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Quarkonium physics: NRQCD factorization formula for $J/\psi \to e^+e^-$

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Abstract
In this lecture, we briefly review the nonrelativistic QCD (NRQCD) factorization approach to describe the quarkonium production and decay. In the NRQCD factorization formula, the long-distance nature of a heavy quarkonium is factorized into the NRQCD long-distance matrix elements (LDMEs) and a physical measurable is expressed as a linear combination of the LDMEs. If we apply the perturbation theory to enhance the theoretical accuracies in the short-distance contributions there is no way to avoid non-analytic Coulomb divergences in the static limit of the heavy quark and infrared divergences. A systematic procedure to isolate such long-distance contributions out of the correction terms in the short-distance coefficients is called matching. As a heuristic example of finding the NRQCD factorization formula for a specific process, we demonstrate the matching procedure of determining the short-distance coefficients involving the leptonic decay of the S-wave spin-triplet state.

Keywords
NRQCD factorization formula; short-distance coefficient; long-distance NRQCD matrix elements; matching.

1 Introduction

The heavy quarkonium is the bound system of a heavy quark (Q) and a heavy antiquark ($\bar{Q}$), where $Q = c$ or b. If $Q = c$ (b), it is called a charmonium (bottomonium). As we do for a hydrogen atom, we allocate the following quantum numbers for a quarkonium system: In order to describe the radial excitation we use the principal quantum number n. The orbital angular momentum quantum number L ($= 0,\ 1,\ 2, \cdots$) is used to identify the relative motion between Q and $\bar{Q}$. The states with $L = 0,\ 1,\ 2$ are also called the S-, P-, D-wave states, respectively. The spin angular momentum quantum number S indicates whether the pair is in the spin-singlet ($S = 0$) or in the spin-triplet ($S = 1$) state. The spectroscopic notation $^{2S+1}L_J$ is used to represent a physical quarkonium state, where J is the total angular momentum quantum number. A physical quarkonium system is in a color singlet state. However, we can also think of a $Q\bar{Q}$ pair created or annihilated at short distances in a certain color combination. The notations $^{2S+1}L_J^{[1]}$ and $^{2S+1}L_J^{[8]}$ are used for the color-singlet and -octet states, respectively. A physical quarkonium system of $^{2S+1}L_J^{[1]}$ state is a simultaneous eigenstate of both the parity P and the charge conjugation parity C. If we consider the fact that the spin wave function for a $Q\bar{Q}$ pair is symmetric (antisymmetric) when $S = 0$ ($S = 1$), then we find that the corresponding symmetry factor in the spin wave function for the exchange of Q and $\bar{Q}$ is $(-1)^{S+1}$. The orbital angular momentum wave function has the parity $(-1)^L$ and the intrinsic parities of Q and $\bar{Q}$ differ. Thus the parity of the $Q\bar{Q}$ bound state is $P = (-1)^{L+1}$. Because the charge conjugation is equivalent to the interchange of the positions and spins for the Q and $\bar{Q}$, $C = (-1)^{S+1}$ and $P = (-1)^{L+S}$. In Table 1 we list the quantum numbers of various quarkonia.

Among various quarkonia, the $^3S_1^{[1]}$ charmonium J/ψ of mass about 3.1 GeV was discovered first in 1974 [1, 2]. This state has a very long life time and, therefore, it has a very narrow width so that it

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Table 1: Quantum numbers of quarkonia with $n = 1$.

$^{2S+1}L_J^{[1]}$	**charmonium**	**bottomonium**	$\boldsymbol{S}$	$\boldsymbol{L}$	$\boldsymbol{J^{PC}}$
$^1S_0^{[1]}$	η_c	η_b	0	0	0^{-+}
$^3S_1^{[1]}$	J/ψ	Υ	1	0	1^{--}
$^1P_1^{[1]}$	h_c	h_b	0	1	1^{+-}
$^3P_{J(=0,1,2)}^{[1]}$	χ_{cJ}	χ_{bJ}	1	1	J^{++}

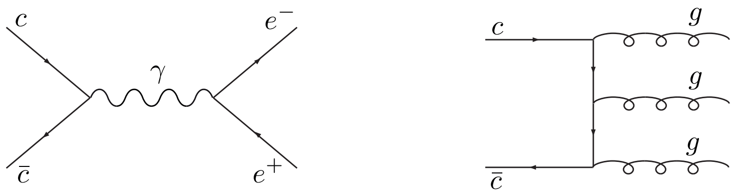

Fig. 1: The Feynman diagrams involving the decay modes $J/\psi \to e^+e^-$ (left) and $J/\psi \to$ light hadrons (right) at LO in α_s, respectively. There are 5 more diagrams for the light-hadronic decay that can be obtained by permutation of three gluons.

Fig. 2: The Feynman diagrams involving the decay modes η_c or $\chi_{c0} \to \gamma\gamma$ (left) and η_c or $\chi_{c0} \to$ light hadrons (right) at LO in α_s, respectively. There is another diagram that can be obtained by exchanging the two final-state particles for each diagram.

can be detected as a sharp resonance in the invariant mass distribution of the lepton pair in the decay mode $J/\psi \to e^+e^-$ or $\mu^+\mu^-$. The annihilation of J/ψ involves the annihilation of a $Q\bar{Q}$ pair because it cannot decay into two charmed mesons ($D\bar{D}$) whose invariant mass is greater than the J/ψ mass. At leading order (LO) in the strong coupling constant α_s the light-hadronic decay mode is dominated by the three-gluon final state of the color-singlet combination whose decay rate is of order α_s^3. This suppression of the hadronic decay mode of the J/ψ makes its width sharper than other quarkonium states such as η_c or χ_{c0} whose leading decay modes are into two gluons. This elementary interpretation is based on the conservation of the charge conjugation parity C as well as the parity P of QCD. Because a physical S-wave spin-triplet $Q\bar{Q}$ bound state has $J^{PC} = 1^{--}$, it may only decay into an odd number of gluons that make a color singlet combination. Thus the color-singlet combination of three gluons appears at LO in α_s so that the dominance of the hadronic decay mode is less severe for the S-wave spin-triplet state in comparison with the cases of the S-wave spin-singlet and the P-wave spin-triplet states whose charge conjugation parities are $+1$. Hence, the leading contributions to the $^1S_0^{[1]}$ or $^3P_J^{[1]}$ hadronic decay rates are color-singlet combination of two gluons that are of order α_s^2. Therefore, the narrow width of $^3S_1^{[1]}$ quarkonium in comparison with other states is well understood in QCD and allows clean signals. In Table 2 we list the total decay rates, branching fractions for the electromagnetic decay modes, and those for the light-hadronic modes for J/ψ, η_c, and χ_{c0}. In Figs. 1 and 2, we list Feynman diagrams for

Table 2: The total decay widths $\Gamma_{\rm tot}$, the branching fractions $\text{Br}_{\rm EM}$ for the electromagnetic decay modes, and the branching fractions $\text{Br}_{\rm had}$ for the light-hadronic decay modes for J/ψ, η_c, and χ_{c0}.

	$\Gamma_{\rm tot}$	$\text{Br}_{\rm EM}$	$\text{Br}_{\rm had}$
J/ψ	93 keV	$2 \times 6\%$	88%
η_c	32 MeV	1.6×10^{-4}	~ 1
χ_{c0}	11 MeV	2.2×10^{-4}	~ 1

the electromagnetic and light-hadronic decays of J/ψ and η_c (χ_{c0}), respectively, at LO in α_s.

An S-wave spin-triplet charmonium H is usually detected at colliders through the muon-antimuon or electron-positron pair final states. However, the original hadron H' that results in these final states is not unique. It can be the ${}^3S_1^{[1]}$ charmonium itself or other hadrons that decay into particles including a ${}^3S_1^{[1]}$ charmonium. The mother hadron H' is mostly another charmonium resonance or B mesons. If H' is a higher quarkonium resonance, then the main decay mechanism is through the strong or electromagnetic interaction. If H' is a b hadron, then its decay is governed by the weak interaction so that the vertex of $\ell^+\ell^-$ is secondary and it is largely separated from the primary vertex (collision point) at which b hadrons are created. The CDF Collaboration of the Fermilab Tevatron installed the Silicon Vertex Detector (SVX) [3] to achieve the asymptotic impact parameter resolutions of order 10 μm. The SVX enabled one to reconstruct b hadrons effectively so that one can easily separate the signals coming from b hadron decays and isolate the signals from the directly produced charmonium or higher charmonium resonances that we call *prompt* charmonium. Although the SVX is useful to separate the prompt charmonium signal from the charmonium coming from the decay of b hadrons, it is unable to distinguish the signals of the directly produced charmonium from those coming from the feeddowns of higher charmonium resonances. The prompt $\psi(2S)$ is the same as the *direct* $\psi(2S)$ because it does not have any feeddowns from higher resonances. In the case of J/ψ, the prompt signal contains the direct J/ψ, J/ψ coming from the feeddowns $\chi_{cJ} \to J/\psi + \gamma$ and $\psi(2S) \to J/\psi\pi\pi$. Here, the χ_{cJ} is either the direct χ_{cJ} or that coming from $\psi(2S) \to \chi_{cJ} + \gamma$.

The advent of the SVX sped up the progress of the phenomenological studies of prompt J/ψ and $\psi(2S)$. This has eliminated the non-prompt samples whose theoretical prediction has large uncertainties. The enormous surplus of these charmonia measured by the CDF Collaboration at the Fermilab Tevatron in comparison with the prediction based on the conventional phenomenological model so called the color-singlet model (CSM) was successfully explained in Ref. [4] by introduction of the color-octet mechanism [5] of the nonrelativistic QCD (NRQCD) factorization approach [6]. This success of explaining the production rate was followed by a puzzling contradiction that the measurements of the J/ψ polarization at the Tevatron and at the LHC are against the theoretical prediction that the prompt J/ψ produced with large transverse momentum p_T should be transversely polarized [7–13]. In 2014, this puzzle has indeed been resolved by considering the leading-power factorization [14].

In this lecture, we briefly review the NRQCD factorization approach to describe the production and decay of the heavy quarkonium. In Sec. 2 we briefly review the quarkonium theory by considering the basic nature of a quarkonium, an old phenomenological model CSM, the origin of the soft singularities that appear in the perturbative computation of a measurable involving a quarkonium. In the latter part of this section we review the NRQCD factorization approach that provides a systematic procedure to isolate such infrared-sensitive factors out of the short-distance coefficients in the factorization formula. In Sec. 3 we provide a demonstration of matching that determines the short-distance coefficients of the NRQCD factorization formula for a specific process by considering the leptonic decay of the S-wave spin-triplet state and we summarize in Sec. 4.

2 Quarkonium theory

In this section, we briefly review theories regarding heavy quarkonium production and decay. We first summarize the fundamental nature of heavy quarkonia. Next we introduce a phenomenological model so called the color-singlet model that had been resorted until mid 1990's. At the end of this section, we summarize basic ideas of the NRQCD factorization approach which is the most rigorous theoretical approach at present.

2.1 Basic nature of quarkonia

The heavy quarkonium H is a bound state of a heavy quark Q and an antiquark $\bar{Q}$ whose interaction is dominated by the strong interaction. Because the Q and $\bar{Q}$ are heavy, the speed of Q or $\bar{Q}$ in the meson rest frame is assumed to be much smaller than the speed of light. For example, the masses of the spin-triplet states J/ψ and $\psi(2S)$ are about 3.1 GeV and 3.7 GeV, respectively. In the case of bottomonia $\Upsilon(1S)$ and $\Upsilon(2S)$, they are about 9.5 GeV and 10 GeV, respectively. The mass difference between the $1S$ and $2S$ states are about 500 – 600 MeV for both $c\bar{c}$ and $b\bar{b}$ states. This mass difference can be scaled by $\sim m_Q v_Q^2$ roughly based on the Virial theorem, where m_Q and v_Q are the mass and the speed of the Q in the meson rest frame, respectively. Thus one can guess that $v_c^2 \sim 0.3$ and $v_b^2 \sim 0.1$. There are typical scales that involve the strong interaction in a quarkonium: The heavy-quark mass $\sim m_Q$ is the hard scale that governs the creation or annihilation of H. The heavy-quark momentum $\sim m_Q v_Q$ in the meson rest frame is the scale that determines the typical size $\sim 1/(m_Q v_Q)$ of the bound state. The life time of the bound state $\sim 1/(m_Q v_Q^2)$ is scaled by the reciprocal of the binding energy $\sim m_Q v_Q^2$. If we can take the approximation that $v_Q \ll 1$, then these scales are well separated as $m_Q \gg m_Q v_Q \gg m_Q v_Q^2$ [6].

The complete production or decay process of a heavy quarkonium H includes all of these interactions over a wide range of scales simultaneously. The annihilation or creation of a $Q\bar{Q}$ pair involving a hard scale of order m_Q or higher can be treated perturbatively because $\alpha_s(m_Q)$ is assumed to be small. The interaction of Q and $\bar{Q}$ that makes the bound state is governed by the strong interaction of scale similar to or less than $m_Q v_Q$ or $m_Q v_Q^2$ are nonperturbative. One can make a guess that the perturbative factor and the nonperturbative factor are separable. The CSM assumes that the two factors are factorized as the product of the short-distance factor that is controlled by the hard scale and the long-distance factor that is responsible for the quarkonium wave function. The NRQCD factorization is a rigorous framework that factorizes the two factors in a systematic double power expansions in $\alpha_s(m_Q)$ and v_Q.

2.2 Color-singlet model

The CSM assumes that the short-distance factor and the long-distance factor are factorized. In addition, it allows only the color-singlet $Q\bar{Q}$ pair with the spectroscopic state identical to the physical quarkonium state in the short-distance contributions [15–27]. According to the CSM, the decay rates for $J/\psi \to e^+e^-$ and $\chi_{c0} \to$ light hadrons are factorized as

$$\Gamma(J/\psi \to e^+e^-) = \hat{\Gamma}\left[c\bar{c}({}^3S_1^{[1]}) \to e^+e^-\right] \times |R(0)|^2, \tag{1a}$$

$$\Gamma(\chi_{c0} \to \text{hadrons}) = \hat{\Gamma}\left[c\bar{c}({}^3P_0^{[1]}) \to gg\right] \times |R'(0)|^2, \tag{1b}$$

where $\hat{\Gamma}$'s are the corresponding short-distance factors that are assumed not to be sensitive to long-distance interactions. $R(0)$ and $R'(0)$ are radial wave function at the origin and the first-order derivative of the radial wave function at the origin, respectively. The J/ψ decay rate formula is an immediate application of the Van Royen-Weisskopf formula for a meson decay into a lepton pair [28]. However, in the CSM, the infrared (IR) insensitivity of the short-distance coefficient $\hat{\Gamma}$ is not guaranteed at higher orders in α_s. For example, as shown in Fig. 3, the short-distance coefficient for the light-hadronic decay of χ_{c0} contains the three-body decay mode $\chi_{c0} \to gq\bar{q}$, where q is a light quark, at the next-to-leading order (NLO) in α_s. In the end point limit of the phase space that the q and $\bar{q}$ are back to back, the gluon

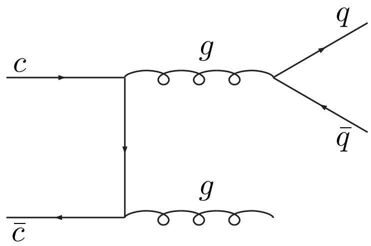

Fig. 3: One of the Feynman diagrams involving $\chi_{c0} \to$ light hadrons at NLO in α_s.

becomes soft. The IR divergence due to the attachment of a soft gluon to c and $\bar{c}$ does not cancel if one cannot ignore the relative momentum between c and $\bar{c}$. This remaining IR divergence of the short-distance coefficient at this order leads to the failure in factorization of the CSM [5]. Such a failure of factorization in the CSM appears also in the S-wave case from order v_Q^4 [29, 30] even at LO in α_s. The order-α_s correction to the short-distance process for $J/\psi \to e^+e^-$ also brings in the IR sensitivity to all orders in v_Q as shown in Ref. [31].

2.3 Infrared divergence

The cancellation of the IR divergence when we attach a soft gluon to the $Q\bar{Q}$ pair is exact as long as the two momenta for the Q and $\bar{Q}$ are identical to each other. However, such cancellation does not hold once the two momenta are different. We demonstrate this phenomenon in a schematic way. Consider the amplitude for the production of a $Q\bar{Q}$ pair

$$i\mathcal{M}_0 = \bar{u}(p_1)\mathcal{A}v(p_2), \tag{2}$$

where $\mathcal{A}$ is the amputated amplitude that excludes the external lines for the $Q\bar{Q}$ pair, p_1 and p_2 are the momenta for the on-shell Q and $\bar{Q}$, respectively, with $p_1^2 = p_2^2 = m^2$. They can be expressed as linear combinations of the momentum $P = p_1 + p_2$ for the $Q\bar{Q}$ pair and half their relative momentum $q = (p_1 - p_2)/2$ as

$$p_1 = \frac{1}{2}P + q, \tag{3a}$$
$$p_2 = \frac{1}{2}P - q. \tag{3b}$$

Analogously one can apply a similar method to the annihilation decay process.

If we attach a gluon with momentum k to the external line of the heavy quark Q, then the amplitude becomes

$$\begin{aligned} i\mathcal{M}_1 &= g_s\mu^\epsilon \bar{u}(p_1)[-i\not{\epsilon}^*]\frac{i(\not{p}_1 + \not{k} + m)}{(p_1+k)^2 - m^2}T^a\mathcal{A}v(p_2) \\ &= g_s\mu^\epsilon \bar{u}(p_1)\frac{\not{\epsilon}^*(\not{p}_1 + \not{k} + m)}{(p_1+k)^2 - m^2}T^a\mathcal{A}v(p_2) \\ &= g_s\mu^\epsilon \bar{u}(p_1)\frac{2\epsilon^*\cdot(p_1+k) - (\not{p}_1 + \not{k} - m)\not{\epsilon}^*}{2p_1\cdot k}T^a\mathcal{A}v(p_2), \end{aligned} \tag{4}$$

where ϵ^* and a are the polarization vector and color index of the gluon, T^a is the generator of the fundamental representation of color SU(N_c) with $N_c = 3$, $g_s = \sqrt{4\pi\alpha_s}$, μ is the renormalization scale, and we use dimensional regularization in $d = 4 - 2\epsilon$ space-time dimensions. By making use of the equation of motion for the external quark,

$$\bar{u}(p_1)(\not{p}_1 - m) = 0, \tag{5}$$

and the transverse condition for the external gluon

$$\epsilon^* \cdot k = 0, \tag{6}$$

we find that

$$i\mathcal{M}_1 = g_s\mu^\epsilon \bar{u}(p_1)\frac{2\epsilon^*\cdot p_1 - \not{k}\not{\epsilon}^*}{2p_1\cdot k}T^a\mathcal{A}v(p_2). \tag{7}$$

Taking the leading contribution as $k \to 0$ to collect the IR sensitive part, we find that

$$i\mathcal{M}_1^{\rm IR} = g_s\mu^\epsilon\frac{p_1\cdot\epsilon^*}{p_1\cdot k}\bar{u}(p_1)T^a\Gamma v(p_2). \tag{8}$$

If we attach a gluon with momentum k to the external line of the heavy antiquark $\bar{Q}$, then

$$\begin{aligned} i\mathcal{M}_2 &= \bar{u}(p_1)\mathcal{A}T^a\frac{i(-\not{p}_2-\not{k}+m)}{(-p_2-k)^2-m^2}[-i\not{\epsilon}^*]v(p_2) \\ &= g_s\mu^\epsilon\bar{u}(p_1)\mathcal{A}T^a\frac{\not{\epsilon}^*(-\not{p}_2-\not{k}+m)}{(p_2+k)^2-m^2}v(p_2) \\ &= g_s\mu^\epsilon\bar{u}(p_1)\mathcal{A}T^a\frac{-2\epsilon^*\cdot(p_2+k)+(\not{p}_2+\not{k}+m)\not{\epsilon}^*}{2p_2\cdot k}v(p_2). \end{aligned} \tag{9}$$

By making use of the equation of motion for the external antiquark,

$$(\not{p}_2+m)v(p_2) = 0, \tag{10}$$

and the transverse condition

$$\epsilon^*\cdot k = 0, \tag{11}$$

we find that

$$i\mathcal{M}_2 = g_s\mu^\epsilon\bar{u}(p_1)\mathcal{A}T^a\frac{-2\epsilon^*\cdot(p_2+k)+\not{k}\not{\epsilon}^*}{2p_2\cdot k}v(p_2). \tag{12}$$

Taking the leading contribution as $k \to 0$ to collect the IR sensitive part, we find that

$$i\mathcal{M}_2 = -g_s\mu^\epsilon\frac{p_2\cdot\epsilon^*}{p_2\cdot k}\bar{u}(p_1)\mathcal{A}T^a v(p_2). \tag{13}$$

In summary, the approximation for the final-state quark and antiquark amplitude emitting a soft gluon with momentum k and the polarization vector ϵ^* can be expressed as

$$i(\mathcal{M}_1+\mathcal{M}_2) = i\mathcal{M}_0|_{\mathcal{A}\to\mathcal{A}'}, \tag{14}$$

where

$$\mathcal{A}' = g_s\mu^\epsilon\left[T^a\frac{p_1\cdot\epsilon^*}{p_1\cdot k}\mathcal{A} - \mathcal{A}T^a\frac{p_2\cdot\epsilon^*}{p_2\cdot k}\right]. \tag{15}$$

We consider the simplest case like the amputated amplitude for $\gamma^*\to Q\bar{Q}$ that is free of color structure. In such a case, the soft-gluon attachment leads to the replacement $\mathcal{A}\to\mathcal{A}' = C_{\rm soft}\mathcal{A}$, where

$$\begin{aligned} C_{\rm soft} &= g_s\mu^\epsilon T^a\left[\frac{p_1\cdot\epsilon^*}{p_1\cdot k} - \frac{p_2\cdot\epsilon^*}{p_2\cdot k}\right] \\ &= g_s\mu^\epsilon T^a\left[\frac{(\frac{1}{2}P+q)\cdot\epsilon^*}{(\frac{1}{2}P+q)\cdot k} - \frac{(\frac{1}{2}P-q)\cdot\epsilon^*}{(\frac{1}{2}P-q)\cdot k}\right]. \end{aligned} \tag{16}$$

It is manifest that the factor $C_{\rm soft}$ vanishes in the limit $q\to 0$. Thus the soft-gluon attachment is free of IR divergence as long as we consider the S-wave case with vanishing relative momentum $q = 0$. In case of the P-wave contribution, the cancellation does not hold and the attachment of a soft gluon leads to the IR divergence. Even in the S-wave case, the relativistic correction of order v_Q^2 at the amplitude level brings in the IR divergence. Therefore, in the cross section or the decay rate, the S-wave process acquires the IR divergence from relative order v_Q^4 or higher. Note that the soft gluon does not change the spin state of the $Q\bar{Q}$ pair while it carries the orbital angular momentum resulting in the transition with $\Delta L = \pm 1$. Because the gluon carries color, the color state of the $Q\bar{Q}$ state changes. This is a chromoelectric dipole transition. As a result, the perturbative corrections to the short-distance coefficient for the amplitude of the color-singlet $Q\bar{Q}$ pair acquires the sensitivity to the long-distance contribution. Without introducing a systematic separation of this long-distance contribution of QCD corrections, the factorization is not achieved in the CSM.

2.4 NRQCD factorization approach

We have observed that the perturbative QCD correction to the short-distance coefficient involving the annihilation decay or production of a color-singlet $Q\bar{Q}$ pair with the spectroscopic state ${}^{2S+1}L_J^{[1]}$ introduces IR sensitive contributions if one does not neglect v_Q. The divergent contribution involves the $Q\bar{Q}$ pair with the spectroscopic state ${}^{2S+1}L'^{[8]}_{J'}$, where $L' - L = \pm 1$ and S is invariant. Actually this contribution can be understood as the QCD correction to the state ${}^{2S+1}L'^{[8]}_{J'}$ through the long-distance QCD interactions. Based on this, the Fock-state expansion can be made to express a physical quarkonium state such as χ_{c0} as $|\chi_{c0}\rangle = |c\bar{c}({}^3P_0^{[1]})\rangle + |c\bar{c}({}^3S_1^{[8]}) + g_{\text{soft}}\rangle + \cdots$, while the CSM only allows $|\chi_{c0}\rangle = |c\bar{c}({}^3P_0^{[1]})\rangle$. The higher-order Fock state components are scaled with powers of v_Q that involves the strength of the corresponding long-distance interactions. The NRQCD factorization approach is a systematic theoretical formalism to treat these long-distance transition of a $Q\bar{Q}$ state to another as a power series of v_Q that are assumed to be small like $v_c^2 \sim 0.3$ and $v_b^2 \sim 0.1$. By making use of this formalism, the separation of the long-distance contribution out of the perturbative corrections to the short-distance coefficient is carried out. If such a separation is proved to be valid to all orders in α_s, then we call it the factorization theorem. A rigorous proof of the factorization theorem has been made for the electromagnetic annihilation decay and the annihilation decay into light hadrons [6] as a generalization of the work on the P-wave quarkonium decay [5]. The rigorousness of the proof is similar to that for the Drell-Yan process [32]. Since the introduction of the factorization conjecture for the inclusive quarkonium production in Ref. [6], a bunch of phenomenological studies and the corresponding experimental verifications have been carried out. We refer the readers to a recent review paper [33] for more details. The rigorous construction of the factorization theorem for the inclusive quarkonium production is still under way. Such efforts can be found, for example, in Refs. [34–37].

2.4.1 NRQCD Lagrangian

In order to describe the long-distance QCD interactions of the $Q\bar{Q}$ pair, it is convenient to reformulate the full QCD in terms of an effective theory called the NRQCD. The NRQCD Lagrangian is therefore equivalent to the full QCD except that in NRQCD the annihilation and/or decay of the $Q\bar{Q}$ pair is forbidden because they are of the scale above the ultraviolet (UV) cutoff $\sim m_Q$ of this effective theory. Thus the full QCD Lagrangian can be expressed as the sum of the Lagrangian $\mathcal{L}_{\text{light}}$ for the light degrees of freedom and that ($\mathcal{L}_{\text{heavy}}$) for the heavy quark as

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{light}} + \mathcal{L}_{\text{heavy}} + \delta\mathcal{L}. \tag{17}$$

$\mathcal{L}_{\text{light}}$ is identical to that for the full QCD as

$$\mathcal{L}_{\text{light}} = -\frac{1}{2}\text{tr}\, G_{\mu\nu}G^{\mu\nu} + \sum_q \bar{\Psi}_q(i\not{D} - m_q)\Psi_q, \tag{18}$$

where $G_{\mu\nu} = G_a^{\mu\nu} T^a$ and $G_a^{\mu\nu}$ is the field strength tensor of the gluon with color index a, Ψ_q is the Dirac spinor field for the light quark q, m_q is the mass of the light quark q, $D^\mu = (D_t, -\boldsymbol{D}) = \partial^\mu + ig_s A^\mu$ is the gauge-covariant derivative, and $A^\mu = (\phi, \boldsymbol{A}) = A_a^\mu T^a$ is the SU(3) color matrix valued gauge field. $\mathcal{L}_{\text{heavy}}$ is the contribution to the heavy quark and antiquark at LO in v_Q:

$$\mathcal{L}_{\text{heavy}} = \psi^\dagger \left(iD_t + \frac{\boldsymbol{D}^2}{2m_Q} \right) \psi + \chi^\dagger \left(iD_t - \frac{\boldsymbol{D}^2}{2m_Q} \right) \chi, \tag{19}$$

where ψ (χ) is the Pauli spinor field that annihilates (creates) a heavy quark (antiquark). The leading contribution $\mathcal{L}_{\text{heavy}}$ and the relativistic corrections $\delta\mathcal{L}_{\text{bilinear}}$ can be obtained by block-diagonalizing the full QCD Lagrangian for the heavy quark by making use of the Foldy-Wouthuysen-Tani transformation [38] except that we have subtracted the mass terms. Explicit expressions for the bilinear contributions can be found in Eq. (25) of Ref. [5]. The remaining contribution $\delta\mathcal{L}$ is expressed as

$$\delta\mathcal{L} = \delta\mathcal{L}_{\text{bilinear}} + \delta\mathcal{L}_{\text{four-fermion}}. \tag{20}$$

Here, $\delta\mathcal{L}_{\text{bilinear}}$ is bilinear in either the quark field or the antiquark field that can be read off from the block-diagonalized expression of the full-QCD Lagrangian for the heavy quark. A convenient set of Feynman rules for the NRQCD perturbation theory involving the bilinear contribution can be found in Table I of Ref. [39]. The annihilation decay of a $Q\bar{Q}$ pair cannot be reproduced in terms of $\mathcal{L}_{\text{heavy}}$ and $\delta\mathcal{L}_{\text{bilinear}}$. Such contribution can only be reproduced by the four-fermion terms,

$$\delta\mathcal{L}_{\text{four-fermion}} = \sum_n \frac{f_n(\Lambda)}{m_Q^{d_n-4}} \mathcal{O}_n(\Lambda), \tag{21}$$

where $f_n(\Lambda)$ is the short-distance coefficient that is insensitive to the long-distance interactions, $\mathcal{O}_n(\Lambda)$ is a higher dimensional operator, and d_n is the dimension of the operator $\mathcal{O}_n(\Lambda)$. Because the scale involving the creation or annihilation of a $Q\bar{Q}$ pair is above the UV cutoff Λ, such a relativistic effects can only be reproduced by adding four-fermion operators like $\psi^\dagger\chi\chi^\dagger\psi$. In the following section, we list definitions of four-quark operators and summarize a way to determine the short-distance coefficient by matching NRQCD Lagrangian on to the full QCD counterpart.

2.4.2 NRQCD operators

We list frequently used four-fermion NRQCD operators $\mathcal{O}_n(\Lambda)$ that appear in $\delta\mathcal{L}_{\text{four-fermion}}$. The dimension-6 operators of $\mathcal{O}_n(\Lambda)$ that involve light-hadronic decays are given by

$$\begin{aligned}
\mathcal{O}(^1S_0^{[1]}) &= \psi^\dagger\chi\chi^\dagger\psi, &(22\text{a})\\
\mathcal{O}(^3S_1^{[1]}) &= \psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi, &(22\text{b})\\
\mathcal{O}(^1S_0^{[8]}) &= \psi^\dagger T^a\chi\chi^\dagger T^a\psi, &(22\text{c})\\
\mathcal{O}(^3S_1^{[8]}) &= \psi^\dagger\boldsymbol{\sigma}T^a\chi\cdot\chi^\dagger\boldsymbol{\sigma}T^a\psi. &(22\text{d})
\end{aligned}$$

The color-singlet dimension-8 operators that involve light-hadronic decays are given by

$$\begin{aligned}
\mathcal{O}(^1P_1^{[1]}) &= \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\right)\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\right)\psi, &(23\text{a})\\
\mathcal{O}(^3P_0^{[1]}) &= \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma}\right)\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma}\right)\psi, &(23\text{b})\\
\mathcal{O}(^3P_1^{[1]}) &= \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma}\right)\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma}\right)\psi, &(23\text{c})\\
\mathcal{O}(^3P_2^{[1]}) &= \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\psi, &(23\text{d})
\end{aligned}$$

$$\mathcal{O}_2({}^1S_0^{[1]}) = \frac{1}{2}\left[\psi^\dagger\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\right)^2\psi + \text{h.c.}\right], \tag{23e}$$

$$\mathcal{O}_2({}^3S_1^{[1]}) = \frac{1}{2}\left[\psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{D}}\right)^2\psi + \text{h.c.}\right], \tag{23f}$$

$$\mathcal{O}_2({}^3S_1^{[1]},{}^3D_1^{[1]}) = \frac{1}{2}\left[\psi^\dagger\chi\chi^\dagger\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\psi + \text{h.c.}\right], \tag{23g}$$

where $\mathcal{O}_2({}^{2S+1}L_J^{[1]})$ represents the relativistic corrections of relative order v_Q^2 to the LO operator $\mathcal{O}({}^{2S+1}L_J^{[1]})$, and $A^{(i}B^{j)} = \frac{1}{2}\left(A^iB^j + A^jB^i\right) - \frac{1}{3}\delta^{ij}A^kB^k$ is the traceless symmetric component of the Cartesian tensor A^iB^j. The corresponding color-octet operators can be obtained in a similar manner as those for the dimension-6 case that are listed in Eq. (22).

The four-quark operators involving the electromagnetic annihilation decay of a $Q\bar{Q}$ pair is very similar to those for the light-hadronic decay except that the virtual state is the QCD vacuum. This replacement can be achieved by inserting the projection operator $|0\rangle\langle 0|$ that projects out the QCD vacuum. In addition, the color-octet operators are omitted because they have vanishing contribution to the vacuum-to-quarkonium matrix element. For example, the dimension-6 operators for electromagnetic annihilation decay of a $Q\bar{Q}$ pair are given by

$$\mathcal{O}^{\text{EM}}({}^1S_0^{[1]}) = \psi^\dagger\chi|0\rangle\langle 0|\chi^\dagger\psi, \tag{24a}$$

$$\mathcal{O}^{\text{EM}}({}^3S_1^{[1]}) = \psi^\dagger\boldsymbol{\sigma}\chi|0\rangle\cdot\langle 0|\chi^\dagger\boldsymbol{\sigma}\psi. \tag{24b}$$

2.4.3 NRQCD factorization formula

By making use of the effective Lagrangian $\delta L_{\text{four-fermion}}$ and the optical theorem, one can evaluate the inclusive decay rate of H as [6]

$$\begin{aligned}\Gamma(H\to X) &= 2\text{Im}\langle H|\delta\mathcal{L}_{\text{four-fermion}}|H\rangle \\ &= \sum_n \frac{2\text{Im}f_n(\Lambda)}{M^{d_n-4}}\langle H|\mathcal{O}_n(\Lambda)|H\rangle,\end{aligned} \tag{25}$$

where $|H\rangle$ is an eigenstate of the NRQCD Hamiltonian with the standard nonrelativistic normalization $\langle H(\boldsymbol{P}')|H(\boldsymbol{P})\rangle = (2\pi)^3\delta^{(3)}(\boldsymbol{P}'-\boldsymbol{P})$. In Appendix A of Ref. [6], one can find a systematic way to read off the short-distance coefficients $f_n(\Lambda)$ and $f_n^{\text{EM}}(\Lambda)$ by evaluating the $Q\bar{Q}\to Q\bar{Q}$ scattering amplitude in full QCD, taking the imaginary part, and comparing this expression with the linear combination of the corresponding perturbative NRQCD matrix elements $\langle Q\bar{Q}|\mathcal{O}_n(\Lambda)|Q\bar{Q}\rangle$ and $\langle Q\bar{Q}|\mathcal{O}_n^{\text{EM}}(\Lambda)|Q\bar{Q}\rangle$, respectively. Because the NRQCD matrix elements for H and $Q\bar{Q}$ have a common structure in the short-distance limit, the corresponding short-distance coefficients must be equal. This step is called the perturbative matching. The NRQCD long-distance matrix element (LDME) $\langle H|\mathcal{O}_n(\Lambda)|H\rangle$ cannot be computed perturbatively. Instead, some of them can be computed on the lattice [40–42] or they must be determined phenomenologically against experimental data involving the LDME. The numerical accuracy of the NRQCD factorization formula can be improved by extending the perturbative corrections to the short-distance coefficients and by considering as many NRQCD LDMEs that are suppressed in powers of v_Q as possible. In practice, it is impossible to extend the series in v_Q to all orders because it brings in too many LDMEs in comparison with the number of independent measurables. There is no way but to terminate the series at a certain order as long as the terminated factorization formula has uncertainties within desirable accuracies. The velocity scaling rules of NRQCD [6,43], that are derived in the Coulomb gauge on which the NRQCD LDMEs are formulated, can be used to determine the relative importance of the NRQCD LDMEs.

2.4.4 Matching

The short-distance coefficients $f_n(\Lambda)$ in Eq. (25) are insensitive to the long-distance nature of the quarkonium state and can be computed perturbatively. In order to determine the coefficients for the light-hadronic decays, we can consider the full-QCD amplitude $\mathcal{A}(Q\bar{Q} \to Q\bar{Q})$ for the scattering process $Q\bar{Q} \to Q\bar{Q}$ whose intermediate state consists of only light degrees of freedom. This full-QCD expression can be expanded as a linear combination of perturbative NRQCD matrix elements $\langle Q\bar{Q}|\mathcal{O}_n(\Lambda)|Q\bar{Q}\rangle$ as

$$\mathcal{A}(Q\bar{Q} \to Q\bar{Q}) = \sum_n \frac{f_n(\Lambda)}{M^{d_n-4}} \langle Q\bar{Q}|\mathcal{O}_n(\Lambda)|Q\bar{Q}\rangle, \tag{26}$$

which is given in Ref. [6]. Thus the determination of $f_n(\Lambda)$ is straightforward by reading off the coefficient of $\langle Q\bar{Q}|\mathcal{O}_n(\Lambda)|Q\bar{Q}\rangle$ in this expansion. This process is called the matching. Substituting the coefficients $f_n(\Lambda)$ into the NRQCD factorization formula in Eq. (25), one can predict the light-hadronic decay rate as long as the nonperturbative NRQCD LDMEs $\langle H|\mathcal{O}_n(\Lambda)|H\rangle$ are known.

The determination of the short-distance coefficients for the NRQCD factorization formula for the electromagnetic decay is quite similar to that for the light-hadronic decay. However, in the electromagnetic decay, the intermediate state for the scattering amplitude $Q\bar{Q} \to Q\bar{Q}$ must not include any colored particles.

3 Application to $J/\psi \to e^+e^-$

While the coefficients at LO in α_s are intrinsically free of IR divergence, higher-order corrections acquire soft singularities. The NRQCD factorization approach provides a systematic procedure to isolate such a long-distance contribution. This long-distance contribution will be identified as the α_s corrections to the NRQCD LDMEs and eventually we can obtain the short-distance coefficients that are insensitive to the long-distance interactions. We provide only a schematic description and further details can be found in Refs. [31, 44–46]. In this section, we demonstrate how to carry out the matching procedure to determine the short-distance coefficients of the NRQCD factorization formula by considering the relativistic corrections to $J/\psi \to e^+e^-$ at NLO in α_s. The relativistic corrections are to be computed to all orders in v_c keeping the ${}^3S_1^{[1]}$ contribution only.

The NRQCD factorization formula for $J/\psi \to e^+e^-$ at LO in v_c and at order α_s is given in Ref. [6] that can be read off from the result in Ref. [47]:

$$\Gamma[J/\psi \to e^+e^-] \;=\; \frac{8\pi e_c^2\alpha^2}{3M_{J/\Psi}^2}\left(1 - 4C_F\frac{\alpha_s}{\pi}\right)\langle J/\psi|\mathcal{O}^{\rm EM}({}^3S_1^{[1]})|J/\psi\rangle, \tag{27}$$

where e_c is the fractional electric charge of the charm quark, α is the electromagnetic fine structure constant, $C_F = (N_c^2 - 1)/(2N_c) = 4/3$, and the J/ψ mass can be set $M_{J/\psi} = 2m_c$ at order v_c^0. The order-α_s^2 and -α_s^3 corrections are also available in Ref. [48] and [49], respectively. The perturbative matching introduced in the previous section involves the analysis of the scattering amplitude $Q\bar{Q} \to Q\bar{Q}$. In case of the electromagnetic decay that omits QCD states in the final state, it is convenient to carry out perturbative matching at the amplitude level.

In quite a few processes involving charmonium production and decay, the corresponding predictions at LO in α_s and at LO in v_c have poor accuracies in comparison with the empirical data. It is partly due to the fact that both α_s and v_c for a charmonium process are not sufficiently small. For example, the prediction of the cross section for the exclusive process $e^+e^- \to J/\psi + \eta_c$ at B factories is smaller than the empirical values [50] by about an order of magnitude at LO in both α_s and v_c [51, 52]. A reasonable agreement between theory and experiment for this process was reached only after including the order-α_s corrections and the relativistic corrections resummed to all orders in v_c [53–55]. As is stated in Refs. [45], the NRQCD LDMEs involving relativistic corrections has intrinsic power UV divergence that must be subtracted in dimensional regularization, that is the most commonly used in phenomenology. This leads to large numerical uncertainties in determining the LDMEs even in the signs when we compute the LDMEs on the lattice which employs a hard cutoff regulator [45, 56]. The generalized Gremm-Kapustin relation [45, 57] can be used to resum relativistic corrections to a color-singlet contribution resulting in a significant improvement of the numerical accuracies in the theoretical predictions. We regularize the UV and IR singularities using dimensional regularization in $d = 4 - 2\epsilon$ space-time dimensions.

3.1 1-loop matching

The amplitude for $J/\psi \to e^+e^-$ can be factorized into the leptonic current, which is free of colored particles, and the hadronic part $\mathcal{A}_H^\mu$ [31],

$$-iee_c\, i\mathcal{A}_H^\mu = \langle 0|\mathcal{J}_{\rm EM}^\mu|H\rangle, \tag{28}$$

where $-e$ is the electric charge, H is the spin-triplet S-wave charmonium (J/ψ), and $\mathcal{J}_{\rm EM}^\mu$ is the heavy-quark electromagnetic current

$$\mathcal{J}_{\rm EM}^\mu = (-iee_c)\bar\psi\gamma^\mu\psi. \tag{29}$$

The conservation of the electromagnetic current restricts that $i\mathcal{A}_H^0 = 0$ in the quarkonium rest frame. In that frame, the spatial component must be a linear combination of the NRQCD matrix elements as

$$i\mathcal{A}_H^i = \sqrt{2m_H}\sum_n c_n\langle 0|\mathcal{O}_n^i|H\rangle, \tag{30}$$

where c_n are short-distance coefficients, $\mathcal{O}_n^i$ are the NRQCD operators, and m_H is the quarkonium mass. The overall factor $\sqrt{2m_H}$ is introduced because the quarkonium state $|H\rangle$ on the right side is normalized nonrelativistically while the full expression is normalized relativistically. We can think of the corresponding amplitude for the free $Q\bar Q$ pair, where $Q = c$. The short-distance nature of the process for the $Q\bar Q$ pair must be identical to that of the H. Therefore, $Q\bar Q$ must be in a state $Q\bar Q(^3S_1^{[1]})$. Thus the two processes have a common set of short-distance coefficients c_n. Then the coefficients c_n can be determined from the following matching formula:

$$i\mathcal{A}_{Q\bar Q}^i = \sum_n c_n\langle 0|\mathcal{O}_n^i|Q\bar Q\rangle, \tag{31}$$

where $\mathcal{A}_{Q\bar Q}^i$ is the amplitude for the on-shell $Q\bar Q$ pair that has the same short-distance process as $\mathcal{A}_H^i$ and $|Q\bar Q\rangle$ is the on-shell $Q\bar Q$ pair state. At LO in α_s, the short-distance coefficients are free of UV and IR divergences. If we consider the perturbation of the amplitude corrected to order α_s, then the matching formula is modified as

$$i\mathcal{A}_{Q\bar Q}^{i(0)} = \sum_n c_n^{(0)}\langle 0|\mathcal{O}_n^i|Q\bar Q\rangle^{(0)}, \tag{32}$$

$$i\mathcal{A}_{Q\bar Q}^{i(0)} + i\mathcal{A}_{Q\bar Q}^{i(1)} = \sum_n (c_n^{(0)} + c_n^{(1)})\langle 0|\mathcal{O}_n^i|Q\bar Q\rangle^{(0)} + \sum_n c_n^{(0)}\langle 0|\mathcal{O}_n^i|Q\bar Q\rangle^{(1)}, \tag{33}$$

where the superscript (i) stands for the order in α_s. As the first step, we can read off the coefficients $c_n^{(0)}$ from the expression for $i\mathcal{A}_{Q\bar Q}^{i(0)}$. Note that the NRQCD LDME $\langle 0|\mathcal{O}_n^i|Q\bar Q\rangle^{(i)}$ can be computed perturbatively although the LDME $\langle 0|\mathcal{O}_n^i|H\rangle$ for a quarkonium state is nonperturbative. As the next step, we subtract Eq. (32) from Eq. (33).

$$i\mathcal{A}_{Q\bar Q}^{i(1)} = \sum_n c_n^{(1)}\langle 0|\mathcal{O}_n^i|Q\bar Q\rangle^{(0)} + \sum_n c_n^{(0)}\langle 0|\mathcal{O}_n^i|Q\bar Q\rangle^{(1)}. \tag{34}$$

The first term is the order-α_s correction to the short-distance coefficient multiplied by the NRQCD ME at LO. The second term is the LO short-distance coefficient multiplied by the NRQCD ME at NLO. The IR divergent contribution of the short-distance coefficient, for example in the CSM, at NLO in α_s is to be identified as the second term in this expression. In this manner, the separation of long-distance contribution from the short-distance factor is being carried out order by order systematically.

In order to find the relativistic corrections, it is convenient to define the following perturbative NRQCD LDMEs for the $Q\bar{Q}$ pair:

$$\begin{aligned}
\langle 0|\mathcal{O}^i_{1n}|Q\bar{Q}\rangle^{(0)} &= \langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\right)^{2n}\sigma^i\psi|Q\bar{Q}\rangle^{(0)} \\
&= \boldsymbol{q}^{2n}\eta^\dagger\sigma^i\xi, \qquad (35a)\\
\langle 0|\mathcal{O}^i_{2n}|Q\bar{Q}\rangle^{(0)} &= \langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\right)^{2n-2}\left(-\frac{i}{2}\overleftrightarrow{\nabla}^i\right)\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\cdot\boldsymbol{\sigma}\right)\psi|Q\bar{Q}\rangle^{(0)} \\
&= \boldsymbol{q}^{2n-2}q^i\eta^\dagger\boldsymbol{q}\cdot\boldsymbol{\sigma}\xi, \qquad (35b)
\end{aligned}$$

where ξ and η are Pauli spinors for the Q and $\bar{Q}$, respectively, $q = (p_1 - p_2)/2$, p_1 and p_2 are the momenta for Q and $\bar{Q}$, respectively, and $\boldsymbol{q}$ is the spatial component of q in the $Q\bar{Q}$ rest frame.

Then the one-loop matching formula reduces into

$$i\mathcal{A}^{i(1)}_{Q\bar{Q}} = \sum_n c^{(1)}_{1n}\langle 0|\mathcal{O}^i_{1n}|Q\bar{Q}\rangle^{(0)} + \sum_n c^{(1)}_{2n}\langle 0|\mathcal{O}^i_{2n}|Q\bar{Q}\rangle^{(0)} + i\mathcal{A}^{\mathrm{NRQCD},i}_{Q\bar{Q}}, \qquad (36)$$

where $i\mathcal{A}^{\mathrm{NRQCD},i}_{Q\bar{Q}}$ represents the contribution that is a product of the short-distance coefficient at LO in α_s and the perturbative NRQCD MEs at NLO in α_s. For further information on the matching procedure, refer to Ref. [31].

3.2 computation of $\mathcal{A}^{\mu}_{Q\bar{Q}}$

To any order in α_s, the $Q\bar{Q}$ amplitude corresponding to the hadronic current $\mathcal{A}^\mu_H$ is of the form [31]

$$i\mathcal{A}^\mu_{Q\bar{Q}} = \bar{v}(p_2)\left[Z_Q(1+\Lambda)\gamma^\mu + Bq^\mu\right]u(p_1), \qquad (37)$$

where Z_Q is the heavy-quark wave function renormalization factor, Λ is the multiplicative factor for the vertex correction, and B is the multiplicative correction factor coming from the magnetic moment contribution that appears from order α_s corrections to the vertex. At LO in α_s, $Z_Q = 1$ and $\Lambda = 0$ and $B = 0$. The B term contributes to the S-wave spin-triplet contribution only from the corrections of order α_s with relativistic corrections. One can find the values for Z_Q and Λ at order α_s keeping only the leading contributions in powers in v_c, for example, in Refs. [31] and [58] and as

$$\begin{aligned}
Z_Q &= 1 + \frac{\alpha_s C_F}{4\pi}\left(-\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{2}{\epsilon_{\mathrm{IR}}} - 3\log\frac{4\pi\mu^2 e^{-\gamma_{\mathrm{E}}}}{m_c^2} - 4\right), \qquad (38a)\\
\Lambda &= \frac{\alpha_s C_F}{4\pi}\left\{\frac{1}{\epsilon_{\mathrm{UV}}} + \frac{2}{\epsilon_{\mathrm{IR}}} + 3\log\frac{4\pi\mu^2 e^{-\gamma_{\mathrm{E}}}}{m_c^2} - 4 + \frac{1}{v_c}\left(\pi^2 - \frac{i\pi}{\epsilon_{\mathrm{IR}}} - i\pi\log\frac{\pi\mu^2 e^{-\gamma_{\mathrm{E}}}}{\boldsymbol{q}^2}\right)\right\}, \qquad (38b)
\end{aligned}$$

where μ is the dimensional regularization scale, γ_{E} is the Euler-Marscheroni constant, and the subscript of $1/\epsilon$ indicates the origin of divergence.

If we keep the leading contributions in v_c only at this order, then the contribution proportianal to q^μ in the current in Eq. (37) does not appear because the corresponding NRQCD ME is to be neglected. Then, it is sufficient to know

$$Z_Q(1+\Lambda) = 1 + \frac{\alpha_s C_F}{4\pi}\left\{-8 + \frac{1}{v_c}\left[\pi^2 - \frac{i\pi}{\epsilon_{\mathrm{IR}}} - i\pi\log\frac{\pi\mu^2 e^{-\gamma_{\mathrm{E}}}}{\boldsymbol{q}^2}\right]\right\} + O(\alpha_s^2). \qquad (39)$$

The result shows that the IR divergence survives while the UV divergence cancels. In addition, there is a non-analytic contribution as $v_c \to 0$ which is originated from the Coulomb interaction.

According to Ref. [31], the correction factors contributing to the S-wave spin-triplet state resummed to all orders in v_c are given by

$$\begin{aligned} Z_Q(1+\Lambda) &= 1 + \frac{\alpha_s C_F}{4\pi}\Big\{ 2[(1+\delta^2)L(\delta)-1]\left(\frac{1}{\epsilon_{\text{IR}}} + \log\frac{4\pi\mu^2 e^{-\gamma_{\text{E}}}}{m_c^2}\right) + 6\delta^2 L(\delta) \\ &\quad -4(1+\delta^2)K(\delta) - 4 + (1+\delta^2)\left[\frac{\pi^2}{\delta} - \frac{i\pi}{\delta}\left(\frac{1}{\epsilon_{\text{IR}}} + \log\frac{\pi\mu^2 e^{-\gamma_{\text{E}}}}{\boldsymbol{q}^2} + \frac{3\delta^2}{1+\delta^2}\right)\right]\Big\}, \end{aligned} \tag{40}$$

$$B = \frac{\alpha_s C_F}{4\pi}\frac{1-\delta^2}{m_c}\left[2L(\delta) - \frac{i\pi}{\delta}\right], \tag{41}$$

where

$$\delta = \frac{v_c}{\sqrt{1+v_c^2}}, \tag{42a}$$

$$L(\delta) = \frac{1}{2\delta}\log\frac{1+\delta}{1-\delta}, \tag{42b}$$

$$K(\delta) = \frac{1}{4\delta}\left[\text{Li}_2\left(\frac{2\delta}{1+\delta}\right) - \text{Li}_2\left(-\frac{2\delta}{1+\delta}\right)\right], \tag{42c}$$

and Li_2 is the Spence function:

$$\text{Li}_2(x) = \int_x^0 dt\,\frac{\log(1-t)}{t}. \tag{42d}$$

We use the nonrelativistic normalization for the spinors to find that

$$\bar{v}(p_2)\gamma^i u(p_1) = \eta^\dagger\sigma^i\xi - \frac{q^i\eta^\dagger\boldsymbol{q}\cdot\boldsymbol{\sigma}\xi}{E(E+m_c)}, \tag{43a}$$

$$\bar{v}(p_2)q^i u(p_1) = -\frac{q^i\eta^\dagger\boldsymbol{q}\cdot\boldsymbol{\sigma}\xi}{E}, \tag{43b}$$

where $E = \sqrt{m_c^2 + \boldsymbol{q}^2}$ that is the energy of the quark or antiquark in the $Q\bar{Q}$ rest frame. Substituting Eq. (43) into Eq. (37) and expanding in powers of v_c, we obtain

$$\begin{aligned} i\mathcal{A}^i_{Q\bar{Q}} &= \eta^\dagger\sigma^i\xi\Big[1 + \frac{\alpha_s C_F}{4\pi}\Big\{\frac{8v_c^2}{3}\left(\frac{1}{\epsilon_{\text{IR}}} + \log\frac{4\pi\mu^2 e^{-\gamma_{\text{E}}}}{m_c^2}\right) - 8 + \frac{2v_c^2}{9} \\ &\quad + \left(1 + \frac{3v_c^2}{2}\right)\left[\frac{\pi^2}{v_c} - \frac{i\pi}{v_c}\left(\frac{1}{\epsilon_{\text{IR}}} + \log\frac{\pi\mu^2 e^{-\gamma_{\text{E}}}}{\boldsymbol{q}^2}\right)\right] - 3i\pi v_c\Big] \\ &\quad - \frac{q^i\eta^\dagger\boldsymbol{q}\cdot\boldsymbol{\sigma}\xi}{2m_c^2}\left\{1 + \frac{\alpha_s C_F}{4\pi}\left[-4 + \frac{\pi^2}{v_c} - \frac{i\pi}{v_c}\left(\frac{1}{\epsilon_{\text{IR}}} + \log\frac{\pi\mu^2 e^{-\gamma_{\text{E}}}}{\boldsymbol{q}^2} + 2\right)\right]\right\} + O(v_c^3). \end{aligned} \tag{44}$$

In comparison with the case in Eq. (39), there are a lot of terms that carries IR divergences and non-analytic Coulomb contributions. Note that the term proportional to $q^i\eta^\dagger\boldsymbol{q}\cdot\boldsymbol{\sigma}\xi$ contributes to both the S-wave spin-triplet state and D-wave state.

3.3 computation of $\mathcal{A}^{\text{NRQCD},\mu}_{Q\bar{Q}}$

Next we compute the amplitude $i\mathcal{A}^{\text{NRQCD},\mu}_{Q\bar{Q}}$ by making use of the NRQCD perturbation theory rather than the full QCD. Again, the amplitude must be of the form

$$i\mathcal{A}^{\text{NRQCD},\mu}_{Q\bar{Q}} = \bar{v}(p_2)\left[Z_Q^{\text{NRQCD}}(1+\Lambda^{\text{NRQCD}})\gamma^\mu + B^{\text{NRQCD}}q^\mu\right]u(p_1). \tag{45}$$

If we follow the standard approach that is described in Ref. [5], it is practically impossible to carry out the NRQCD perturbation because there are infinite number of Feynman rules that involve the relativistic corrections to all orders in v_c. Fortunately, the authors of Ref. [31] have devised a very convenient way to achieve this goal without applying the infinite number of Feynman rules. Instead, they have introduced an equivalent way in which they evaluate the loop integral to remove the UV power divergent scaleless integrals. Here, we quote the results of Ref. [31]:

$$\begin{aligned} Z_Q^{\text{NRQCD}}(1+\Lambda^{\text{NRQCD}}) &= 1+\frac{\alpha_s C_F}{4\pi}\Bigg\{2[(1+\delta^2)L(\delta)-1]\left(\frac{1}{\epsilon_{\text{IR}}}-\frac{1}{\epsilon_{\text{UV}}}\right) \\ &\quad +(1+\delta^2)\left[\frac{\pi^2}{\delta}-\frac{i\pi}{\delta}\left(\frac{1}{\epsilon_{\text{IR}}}+\log\frac{\pi\mu^2 e^{-\gamma_{\text{E}}}}{\boldsymbol{q}^2}+\frac{3\delta^2}{1+\delta^2}\right)\right]\Bigg\}, \qquad (46a) \\ B^{\text{NRQCD}} &= \frac{\alpha_s C_F}{4\pi}\frac{1-\delta^2}{m_c}\left[-\frac{i\pi}{\delta}\right]. \qquad (46b) \end{aligned}$$

If we apply Eqs. (43) and (46) into Eq. (45) and expand in powers of v_c, then we find that

$$\begin{aligned} i\mathcal{A}_{Q\bar{Q}}^{\text{NRQCD},i} &= \eta^\dagger\sigma^i\xi\Bigg\{1+\frac{\alpha_s C_F}{4\pi}\Bigg\{\frac{8v_c^2}{3}\left(\frac{1}{\epsilon_{\text{IR}}}-\frac{1}{\epsilon_{\text{UV}}}\right)+ \\ &\quad \left(1+\frac{3v_c^2}{2}\right)\left[\frac{\pi^2}{v_c}-\frac{i\pi}{v_c}\left(\frac{1}{\epsilon_{\text{IR}}}+\log\frac{\pi\mu^2 e^{-\gamma_{\text{E}}}}{\boldsymbol{q}^2}\right)\right]-3i\pi v_c\Bigg\} \\ &\quad -\frac{q^i\eta^\dagger\boldsymbol{q}\cdot\boldsymbol{\sigma}\xi}{2m_c^2}\left\{1+\frac{\alpha_s C_F}{4\pi}\left[\frac{\pi^2}{v_c}-\frac{i\pi}{v_c}\left(\frac{1}{\epsilon_{\text{IR}}}+\log\frac{\pi\mu^2 e^{-\gamma_{\text{E}}}}{\boldsymbol{q}^2}+2\right)\right]\right\}+O(v_c^3). \qquad (47) \end{aligned}$$

3.4 determination of short-distance coefficients

Now we are ready to apply the matching condition in Eq. (36) to determine the short-distance coefficients. In this step, the IR divergent contribution and the non-analytic Coulomb divergent terms in Eqs. (44) and (47) cancel. The remaining UV divergence is to be removed by renormalizing the NRQCD operator in the modified minimal subtraction ($\overline{\text{MS}}$) scheme as

$$\chi^\dagger\sigma^i\psi = \left[\chi^\dagger\sigma^i\psi\right]_{\overline{\text{MS}}}-\frac{(4\pi e^{-\gamma_{\text{E}}})^\epsilon}{\epsilon_{\text{UV}}}\frac{2\alpha_s C_F}{3\pi m_c^2}\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\right)^2\sigma^i\psi, \qquad (48)$$

where the subscript $\overline{\text{MS}}$ indicates that the corresponding operator is renormalized in the $\overline{\text{MS}}$ scheme. The resultant short-distance coefficients are free of long-distance sensitivities:

$$\begin{aligned} c_{10}^{(1)} &= \frac{\alpha_s C_F}{4\pi}(-8), \qquad (49a) \\ c_{11}^{(1)} &= \frac{\alpha_s C_F}{4\pi}\frac{1}{m_c^2}\left(\frac{2}{9}+\frac{8}{3}\log\frac{\mu_{\text{NRQCD}}^2}{m_c^2}\right), \qquad (49b) \\ c_{21}^{(1)} &= \frac{\alpha_s C_F}{4\pi}\frac{2}{m_c^2}, \qquad (49c) \\ & \qquad (49d) \end{aligned}$$

where μ_{NRQCD} is the NRQCD factorization scale. At LO in α_s through order v_c^2, the short-distance coefficients are given by [29, 31, 59]

$$\begin{aligned} c_{1n}^{(0)} &= \delta_{n0}, \qquad (50a) \\ c_{21}^{(0)} &= -\frac{1}{2m_c^2}. \qquad (50b) \end{aligned}$$

In order to apply this factorization formula for an S-wave spin-triplet state like J/ψ, we need to project out the S-wave contribution out of $i\mathcal{A}^i_{Q\bar{Q}}$. Thus we decompose the operator $\mathcal{O}^i_{2n}$ into

$$\mathcal{O}^i_{2n} = \frac{\mathcal{O}^i_{1n}}{d-1} + \mathcal{O}^i_{Dn}, \tag{51}$$

where the D-wave operator $\mathcal{O}^i_{Dn}$ is defined by

$$\mathcal{O}^i_{Dn} = \chi^\dagger \left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\right)^{2n-2} \left[\left(-\frac{i}{2}\overleftrightarrow{\nabla}^i\right)\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\cdot\boldsymbol{\sigma}\right) - \frac{1}{d-1}\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\right)^2\sigma^i\right]\psi. \tag{52}$$

The S-wave component in $i\mathcal{A}^i_{Q\bar{Q}}$ through order $\alpha_s v^2$ is

$$\begin{aligned}
i\mathcal{A}^i_{Q\bar{Q}}\Big|_{S-\mathrm{wave}} &= (c^{(0)}_{10} + c^{(1)}_{10})\langle 0|\mathcal{O}^i_{10}|Q\bar{Q}\rangle + \left(c^{(1)}_{11} + \frac{c^{(0)}_{21} + c^{(1)}_{21}}{d-1}\right)\langle 0|\mathcal{O}^i_{11}|Q\bar{Q}\rangle \\
&= \left(1 - \frac{8\alpha_s C_F}{4\pi}\right)\langle 0|\chi^\dagger\sigma^i\psi|Q\bar{Q}\rangle + \left[\frac{\alpha_s C_F}{4\pi}\left(\frac{2}{9} + \frac{8}{3}\log\frac{\mu^2_{\mathrm{NRQCD}}}{m_c^2}\right)\right. \\
&\quad \left. + \frac{1}{d-1}\left(-\frac{1}{2} + \frac{2\alpha_s C_F}{4\pi}\right)\right]\frac{\langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\right)^2\sigma^i\psi|Q\bar{Q}\rangle}{m_c^2} \\
&= \left(1 - \frac{8\alpha_s C_F}{4\pi}\right)\langle 0|\chi^\dagger\sigma^i\psi|Q\bar{Q}\rangle \\
&\quad + \left[-\frac{1}{6} + \frac{\alpha_s C_F}{4\pi}\left(\frac{8}{9} + \frac{8}{3}\log\frac{\mu^2_{\mathrm{NRQCD}}}{m_c^2}\right)\right]\frac{\langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\right)^2\sigma^i\psi|Q\bar{Q}\rangle}{m_c^2}.
\end{aligned} \tag{53}$$

Our final results for the NRQCD factorization formula for the hadronic current that contributes to the leptonic decay of the S-wave spin-triplet state is

$$\begin{aligned}
i\mathcal{A}^i_{H}\Big|_{S-\mathrm{wave}} &= \sqrt{2m_H}\left(1 - \frac{8\alpha_s C_F}{4\pi}\right)\langle 0|\chi^\dagger\sigma^i\psi|H\rangle \\
&\quad + \sqrt{2m_H}\left[-\frac{1}{6} + \frac{\alpha_s C_F}{4\pi}\left(\frac{8}{9} + \frac{8}{3}\log\frac{\mu^2_{\mathrm{NRQCD}}}{m_c^2}\right)\right]\frac{\langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\boldsymbol{\nabla}}\right)^2\sigma^i\psi|H\rangle}{m_c^2}.
\end{aligned} \tag{54}$$

4 Summary

In this lecture, we have briefly reviewed the NRQCD factorization approach to describe the quarkonium production and decay. In the NRQCD factorization formula, the long-distance nature of heavy quarkonium is factorized into the NRQCD long-distance matrix elements (LDMEs) and a physical measurable is expressed as a linear combination of LDMEs. We have reviewed that the infrared sensitivity emerges if we apply the perturbation theory to enhance the theoretical accuracies in the short-distance contributions. This infrared divergence always appears as long as one does not neglect v_Q and there exists the non-analytic Coulomb singularities as $v_Q \to 0$. The color-singlet model breaks down because this theory is not equipped with a systematic procedure to cure this problem. In the NRQCD factorization approach, the corresponding short-distance factors are free of infrared sensitivities. In order to improve the numerical accuracies of such a measurable, one can compute corrections in powers of α_s and v_Q. One can truncate the series in v_Q by considering the velocity scaling rules that estimate the relative numerical importance of a long-distance process in powers of v_Q.

A systematic procedure to isolate such a long-distance interactions out of the correction terms in the short-distance coefficients is called matching. The matching procedure makes use of the fact that the short-distance coefficients for the NRQCD factorization formula for a quarkonium must be identical to those for the on-shell $Q\bar{Q}$ counterparts. While one cannot compute the NRQCD LDMEs for a quarkonium, one can compute the NRQCD LDMEs for an on-shell $Q\bar{Q}$ pair under the NRQCD perturbation theory. By comparing the $Q\bar{Q}$ amplitude computed in the full QCD with that computed in the NRQCD perturbation theory, one can isolate the infrared sensitive contributions and absorb them into the NRQCD LDMEs. The resultant short-distance factors are free of infrared divergences. A standard renormalization procedure can be applied to absorb the remaining UV divergences. As a heuristic example of finding the NRQCD factorization formula for a specific process, we have demonstrated the matching procedure of determining the short-distance coefficients involving the leptonic decay of the S-wave spin-triplet state.

In the latter part of the lecture, we have reviewed the NRQCD factorization approach for J/ψ hadroproduction and polarization. Due to the page limit of the proceedings contribution, we were not able to describe these subjects in the text. We refer the readers to Ref. [60] for details.

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