&\oint_C \vec{E}\cdot \mathrm{d}\vec{s} = -\int_A \left(\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}\right)\cdot \mathrm{d}\vec{A}\,,\\
&\oint_C \vec{B}\cdot \mathrm{d}\vec{s} = \mu_0 \int_A \left(\vec{j} + \epsilon_0\frac{\mathrm{d}\vec{E}}{\mathrm{d}t}\right)\cdot \mathrm{d}\vec{A}\,.
\end{aligned} \tag{28}$$

5.2 Maxwell's equations in material

In material, we have to modify the electromagnetic fields $\vec{E}$ and $\vec{H}$ and relate those to the magnetic induction $\vec{B}$ and electric displacement $\vec{D}$. In vacuum, we had:

$$\vec{D} = \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_0 \cdot \vec{H}\,. \tag{29}$$

In a material, the relations read:

$$\vec{D} = \epsilon_\mathrm{r}\cdot\epsilon_0\cdot\vec{E} = \epsilon_0\vec{E} + \vec{P}\,, \tag{30}$$

$$\vec{B} = \mu_\mathrm{r}\cdot\mu_0\cdot\vec{H} = \mu_0\vec{H} + \vec{M}\,. \tag{31}$$

The origin of these additional contributions are $\vec{P}$olarization and $\vec{M}$agnetization in material.

We can summarize:

$$\epsilon_\mathrm{r}(\vec{E},\vec{r},\omega) \Rightarrow \epsilon_\mathrm{r}\ \text{is}\ \textit{relative}\ \text{permittivity} \approx [1\text{–}10^5]\,;$$

$$\mu_{\mathrm{r}}(\vec{H}, \vec{r}, \omega) \Rightarrow \mu_{\mathrm{r}} \text{ is } \textit{relative} \text{ permeability} \approx [0\text{–}10^6]\,.$$

If $\vec{D}$ and $\vec{B}$ do not depend on the fields $\vec{E}$ and $\vec{H}$, they are linear; if they do not depend on the direction ($\vec{r}$) or frequency (ω), they are isotropic and non-dispersive.

The so-called macroscopic Maxwell's equations become:

$$\begin{aligned} \nabla \vec{D} &= \rho\,, \\ \nabla \vec{B} &= 0\,, \\ \nabla \times \vec{E} &= -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}\,, \\ \nabla \times \vec{H} &= \vec{j} + \frac{\mathrm{d}\vec{D}}{\mathrm{d}t}\,. \end{aligned} \tag{32}$$

6 Electromagnetic potentials

It was shown that electric fields can be derived from a scalar potential ϕ:

$$\vec{E} = -\vec{\nabla}\phi\,. \tag{33}$$

Since div $\vec{B} = 0$, we can write $\vec{B}$ using a vector potential $\vec{A}$:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \mathsf{curl}\vec{A}\,, \tag{34}$$

combining Maxwell (I) and Maxwell (III):

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\,. \tag{35}$$

Fields can be written as derivatives of scalar and vector potentials $\phi(x, y, z)$ and $\vec{A}(x, y, z)$. Knowledge of the potentials allows computation of the fields.

6.1 Gauge invariance

The equations for the potentials can be directly derived from Maxwell's equations:

$$\Delta\phi = \frac{1}{c}\frac{\partial(\nabla \cdot \vec{A})}{\partial t} = -4\pi\rho\,, \tag{36}$$

and

$$\Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial^2 t} - \nabla\left(\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = \frac{4\pi}{c}\vec{j}\,. \tag{37}$$

We have two coupled differential equations for the potentials, which may be difficult to solve for general charge densities and current densities. We shall try to decouple these equations using a particular property of the potentials. While the absolute values of the electric and magnetic fields can be measured, the absolute values of the potentials are not defined. The electromagnetic potentials are merely auxiliary 'constructions', although very important ones, in particular, for the relativistic formulation of the electromagnetic theory.

Without going into the details, the theory should be invariant under a change of scale ('gauge'). The most commonly used is the Lorentz gauge, which yields a condition between the potentials:

$$\vec{A}_{\mathrm{g}} = \vec{A} + \nabla f\,, \tag{38}$$

$$\phi_{\rm g} = \phi + \frac{1}{c}\frac{\partial f}{\partial t}\,, \tag{39}$$

$$\frac{1}{c}\frac{\partial \phi_{\rm g}}{\partial t} + \nabla \vec{A}_{\rm g} = 0\,, \tag{40}$$

where f is an arbitrary function of position and time. These equations lead to the same (measurable) fields and do therefore satisfy Maxwell's equations. This 'gauge' transformation decouples Eq. (36) and Eq. (37) and leads to:

$$\Delta\phi(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi(\vec{r},t) = -4\pi\cdot\rho(\vec{r},t)\,, \tag{41}$$

$$\Delta\vec{A}(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}(\vec{r},t) = -\frac{4\pi}{c}\cdot\vec{j}(\vec{r},t)\,. \tag{42}$$

We observe two consequences: first, the equations for the potentials are decoupled and depend only on the charge density and current density. Second, without charges or current, the equations have the form of a wave equation. The relevance becomes clear later, in particular, when Maxwell's equations are written in a relativistically invariant form [1].

Another very useful gauge is the Coulomb gauge:

$$\nabla\cdot\vec{A} = 0\,. \tag{43}$$

This leads us to a particularly simple expression for the electric potential:

$$\Delta\phi(\vec{r},t) = -4\pi\rho(\vec{r},t)\,. \tag{44}$$

The name 'Coulomb gauge' becomes obvious.

A formal solution can now be written as:

$$\phi = \int \frac{\rho(\vec{r'},t)}{\mid \vec{r}-\vec{r'}\mid}{\rm d}V\,. \tag{45}$$

6.2 Example: Coulomb potential

Equation (45) can immediately be applied to compute the Coulomb potential of a static charge q:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{|\vec{r}-\vec{r}_q|}\,, \tag{46}$$

where $\vec{r}$ is the observation point and $\vec{r}_q$ the location of the charge.

7 Powering and self-induction

There are also induction effects in a single coil. A varying current (e.g., in a transformer or power line) produces a varying magnetic field inside itself and the flux of this field is continually changing, leading to a self-induced electromotive force (Fig.10). This electromotive force (EMF) is acting on any current when it is building up a magnetic field or when the field is changing in any way. This effect is called self-inductance. According to Lenz's rule, this EMF is opposing any flux change. The direction of an induced EMF is always such that it produces a flux of $\vec{B}$ that opposes the change of the flux that produces the EMF. It tries to keep the current constant: it is opposite to the current when the current is increasing and in the direction of the current when it is decreasing.

This effect is particularly important for particle accelerators. A large electromagnet will have a large self-inductance. To change the current I in such a magnet requires a minimum voltage U to overcome this effect. This voltage is computed as:

$$U = -L\frac{\partial I}{\partial t}\,. \tag{47}$$

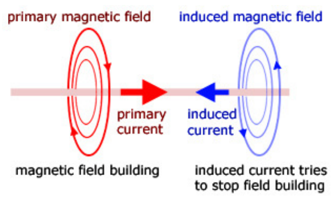

Fig. 10: Self-induction by a changing electric current

The self-inductance L is measured in henrys (H).

The necessary voltage is determined by this self-inductance and the rate of change of the current (Eq. (47)).

As a numerical example, we use the Large Hadron Collider parameters:

- required ramp rate, 10 A/s;
- self-inductance, L = 15.1 H per powering sector;
- required voltage to ramp at this rate, ≈150 V.

7.1 Lorentz force

A charge experiences forces in the presence of electromagnetic fields. This force depends not only on where it is (which determines the electromagnetic fields), but also on how it is moving. Moving ($\vec{v}$) charged (q) particles in electric ($\vec{E}$) and magnetic ($\vec{B}$) fields experience the force $\vec{f}$ (Lorentz force):

$$\vec{f} = q \cdot (\vec{E} + \vec{v} \times \vec{B})\,. \tag{48}$$

The electric force $q \cdot \vec{E}$ is always in the direction of the field $\vec{E}$ and proportional to the magnitude of the field and the charge.

The magnitude of the magnetic force $q \cdot \vec{v} \times \vec{B}$ is proportional to the velocity perpendicular to the direction of the field $\vec{B}$.

The Lorentz force is often treated as an ad-hoc plug-in to Maxwell's equation, but it is a relativistic effect (shown in Ref. [1]).

8 Electromagnetic waves in vacuum

A remarkable success of Maxwell's equations was the prediction of electromagnetic waves. Their existence was proven experimentally for very different wavelengths; in all cases, they were found to satisfy Maxwell's equations.

Starting from $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$, we can apply several mathematical transformations in steps:

$$\begin{aligned}
&\Longrightarrow \nabla \times (\nabla \times \vec{E}) = -\nabla \times \left(\frac{\partial \vec{B}}{\partial t}\right) \\
&\Longrightarrow -(\nabla^2 \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B}) \\
&\Longrightarrow -(\nabla^2 \vec{E}) = -\mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}
\end{aligned} \tag{49}$$

The last equation has the form of a plane wave.

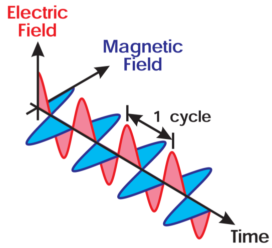

Fig. 11: Propagating electric and magnetic fields

This wave happens to be

$$\mu_0 \cdot \epsilon_0 = \frac{1}{c^2}\,,$$

and we rewrite:

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{E}}{\partial t^2}$$

and

$$\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{B}}{\partial t^2} \tag{50}$$

This is a general form of a wave equation.

As a solution of these equations, we can write:

$$\begin{aligned} \vec{E} &= \vec{E_0} \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)}\,, \\ \vec{B} &= \vec{B_0} \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)}\,, \end{aligned} \tag{51}$$

where we use the following definitions:

$$\begin{array}{ll} \text{propagation vector}: & |\vec{k}| = \dfrac{2\pi}{\lambda} \;=\; \dfrac{\omega}{c}\,, \\ \text{wavelength, 1 cycle}: & \lambda\,, \\ \text{frequency} \cdot 2\pi: & \omega\,, \\ \text{wave velocity}: & c \;=\; \dfrac{\omega}{k}\,. \end{array} \tag{52}$$

Magnetic and electric fields are transverse to the direction of propagation (Fig. 11):

$$\vec{E} \perp \vec{B} \perp \vec{k} \quad \Rightarrow \quad \vec{k} \times \vec{E_0} = \omega \vec{B_0}\,.$$

The speed of the wave in vacuum is exactly the speed of light: $c = 299792458$ m/s. Examples of the spectrum of electromagnetic waves are shown in Fig. 12 and Table 1.

The frequencies and, therefore, energies of existing waves span about 20 orders of magnitude.

9 Polarization of electromagnetic waves

9.1 General features

The solutions of the wave equations imply monochromatic plane waves. The solutions for electric and magnetic fields are:

$$\vec{E} = \vec{E_0} \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)}\,, \tag{53}$$

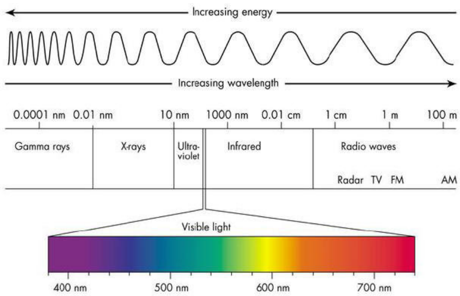

Fig. 12: Electromagnetic spectrum

Table 1: Properties of parts of the electromagnetic spectrum

Type	Frequency	Energy per photon
Radio	as low as 40 Hz	($\lesssim 10^{-13}$ eV)
Cosmic Microwave Background	$\lesssim 3 \cdot 10^{11}$ Hz	($\lesssim 10^{-3}$ eV)
Yellow light	$\approx 5 \cdot 10^{14}$ Hz	(≈ 2 eV)
X rays	$\leq 1 \cdot 10^{18}$ Hz	(≈ 4 keV)
γ rays	$\leq 3 \cdot 10^{21}$ Hz	(≤ 12 MeV)
$\pi^0 \rightarrow \gamma\gamma$	$\geq 2 \cdot 10^{22}$ Hz	(≥ 70 MeV)

$$\vec{B} = \vec{B}_0 \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)} \,. \tag{54}$$

These equations can be rewritten using unit vectors in the plane transverse to propagation. For example, for the electric component:

$$\vec{\epsilon_1} \perp \vec{\epsilon_2} \perp \vec{k} \,.$$

The two orthogonal components are:

$$\vec{E_1} = \vec{\epsilon_1} E_1 \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)} \,,$$
$$\vec{E_2} = \vec{\epsilon_2} E_2 \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)} \,.$$

The general field is a superposition of the two components:

$$\Rightarrow \vec{E} = (\vec{E_1} + \vec{E_2}) = (\vec{\epsilon_1} E_1 + \vec{\epsilon_2} E_2) \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)} \,. \tag{55}$$

For the propagation, we can allow for a phase shift ϕ between the two directions as well as different amplitudes:

$$\vec{E} = \vec{\epsilon_1} E_1 \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)} + \vec{\epsilon_2} E_2 \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t+\phi)} \,.$$

Depending on the amplitudes E_1 and E_2 and the relative phase ϕ, we get different types of polarized light:

$$\begin{array}{ll} \phi = 0: & \text{linearly polarized light}\,; \\ \phi \neq 0 \text{ and } E_1 \neq E_2: & \text{elliptically polarized light}\,; \\ \phi = \pm\dfrac{\pi}{2} \text{ and } E_1 = E_2: & \text{circularly polarized light}\,. \end{array}$$

9.2 Polarized light in accelerators

Polarized light is rather important in accelerators and is produced (amongst others) in synchrotron light machines (linearly and circularly polarized light, adjustable).

Typical applications and phenomena of polarized light are:

- polarized light reacts differently with charged particles;
- material science;
- beam diagnostics, medical diagnostics (blood sugar, ...);
- inverse free electron lasers;
- 3-D motion pictures, LCD display, outdoor activities, cameras (glare), ...

10 Energy of electromagnetic waves

We define the *Poynting vector* (SI units):

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}\,. \tag{56}$$

The vector $\vec{S}$ points in the direction of propagation and describes the 'energy flux', i.e., energy crossing a unit area, per second [J / m^2 s].

In free space, the energy in a plane is shared between the electric and magnetic fields The energy density $\mathcal{H}$ would be:

$$\mathcal{H} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\,. \tag{57}$$

11 Electromagnetic waves in material

We start now with the macroscopic Maxwell's equations (Eq. (32)), using $\mu_0\vec{H} = \vec{B}$ and $\epsilon_0\vec{E} = \vec{D}$:

$$\begin{aligned} \nabla \times \vec{E} &= -\mu_0 \frac{\mathrm{d}\vec{H}}{\mathrm{d}t}\,, \\ \nabla \times \vec{H} &= \vec{j}\; + \;\epsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t}\,. \end{aligned} \tag{58}$$

We assume a material with relative permittivity ϵ and permeability μ, as well as a finite conductivity σ, and get:

$$\begin{aligned} \nabla \times \vec{E} &= -\mu \cdot \mu_0 \cdot \frac{\mathrm{d}\vec{H}}{\mathrm{d}t}\,, \\ \nabla \times \vec{H} &= \sigma\vec{E} + \epsilon \cdot \epsilon_0 \cdot \frac{\mathrm{d}\vec{E}}{\mathrm{d}t}\,, \end{aligned} \tag{59}$$

where the current density $\vec{j}$ is replaced by $\sigma\vec{E}$ (Ohm's law). Following the same procedure as before, we obtain for the wave equation (electric field only):

$$\nabla^2\vec{E} = \sigma \cdot \epsilon \cdot \epsilon_0 \cdot \frac{\partial\vec{E}}{\partial t} + \mu \cdot \mu_0 \cdot \epsilon \cdot \epsilon_0 \cdot \frac{\partial^2\vec{E}}{\partial t^2} \tag{60}$$

For non-conducting media, we can set $\sigma = 0$ and obtain the previous equations.

As a direct consequence of Eq. (60) we see that the speed of this wave in the medium is now:

$$v = \frac{1}{\sqrt{\epsilon_0 \cdot \mu_0 \cdot \epsilon \cdot \mu}}\,, \tag{61}$$

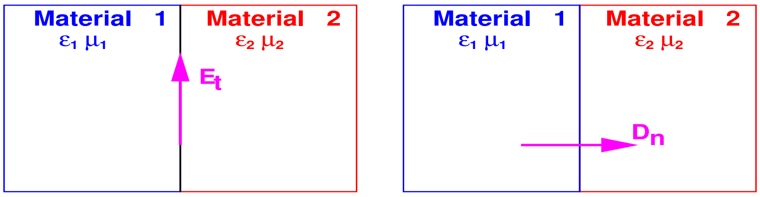

Fig. 13: Boundary conditions for electric fields

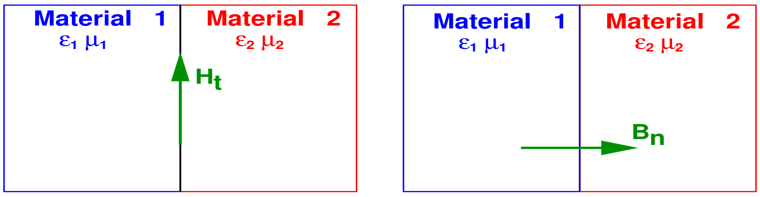

Fig. 14: Boundary conditions for magnetic fields

or, if rewritten using $n = \sqrt{\epsilon \cdot \mu}$,

$$v = \frac{c}{n}\,. \tag{62}$$

The speed of electromagnetic waves in vacuum is c, but reduced by the factor n in a medium with relative permittivity ϵ and permeability μ.

11.1 Boundary conditions

When electromagnetic waves pass through the boundary between two media with different ϵ and μ, we must fulfil some boundary conditions. The results are presented here without proof. For details see Refs. [3, 7]. Assuming *no* surface charges and, from curl$\vec{E} = 0$ we can derive that the *tangential $\vec{E}$-field* is continuous across a boundary ($E_t^1 = E_t^2$) (shown schematically in Fig. 13). Similarly, since we have div$\vec{D} = \rho$, the *normal $\vec{D}$-field* must be continuous across the boundary ($D_n^1 = D_n^2$) (shown schematically in Fig.13).

We follow the same line of reasoning for the boundary conditions for magnetic fields. Assuming *no* surface currents (for a proof, see, e.g., Refs. [3, 7]), we find (see Fig. 14):

From curl$\vec{H} = \vec{j}$,
$\Rightarrow$ *tangential $\vec{H}$-field* continuous across boundary $(H_t^1 = H_t^2)$.
From div $\vec{B} = 0$,
$\Rightarrow$ *normal $\vec{B}$-field* continuous across boundary $(B_n^1 = B_n^2)$.

A short summary for the electromagnetic fields at boundaries between different materials with different permittivity and permeability ($\epsilon_1, \epsilon_2, \mu_1, \mu_2$) is:

$$\begin{aligned} &(E_t^1 = E_t^2) &&(E_n^1 \neq E_n^2)\,, \\ &(D_t^1 \neq D_t^2) &&(D_n^1 = D_n^2)\,, \\ &(H_t^1 = H_t^2) &&(H_n^1 \neq H_n^2)\,, \\ &(B_t^1 \neq B_t^2) &&(B_n^1 = B_n^2)\,. \end{aligned} \tag{63}$$

These conditions lead to reflection and refraction of the waves at the surface; the angles are related to the refraction index $n = \sqrt{\epsilon_1 \mu_1}$ and $n' = \sqrt{\epsilon_2 \mu_2}$.

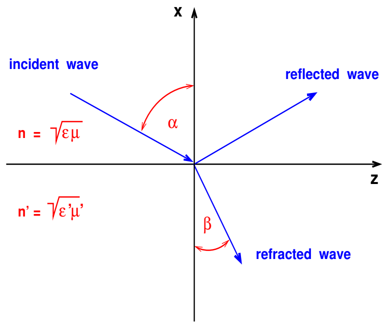

Fig. 15: Reflected and refracted components of an incident wave

The connection between the refraction indices and the scattering and refraction angles shown in Fig. 15 are:

$$\frac{\sin\alpha}{\sin\beta} = \frac{n'}{n} = \tan\alpha_{\mathrm{B}}\,. \tag{64}$$

If ϵ and μ depend on the wave frequency ω, the medium is dispersive and we have to write:

$$\frac{\mathrm{d}n}{\mathrm{d}\lambda} \neq 0\,, \tag{65}$$

i.e., the refraction index and therefore the angles depend on the wavelength.

If light is incident under the special angle α_{B} (the Brewster angle) [3], the reflected light is linearly polarized perpendicular to the plane of incidence.

A popular application is used when fishing, since air–water gives a comfortable angle $\alpha_{\mathrm{B}} \approx 53°$ and reflections can be avoided using polarization glasses.

12 Cavities and waveguides

Of particular interest in accelerator physics and technology is the behaviour and propagation of electromagnetic waves in cavities and waveguides. This behaviour is determined by the boundary conditions and we have to distinguish between material with infinite and finite conductivity. The case of perfectly conducting cavities and wave guides is treated first.

12.1 Rectangular cavities and waveguides

Cavities can be seen as a three-dimensional storage for electromagnetic waves, i.e., photons. The wave functions are contained inside and therefore the dimensions determine the maximum wavelength that can fit inside. This is due to the boundary conditions at the cavity walls.

If the fields are only constrained in two dimensions and allowed to move freely in the third dimension, the fields propagate as waves. The waves are guided through these 'wave guides'. Both are sketched in Fig. 16.

12.2 Cavities and modes

We assume a rectangular RF cavity with dimensions (a, b, c), and as an ideal conductor.

Without derivations (e.g., Refs. [3, 7, 8]), the components of the electric fields are:

$$E_x = E_{x0}\cdot\cos(k_x x)\cdot\sin(k_y y)\cdot\sin(k_z z)\cdot \mathrm{e}^{-\mathrm{i}\omega t}\,,$$
$$E_y = E_{y0}\cdot\sin(k_x x)\cdot\cos(k_y y)\cdot\sin(k_z z)\cdot \mathrm{e}^{-\mathrm{i}\omega t}\,,$$

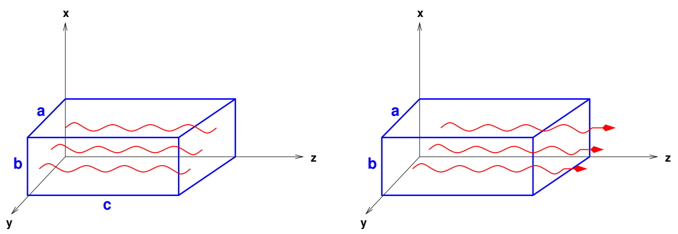

Fig. 16: Boundary conditions for electromagnetic fields. Fields are fully enclosed in a cavity (left-hand side) and can move freely in one dimension in waveguides (right-hand side).

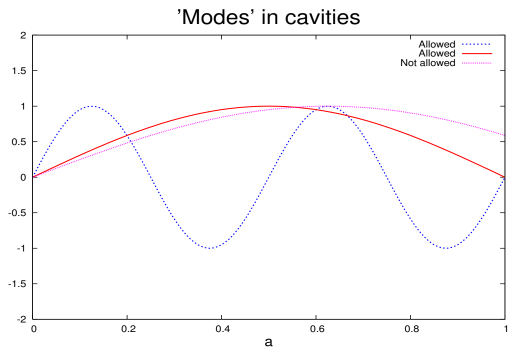

Fig. 17: Boundary conditions for electromagnetic fields

$$E_z = E_{z0} \cdot \sin(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot \mathrm{e}^{-\mathrm{i}\omega t} \,. \tag{66}$$

For the magnetic fields we get immediately, with $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$:

$$\begin{aligned} B_x &= \frac{\mathrm{i}}{\omega}(E_{y0}k_z - E_{z0}k_y) \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \cos(k_z z) \cdot \mathrm{e}^{-\mathrm{i}\omega t} \,, \\ B_y &= \frac{\mathrm{i}}{\omega}(E_{z0}k_x - E_{x0}k_z) \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \cos(k_z z) \cdot \mathrm{e}^{-\mathrm{i}\omega t} \,, \\ B_z &= \frac{\mathrm{i}}{\omega}(E_{x0}k_y - E_{y0}k_x) \cdot \cos(k_x x) \cdot \cos(k_y y) \cdot \sin(k_z z) \cdot \mathrm{e}^{-\mathrm{i}\omega t} \,. \end{aligned} \tag{67}$$

12.3 Consequences for cavities

The fields must be zero at the conductor boundary, as shown before. This is possible only with the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \,, \tag{68}$$

and for k_x, k_y, k_z we can write:

$$k_x = \frac{m_x \pi}{a} \,, \qquad k_y = \frac{m_y \pi}{b} \,, \qquad k_z = \frac{m_z \pi}{c} \,. \tag{69}$$

The integer numbers m_x, m_y, m_z are called the mode numbers of the wave and are directly related to the dimensions of the cavity.

Equations (68) and (69) imply that a half wavelength $\lambda/2$ must always fit exactly the size of the cavity. This is shown in Fig. 17 for different wavelengths compared with the cavity dimensions. Only modes that 'fit' into the cavity are allowed.

We can examine three cases:

$$\frac{\lambda}{2} = \frac{a}{4}, \qquad \frac{\lambda}{2} = \frac{a}{1}, \qquad \frac{\lambda}{2} = \frac{a}{0.8}\,.$$

No electric field at the boundaries requires that the wave must have ‘nodes’ at the boundaries. Only the first two wavelengths fulfil this condition; the third form cannot exist.

12.4 Waveguide modes

Similar considerations lead to (propagating) solutions in (rectangular) waveguides:

$$\begin{aligned}
E_x &= E_{x0} \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \mathrm{e}^{\mathrm{i}(k_z z - \omega t)}\,, \\
E_y &= E_{y0} \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \mathrm{e}^{\mathrm{i}(k_z z - \omega t)}\,, \\
E_z &= \mathrm{i} \cdot E_{z0} \cdot \sin(k_x x) \cdot \sin(k_y y) \cdot \mathrm{e}^{\mathrm{i}(k_z z - \omega t)}\,, \qquad (70) \\
B_x &= \frac{1}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot \sin(k_x x) \cdot \cos(k_y y) \cdot \mathrm{e}^{\mathrm{i}(k_z z - \omega t)}\,, \\
B_y &= \frac{1}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot \cos(k_x x) \cdot \sin(k_y y) \cdot \mathrm{e}^{\mathrm{i}(k_z z - \omega t)}\,, \\
B_z &= \frac{1}{\mathrm{i} \cdot \omega}(E_{x0} k_y - E_{y0} k_x) \cdot \cos(k_x x) \cdot \cos(k_y y) \cdot \mathrm{e}^{\mathrm{i}(k_z z - \omega t)}\,. \qquad (71)
\end{aligned}$$

12.5 Consequences for waveguides

To have no field at the boundary, we must again satisfy the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}\,. \qquad (72)$$

This leads to modes like (no boundaries in direction of propagation z):

$$k_x = \frac{m_x \pi}{a}\,, \qquad k_y = \frac{m_y \pi}{b}\,, \qquad (73)$$

The numbers m_x, m_y are called the mode numbers for planar waves in waveguides.

12.6 Cut-off frequency

One can rewrite Eq. (72) as:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \qquad (74)$$

and

$$k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}\,. \qquad (75)$$

A propagation without losses requires k_z to be real, i.e.,

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2 = \left(\frac{m_x \pi}{a}\right)^2 + \left(\frac{m_y \pi}{b}\right)^2\,. \qquad (76)$$

This defines a cut-off frequency ω_c:

$$\omega_\mathrm{c} = \frac{\pi \cdot c}{a}\,. \qquad (77)$$

For frequencies above this cut-off frequency, we have propagation without losses. At the cut-off frequency, we obtain a standing wave and an attenuated wave for lower frequencies, i.e., k_z becomes complex.

The cut-off frequencies are different for different modes and no modes can propagate below the lowest frequency. The mode of Eq. (77) is assumed to be this lowest frequency mode.

12.7 Circular cavities

Waveguides and cavities used in accelerators are more likely to be circular.

Derivation involves using the Laplace equation in cylindrical co-ordinates; for the derivation see e.g., Refs. [7, 8]:

$$
\begin{aligned}
E_r &= E_0 \frac{k_z}{k_r} J_n'(k_r) \cdot \cos(n\theta) \cdot \sin(k_z z) \cdot \mathrm{e}^{-\mathrm{i}\omega t}\,, \\
E_\theta &= E_0 \frac{n k_z}{k_r^2 r} J_n(k_r) \cdot \sin(n\theta) \cdot \sin(k_z z) \cdot \mathrm{e}^{-\mathrm{i}\omega t}\,, \\
E_z &= E_0 J_n(k_r r) \cdot \cos(n\theta) \cdot \sin(k_z z) \cdot \mathrm{e}^{-\mathrm{i}\omega t}\,, \\
B_r &= \mathrm{i} E_0 \frac{\omega}{c^2 k_r^2 r} J_n(k_r r) \cdot \sin(n\theta) \cdot \cos(k_z z) \cdot \mathrm{e}^{-\mathrm{i}\omega t}\,, \\
B_\theta &= \mathrm{i} E_0 \frac{\omega}{c^2 k_r r} J_n'(k_r r) \cdot \cos(n\theta) \cdot \cos(k_z z) \cdot \mathrm{e}^{-\mathrm{i}\omega t}\,, \\
B_z &= 0\,.
\end{aligned}
\tag{78}
$$

12.8 Accelerating circular cavities

For accelerating cavities, we need a longitudinal electric field component $E_z \neq 0$ and purely transverse magnetic fields:

$$
\begin{aligned}
E_r &= 0\,, \\
E_\theta &= 0\,, \\
E_z &= E_0 J_0 \left(p_{01} \frac{r}{R}\right) \cdot \mathrm{e}^{-\mathrm{i}\omega t}\,, \\
B_r &= 0\,, \\
B_\theta &= -\mathrm{i} \frac{E_0}{c} J_1 \left(p_{01} \frac{r}{R}\right) \cdot \mathrm{e}^{-\mathrm{i}\omega t}\,, \\
B_z &= 0\,.
\end{aligned}
\tag{79}
$$

(p_{nm} is the mth zero of J_n, e.g., $p_{01} \approx 2.405$.)

This would be a cavity with a TM_{010} mode: $\omega_{010} = p_{01} \cdot c/R$.

13 Case of finite conductivity

Starting from Maxwell's equation,

$$
\nabla \times \vec{H} = \vec{j} + \frac{\mathrm{d}\vec{D}}{\mathrm{d}t} = \underbrace{\overbrace{\sigma \cdot \vec{E}}^{\vec{j}}}_{\text{Ohm's law}} + \epsilon \frac{\mathrm{d}\vec{E}}{\mathrm{d}t}\,,
\tag{80}
$$

and the solutions of the wave equations,

$$
\vec{E} = \vec{E_0} \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r} - \omega t)}, \qquad \vec{H} = \vec{H_0} \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r} - \omega t)}\,,
\tag{81}
$$

we want to know k; applying the calculus to the wave equations we have:

$$
\frac{\mathrm{d}\vec{E}}{\mathrm{d}t} = -\mathrm{i}\omega \cdot \vec{E}, \qquad \frac{\mathrm{d}\vec{H}}{\mathrm{d}t} = -\mathrm{i}\omega \cdot \vec{H}, \qquad \nabla \times \vec{E} = \mathrm{i}\vec{k} \times \vec{E}, \qquad \nabla \times \vec{H} = \mathrm{i}\vec{k} \times \vec{H}\,.
\tag{82}
$$

Put these together, using Eqs. (80) and (82):

$$
\vec{k} \times \vec{H} = \mathrm{i}\sigma \cdot \vec{E} - \omega\epsilon \cdot \vec{E} = (-\mathrm{i}\sigma + \omega\epsilon) \cdot \vec{E}\,.
\tag{83}
$$

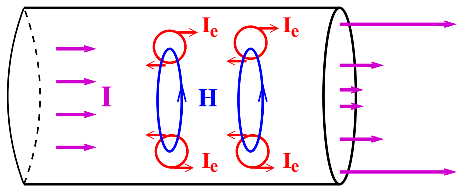

Fig. 18: Flow of current and induced magnetic fields and eddy currents

With $\vec{B} = \mu \vec{H}$:

$$\nabla \times \vec{E} = \mathrm{i}\vec{k} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} = \mathrm{i}\omega\mu\vec{H} \,. \tag{84}$$

Multiplication with $\vec{k}$ and using Eq. (83):

$$\vec{k} \times (\vec{k} \times \vec{E}) = \omega\mu(\vec{k} \times \vec{H}) = \omega\mu(-\mathrm{i}\sigma + \omega\epsilon) \cdot \vec{E} \,. \tag{85}$$

After some calculus and using the property $\vec{E} \perp \vec{H} \perp \vec{k}$:

$$k^2 = \omega\mu(\mathrm{i}\sigma + \omega\epsilon) \,. \tag{86}$$

The propagation vector k now differs from the equation in vacuum by the contributions from the medium and the finite conductivity. This has consequences for the propagation and penetration of waves in material.

13.1 Skin and penetration depth

For a good conductor, $\sigma \gg \omega\epsilon$ (e.g., for Cu we have $\sigma \approx 5.7 \cdot 10^7$ S/m, this value for Cu is also valid for for very high ω):

$$k^2 \approx -\mathrm{i}\omega\mu\sigma \qquad \Rightarrow \qquad k \approx \sqrt{\frac{\omega\mu\sigma}{2}}(1-\mathrm{i}) = \frac{1}{\delta}(1-\mathrm{i}) \,. \tag{87}$$

We define the parameter δ:

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \,. \tag{88}$$

The parameter δ is called the *skin depth*.

From Eq. (88), we deduce that high frequency waves 'avoid' penetrating a conductor, and mainly flow near the surface. One can understand this effect using Fig. 18.

A changing $\vec{H}$-field induces eddy currents in the conductor. These cancel the current flow in the centre of the conductor but enforce current flow at the skin (surface).

13.2 Attenuated waves

Waves incident on conducting material are attenuated. It is basically skin depth, (attenuation to 1/e). The wave form becomes:

$$\mathrm{e}^{\mathrm{i}(kz-\omega t)} = \mathrm{e}^{\mathrm{i}((1+\mathrm{i})z/\delta - \omega t)} = \mathrm{e}^{\frac{-z}{\delta}} \cdot \mathrm{e}^{\mathrm{i}(\frac{z}{\delta} - \omega t)} \,. \tag{89}$$

Some numerical examples:

– Skin depth of copper:
  1 GHz : $\delta \approx 2.1$ μm; 1 kHz : $\delta \approx 2.1$ mm, 50 Hz : $\delta \approx 10$ mm.
  This has important consequences for the design of conducting cables since the high frequency currents propagate at a very thin layer at the surface of the conductor.

– Penetration depth into seawater (σ typically 4 S/m):
  To get $\delta \approx 25$ m, one needs ≈76 Hz.
  Because of the long wavelength and low frequency, communication is very inefficient and has a very low bandwidth (0.03 bps).

14 Summary

Without any attempt to be rigorous or complete, electromagnetic effects most important for the design and operation of particle accelerators have been presented, such as:

– basic concepts;
– Maxwell's equations;
– fields and potentials from charge and current distributions;
– electromagnetic waves in vacuum and media;
– electromagnetic waves in waveguides and cavities;
– polarization of electromagnetic waves and skin effects.

References

[1] W. Herr, Short theory of special relativity and invariant formulation of electrodynamics, these proceedings.

[2] Proceedings of the CAS-CERN Accelerator School on RF for Accelerators, Ebeltoft, Denmark, 8–17 June 2010, edited by R. Bailey, CERN-2011-007 (CERN, Geneva, 2011), https://doi.org/10.5170/CERN-2011-007

[3] J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1998).

[4] L. Landau and E. Lifschitz, *The Classical Theory of Fields* (Butterworth-Heinemann, Oxford, 1975).

[5] J. Slater and N. Frank, *Electromagnetism* (McGraw-Hill, New York, 1947).

[6] W. Herr, in Proc. of the CAS-CERN Accelerator School: Intermediate Course on Accelerator Physics, Zeuthen, Germany, 15–26 September 2003, edited by D. Brandt, CERN-2006-002 (CERN, Geneva, 2006), pp. 379-410. https://doi.org/10.5170/CERN-2006-002.379

[7] R.P. Feynman, *Feynman Lectures on Physics*, (Basic Books, New York, 2001), Vol. 2.

[8] A. Chao *et al.*, *Handbook of Accelerator Physics and Engineering*, World Scientific Publishing, New Jersey (1998).

Appendices

A Electromagnetic units

Formulae use *SI units* throughout.

$\vec{E}(\vec{r},t)$	electric field [V/m]
$\vec{H}(\vec{r},t)$	magnetic field [A/m]
$\vec{D}(\vec{r},t)$	electric displacement [C/m^2]
$\vec{B}(\vec{r},t)$	magnetic flux density [T]
q	electric charge [C]
$\rho(\vec{r},t)$	electric charge density [C/m^3]
$\vec{I},\vec{j}(\vec{r},t)$	current [A], current density [A/m^2]
μ_0	permeability of vacuum, $4\,\pi\cdot 10^{-7}$ [H/m or N/A^2]
ϵ_0	permittivity of vacuum, $8.854\cdot 10^{-12}$ [F/m]

To save typing and space where possible (e.g., equal arguments):
$\vec{E}(\vec{r},t) \Longrightarrow \vec{E}$ and the same for other variables.

B Refresher on vector calculus

B.1 Vector operators

We can define a special vector ∇ (sometimes written as $\vec{\nabla}$):

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) . \tag{B.1}$$

It is called the 'gradient' and invokes 'partial derivatives'.

It can operate on a scalar function $\phi(x,y,z)$:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \vec{G} = (G_x, G_y, G_z)\,, \tag{B.2}$$

and we get a vector $\vec{G}$. It is a kind of 'slope' (steepness) in the three directions.

Example:

$$\phi(x,y,z) = C\cdot\mathbf{ln}(r^2) \qquad \text{with} \qquad r = \sqrt{x^2+y^2+z^2}$$
$$\nabla\phi = (G_x, G_y, G_z) = \left(\frac{2C\cdot x}{r^2}, \frac{2C\cdot y}{r^2}, \frac{2C\cdot z}{r^2}\right)$$

B.1.1 Physical interpretation of the gradient operator

The gradient applied to a scalar field measures the local slope, as shown in Fig. B.1:

- lines of pressure (isobars);
- gradient is large (steep) where lines are close (fast change of pressure).

B.2 Operation on vectors and scalar fields

The gradient ∇ can be used in a scalar or a vector product with a vector $\vec{F}$, sometimes written as $\vec{\nabla}$ and these are used as:

$$\nabla\cdot\vec{F} \qquad \text{or} \qquad \nabla\times\vec{F}\,. \tag{B.3}$$

The definition for products is the same as before; ∇ is treated like a 'normal' vector, but the results depend on how they are applied:

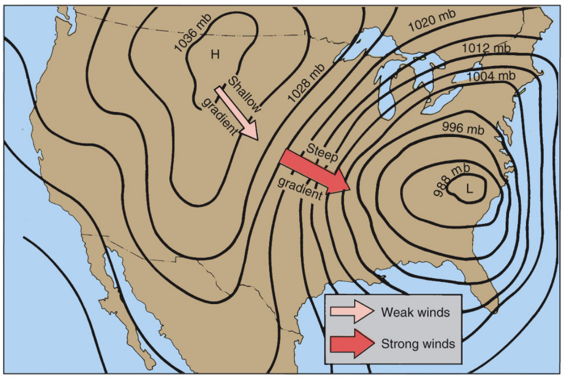

Fig. B.1: Gradient of a scalar field (here air pressure)

- $\nabla\phi$ is a vector;
- $\nabla \cdot \vec{F}$ is a scalar;
- $\nabla \times \vec{F}$ is a pseudo-vector.

B.3 Divergence and curl

Two operations of ∇ have special names.

B.3.1 Divergence (scalar product of gradient with a vector):

$$\mathsf{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\,. \tag{B.4}$$

Physical significance: 'amount of density' (see later).

B.3.2 Curl (vector product of gradient with a vector):

$$\mathsf{curl}(\vec{F}) = \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\,. \tag{B.5}$$

Physical significance: 'amount of rotation' (see later).

B.3.3 Meaning of divergence

Figure B.2 shows the field lines of a vector field $\vec{F}$ seen from some origin.

The divergence (scalar, a single number) characterizes what comes from (or goes to) the origin. How much comes out is measured by the surface integral. For the integrated field vectors passing (perpendicularly) through a surface area A, we obtain the flux:

$$\int\int_A \vec{F} \cdot \mathrm{d}\vec{A}\,. \tag{B.6}$$

It has the meaning of the density of field lines through the surface (Fig. B.3).

For closed surfaces, we can rewrite the integral using Gauss's theorem: the integral through a closed surface (flux) is the integral of the divergence *in the enclosed volume*

$$\int\int_A \vec{F} \cdot \mathrm{d}\vec{A} = \int\int\int_V \nabla \cdot \vec{F} \cdot \mathrm{d}V\,. \tag{B.7}$$

This relates surface integral to volume integrals (Fig. B.4) and is often easier to evaluate.

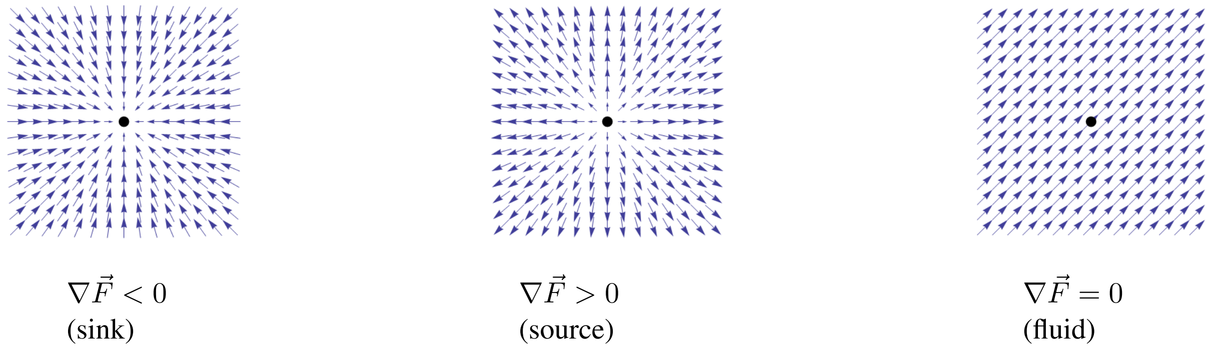

Fig. B.2: Field lines of a vector field $\vec{F}$ seen from some origin

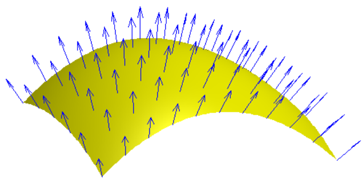

Fig. B.3: Flux

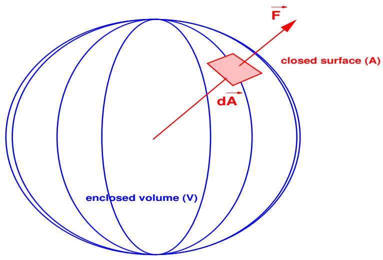

Fig. B.4: Gauss's theorem relates surface integrals to volume integrals

B.3.4 Meaning of curl

The *curl* quantifies a rotation of vectors: it is the integration of (vector-) fields. For two vector fields, we perform a *line integral* along a (closed) line C:

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \int \int_A \nabla \times \vec{F} \cdot \mathrm{d}\vec{A} \tag{B.8}$$

i.e., we 'sum up' vectors (length) in the *direction* of the line C

The line integral for the second vector field in Fig. B.5 vanishes because the field lines are orthogonal to the direction of the integration path along the curve C. The physical significance of this line integral is the work performed along a path.

We can formulate this integral more generally:
Stokes' theorem: *Integral along a closed line is integral of curl in the enclosed area.*

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{s} = \int \int_A \nabla \times \vec{F} \cdot \mathrm{d}\vec{A}\,. \tag{B.9}$$

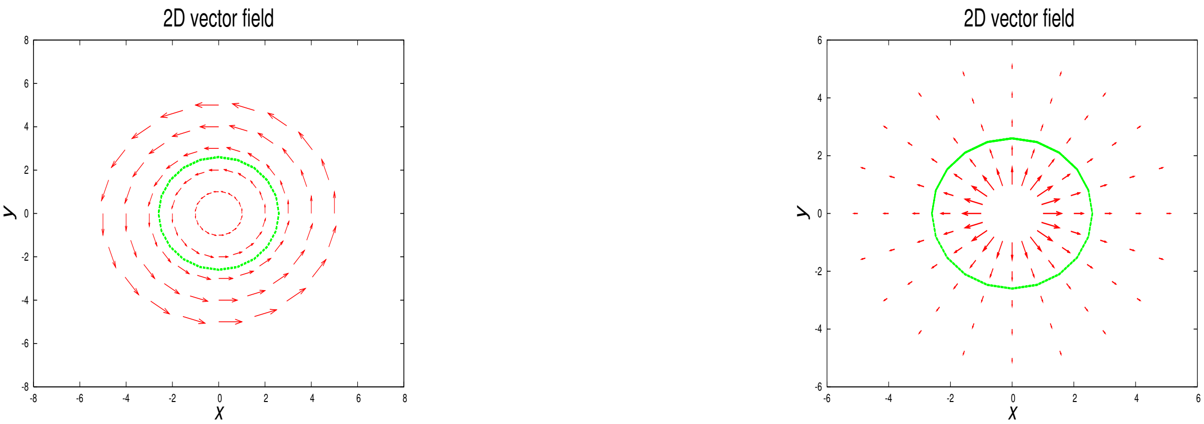

Fig. B.5: Two types of vector field, arbitrary units. For the left field we have: $\nabla\vec{F} = 0 \quad \nabla\times\vec{F} \neq 0$. For the right field: $\nabla\vec{F} \neq 0 \quad \nabla\times\vec{F} = 0$.

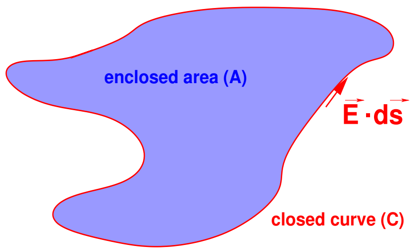

Fig. B.6: Stoke's theorem

B.4 Scalar product

We define a scalar product for (usual) vectors as: $\vec{a}\cdot\vec{b}$,

$$\begin{aligned}
\vec{a} &= (x_a, y_a, z_a)\,,\\
\vec{b} &= (x_b, y_b, z_b)\,,\\
\vec{a}\cdot\vec{b} &= (x_a, y_a, z_a)\;\cdot\;(x_b, y_b, z_b) = (x_a\cdot x_b\;+y_a\cdot y_b\;+\;z_a\cdot z_b)\,.
\end{aligned}$$

This product of two vectors is a *scalar* (number), not a vector.

Example:

$$(-2,2,1)\cdot(2,4,3) = -2\cdot 2 + 2\cdot 4 + 1\cdot 3 = 7\,.$$

B.5 Vector product (sometimes referred to as cross product)

Define a vector product for (usual) vectors as: $\vec{a}\times\vec{b}$,

$$\begin{aligned}
\vec{a} &= (x_a, y_a, z_a)\,,\\
\vec{b} &= (x_b, y_b, z_b)\,,\\
\vec{a}\times\vec{b} &= (x_a, y_a, z_a)\;\times\;(x_b, y_b, z_b)\\
&= (\underbrace{y_a\cdot z_b - z_a\cdot y_b}_{x_{ab}},\;\underbrace{z_a\cdot x_b - x_a\cdot z_b}_{y_{ab}},\;\underbrace{x_a\cdot y_b - y_a\cdot x_b}_{z_{ab}})\,.
\end{aligned}$$

This product of two vectors is a *vector*, not a scalar (number), (on this account: vector product).

Example 1:

$$(-2, 2, 1) \times (2, 4, 3) = (2, 8, -12)\,.$$

Example 2 (two components only in the x–y plane):

$$(-2, 2, 0) \times (2, 4, 0) = (0, 0, -12)\,.$$