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Recapitulation of Electromagnetism

T. Flisgen, J. Heller, and U. van Rienen*
University of Rostock, Germany

Abstract
Electromagnetism plays a vital role in many areas of our modern life, such as telecommunication, navigation, electrical drives, and electrical grids. Moreover, electromagnetism is of special relevance for particle accelerators. Magnetic fields are used to bend and to focus bunches of charged particles, whereas electric fields are used to increase the energy of charged particles. This contribution recapitulates selected fundamentals of the theory of electromagnetic fields and sketches thereby important facts related to vector analysis, electric fields, magnetic fields, and their mutual interaction.

Keywords
Maxwell's equations; vector analysis; conservation principles; continuity constraints of fields; wave equations; field attenuation in conductors.

1 Introduction

Today's daily life would not be as comfortable as it is without a profound understanding of electromagnetic phenomena. This report summarizes the most relevant foundations of electromagnetic theory. However, this article does not discuss electromagnetics in a complete manner, owing to lack of space. The report can be considered as a reminder and as an overview. It is meant to be an appetizer and an invitation for more profound studies with textbooks [1–8]. These textbooks describe electromagnetism in a very detailed way and often comprise hundreds of pages.

Subsection 1.1 presents the quantities describing electromagnetic effects and introduces the famous four equations, which relate these quantities to each other. In this context, important vector algebraic operators, such as the divergence and the curl operator, are introduced and defined. Subsequently, forces acting on charged particles as a result of electromagnetic fields are discussed in Subsection 1.2. Subsections 1.3 and 1.4 discuss conservation principles, such as the conservation of charge and the conservation of energy. Section 2 describes the interaction of electromagnetic fields with matter. A special focus is on continuity constraints of fields at interfaces between different materials. Subsequently, the governing equations for static electric and for static magnetic fields are derived in Sections 3 and 4, respectively. In a next step, wave equations for electric and magnetic waves are deduced from Maxwell's equations in Section 5. A simple example for a solution of the wave equations is provided as an example. Section 6 discusses the attenuation of fields in a conductor and Section 7 concludes the report.

Note that a consistent nomenclature is employed throughout the entire report. Vectors are denoted with boldfaced letters. Where quantities depend on time or space, they are highlighted with t or $\mathbf{r}$, respectively. For instance, a transient spatially dependent vector field is denoted $\mathbf{D}(\mathbf{r},t)$. In the case of harmonic fields, complex-valued quantities arise, which are highlighted by underscores, e.g., $\mathbf{D}(\mathbf{r},t) = \mathrm{Re}\,[\underline{\mathbf{D}}(\mathbf{r},t)]$. The complex-valued quantity $\underline{\mathbf{D}}(\mathbf{r},t)$ can be split up into a phasor $\underline{\mathbf{D}}(\mathbf{r})$ and an exponential function $\exp(\mathrm{j}\omega t)$ accounting for the time dependency, i.e., $\underline{\mathbf{D}}(\mathbf{r},t) = \underline{\mathbf{D}}(\mathbf{r})\exp(\mathrm{j}\omega t)$, where $\mathrm{j}^2 = -1$.

1.1 Maxwell's equations

The following text closely follows the introductory section in Ref. [9]. In the nineteenth century, James Clerk Maxwell summarized and improved important laws on electric and magnetic phenomena [10]

*Currently at Helmholtz Zentrum Berlin.

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originating from experimental observations. In the following decades, Oliver Heaviside and Josiah Willard Gibbs independently reformulated the equations into the now commonly known form. These so-called Maxwell's equations are the foundation of classical electrodynamics. The integral form of Maxwell's equations is given by

$$\oint\!\!\!\oint_{\partial\mathbf{\Omega}} \mathbf{D}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = \iiint_{\mathbf{\Omega}} \rho(\mathbf{r},t)\,\mathrm{d}V\,, \tag{1}$$

$$\oint\!\!\!\oint_{\partial\mathbf{\Omega}} \mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A} = 0\,, \tag{2}$$

$$\oint_{\partial\mathbf{\Gamma}} \mathbf{E}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = -\iint_{\mathbf{\Gamma}} \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{A}\,, \tag{3}$$

$$\oint_{\partial\mathbf{\Gamma}} \mathbf{H}(\mathbf{r},t)\cdot \mathrm{d}\mathbf{s} = \iint_{\mathbf{\Gamma}} \left(\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t)\right)\cdot \mathrm{d}\mathbf{A}\,. \tag{4}$$

These four equations specify the relationship between the *electric flux density* $\mathbf{D}(\mathbf{r},t)$, the *electric charge density* $\rho(\mathbf{r},t)$, the *magnetic flux density* $\mathbf{B}(\mathbf{r},t)$, the *electric field strength* $\mathbf{E}(\mathbf{r},t)$, the *magnetic field strength* $\mathbf{H}(\mathbf{r},t)$, and the *electric current density* $\mathbf{J}(\mathbf{r},t)$. All six vector-valued quantities depend on the spatial co-ordinate $\mathbf{r}$ and the time t. *Gauss's law* (1) states that the total electric flux through the closed boundary $\partial\mathbf{\Omega}$ of a domain $\mathbf{\Omega}$ is equal to the total charge contained in the domain. Figure 1(a) depicts a volume containing the electric charge Q and indicates the electric displacement field as well as the infinitely small area element. *Gauss's law for magnetism* (2) requires the total magnetic flux through the closed boundary $\partial\mathbf{\Omega}$ of a domain $\mathbf{\Omega}$ to be equal to zero. Figure 1(b) shows an example volume, as well as the magnetic flux density and the infinitely small area element. In other words: the field lines of the magnetic flux density have neither a source nor a sink but are closed, owing to the absence of magnetic charges. *Faraday's law of induction* (3) states that the negative time derivative of the total magnetic flux through a surface $\mathbf{\Gamma}$ equals the integration of the electric field along the closed boundary $\partial\mathbf{\Gamma}$ of the surface, as highlighted in Fig. 1(c). Similarly, *Ampère's law with Maxwell's extension* (4) states that the total current through a surface $\mathbf{\Gamma}$ is equal to the integration of the magnetic field along the closed boundary $\partial\mathbf{\Gamma}$ of the surface, as sketched in Fig. 1(d). It is worth noting that Maxwell's equations in their integral form hold for arbitrary surfaces $\mathbf{\Gamma}$ or volumes $\mathbf{\Omega}$. For instance, $\mathbf{\Gamma}$ and $\mathbf{\Omega}$ can be independently chosen from material boundaries.

Maxwell's equations, in their integral representation (Eqs. (1)–(4)) can be transferred to their differential form,

$$\nabla\cdot\mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t)\,, \tag{5}$$

$$\nabla\cdot\mathbf{B}(\mathbf{r},t) = 0\,, \tag{6}$$

$$\nabla\times\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\,, \tag{7}$$

$$\nabla\times\mathbf{H}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{J}(\mathbf{r},t)\,, \tag{8}$$

by considering infinitely small integration surfaces $\mathbf{\Gamma}$ and integration volumes $\mathbf{\Omega}$. Here, $\nabla\cdot$ is the divergence operator, which contains spatial derivatives. The divergence operator acts on a vector field and delivers scalar fields. These scalar fields describe the source strength of the field per unit volume. Note that in some textbooks the divergence is denoted div. In a Cartesian co-ordinate system, the divergence of the vector field $\mathbf{D}(x,y,z)$ is given by

$$\nabla\cdot\mathbf{D}(x,y,z) = \mathrm{div}\,\mathbf{D}(x,y,z) = \frac{\partial}{\partial x}D_x(x,y,z) + \frac{\partial}{\partial y}D_y(x,y,z) + \frac{\partial}{\partial z}D_z(x,y,z)\,, \tag{9}$$

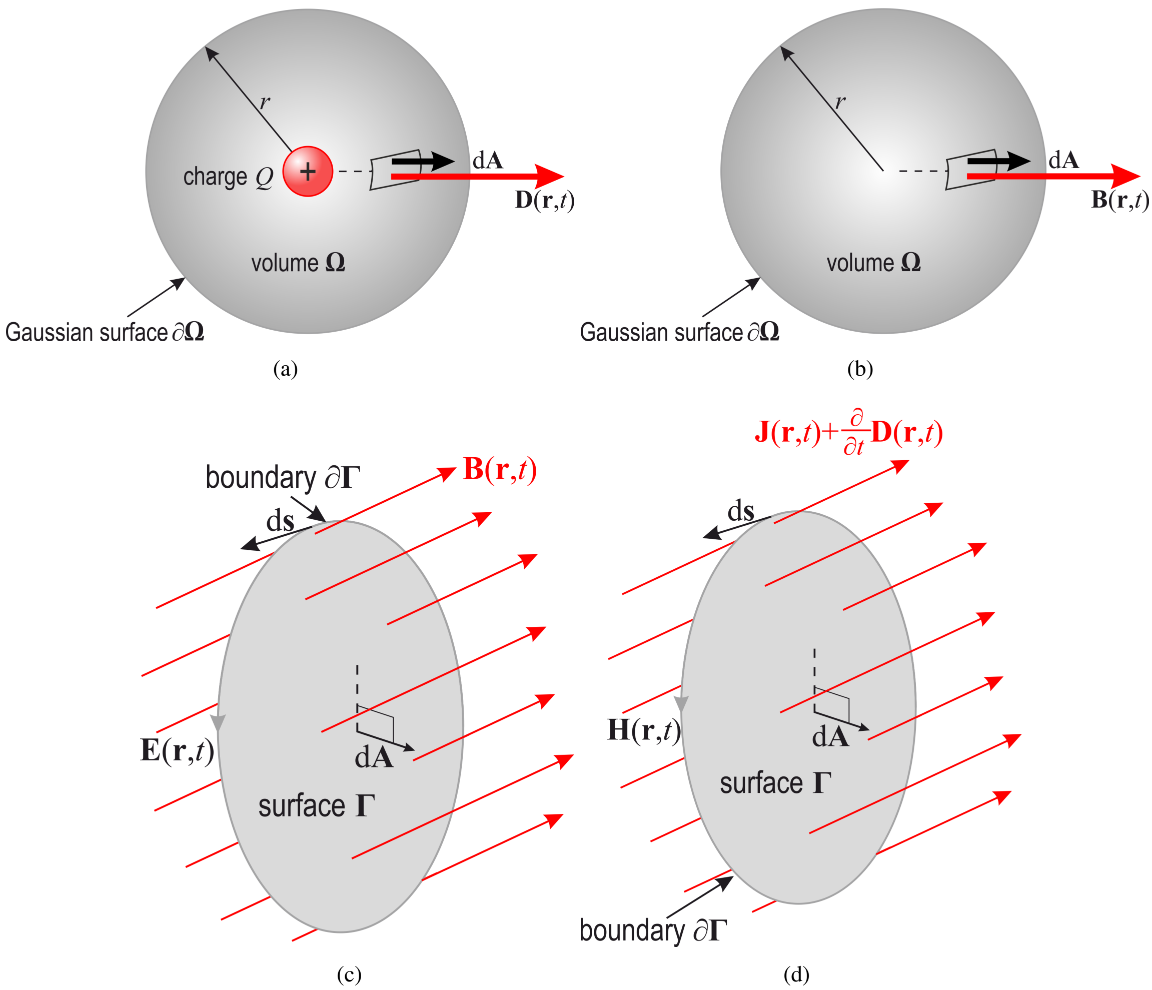
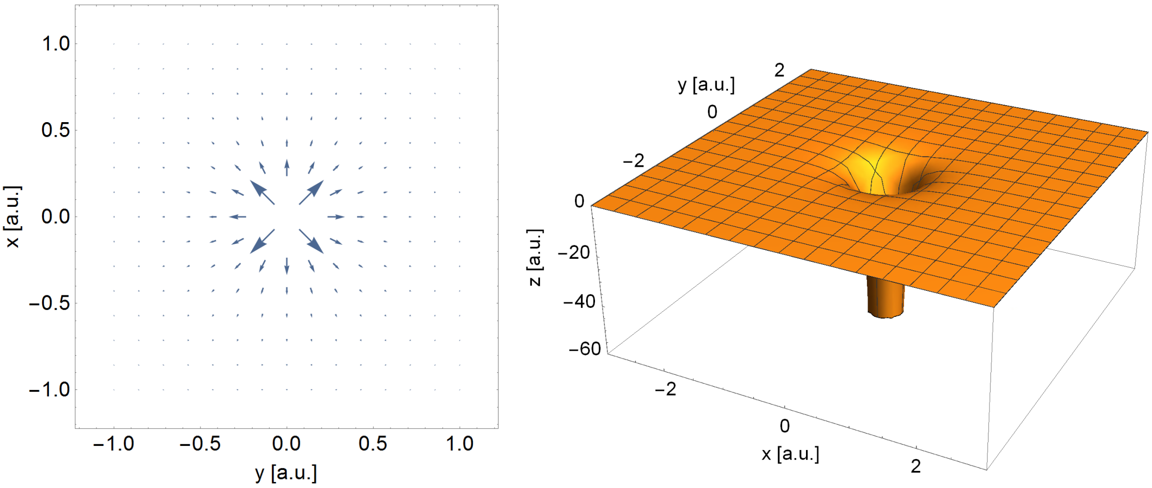


Fig. 1: Relation between electromagnetic quantities expressed by Maxwell's equations: (a) Gauss's law; (b) Gauss's law for magnetism; (c) Faraday's law of induction; (d) Ampère's law with Maxwell's extension. The figures closely are adapted from Ref. [6].

where $D_x(x, y, z)$, $D_y(x, y, z)$, and $D_z(x, y, z)$ are spatially dependent components of the vector field $\mathbf{D}(x, y, z)$. Figure 2(a) depicts an arbitrarily chosen vector field as an example of a divergence field. The divergence of this field, which is a scalar field, as the divergence gives the source strength, is shown in Fig. 2(b).

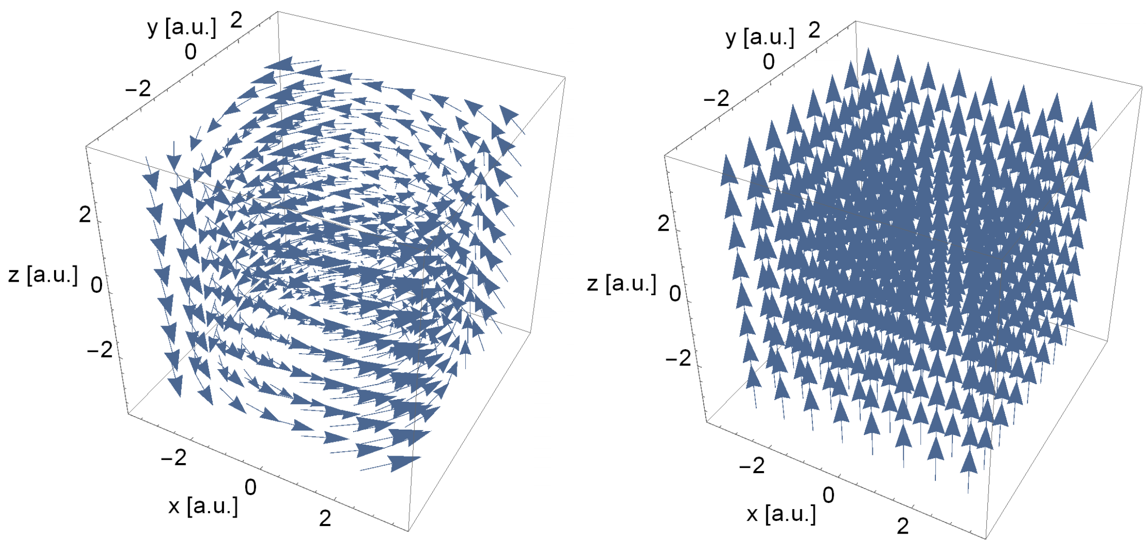
The operator $\nabla\times$ in Eqs. (7)–(8) represents the curl operator, which computes the rotation of a vector field about a point. Like the divergence operator, the curl operator contains spatial derivatives. In contrast with the divergence operator, the curl operator acts on vector fields and delivers vector fields. The vector fields resulting from the curl operator describe the curl strength of the field per unit area. In some textbooks, the curl is abbreviated by curl or rot. In a Cartesian co-ordinate system, the curl of the vector field $\mathbf{E}(x, y, z)$ is determined by

$$\nabla \times \mathbf{E}(x, y, z) = \operatorname{curl} \mathbf{E}(x, y, z) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(x, y, z) & E_y(x, y, z) & E_z(x, y, z) \end{vmatrix}$$



(a) (b)

Fig. 2: (a) Vector field $\mathbf{D}(x,y,z) = [x,y,0]^{\mathrm{T}}/\sqrt{x^2+y^2}^3$ as an example of a divergence field. (b) Divergence of the vector field, $\nabla \cdot \mathbf{D}(x,y,z) = -1/\sqrt{x^2+y^2}^3$, shown in (a).



(a) (b)

Fig. 3: (a) Vector field $\mathbf{E}(x,y,z) = [-y,x,0]^{\mathrm{T}}$ as an example of a curl field. (b) $\nabla \times \mathbf{E}(x,y,z) = [0,0,2]^{\mathrm{T}}$, i.e., the curl of the vector field depicted in (a).

$$= \begin{pmatrix} \frac{\partial}{\partial y}E_z(x,y,z) - \frac{\partial}{\partial z}E_y(x,y,z) \\ \frac{\partial}{\partial z}E_x(x,y,z) - \frac{\partial}{\partial x}E_z(x,y,z) \\ \frac{\partial}{\partial x}E_y(x,y,z) - \frac{\partial}{\partial y}E_x(x,y,z) \end{pmatrix}, \tag{10}$$

with the unit vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$.

Figure 3(a) sketches an arbitrarily chosen vector field which is curled. Figure 3(b) plots the corresponding curl of this field, which is again a vector field.

1.2 Lorentz force

The interaction of electromagnetic fields with a charged particle is described by

$$\mathbf{F}(\mathbf{r},t) = q\left(\mathbf{E}(\mathbf{r},t) + \mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right), \tag{11}$$

with the Lorentz force $\mathbf{F}(\mathbf{r},t)$ acting on a particle moving with velocity $\mathbf{v}$ and carrying charge q. Note that in some textbooks the term Lorentz force exclusively refers to $q\,\mathbf{v}\times\mathbf{B}(\mathbf{r},t)$, whereas the remaining term $q\,\mathbf{E}(\mathbf{r},t)$ is called the Coulomb force.

Independent of naming conventions, Eq. (11) is of crucial relevance for particle accelerators, as electric fields are used to transfer energy to the charged particles and magnetic fields to bend particles and to focus bunches of particles. The force on charged particles due to an electric field is always parallel to the electrical field lines (in fact that is one way to define electric fields). The force on charged particles due to a magnetic field is always orthogonal to the velocity $\mathbf{v}$ of the charged particle. The direction of the force acting on the particle due to magnetic fields can be found using the right-hand rule, where the thumb reflects the velocity $\mathbf{v}$ of the particle and the forefinger the direction of the magnetic flux density $\mathbf{B}(\mathbf{r},t)$. The direction of the forces is indicated by the direction of the middle finger. The Lorentz force is given component-wise in a Cartesian system by

$$F_x(\mathbf{r},t) = q\left(E_x(\mathbf{r},t) + v_y\,B_z(\mathbf{r},t) - v_z\,B_y(\mathbf{r},t)\right), \tag{12}$$

$$F_y(\mathbf{r},t) = q\left(E_y(\mathbf{r},t) + v_z\,B_x(\mathbf{r},t) - v_x\,B_z(\mathbf{r},t)\right), \tag{13}$$

$$F_z(\mathbf{r},t) = q\left(E_z(\mathbf{r},t) + v_x\,B_y(\mathbf{r},t) - v_y\,B_x(\mathbf{r},t)\right). \tag{14}$$

1.3 Conservation of charges

Maxwell's equations inherently comprise the conservation of electrical charges. To show this, the divergence of Ampère's law with Maxwell's extension (4) is determined as

$$\underbrace{\nabla\cdot\Big(\nabla\times\mathbf{H}(\mathbf{r},t)\Big)}_{0} = \nabla\cdot\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \nabla\cdot\mathbf{J}(\mathbf{r},t)\,. \tag{15}$$

The left-hand side is equal to zero as the divergence of any curl operations equals zero. On the right-hand side, the order of the derivation with respect to space and with respect to time can be exchanged by means of Schwarz's theorem, i.e., $\nabla\cdot\partial_t\mathbf{D}(\mathbf{r},t) = \partial_t\nabla\cdot\mathbf{D}(\mathbf{r},t)$, where ∂_t denotes the derivative with respect to time. Using Eq. (5) to replace the divergence of the displacement current delivers a differential form of the charge conservation formula,

$$\nabla\cdot\mathbf{J}(\mathbf{r},t) = -\frac{\partial}{\partial t}\rho(\mathbf{r},t)\,. \tag{16}$$

This formula can be transferred to an integral form through integration over the domain and the application of Gauss' law:

$$\oiint_{\partial\boldsymbol{\Omega}}\mathbf{J}(\mathbf{r},t)\cdot\mathrm{d}\mathbf{A} = -\frac{\partial}{\partial t}\underbrace{\iiint_{\boldsymbol{\Omega}}\rho(\mathbf{r},t)\,\mathrm{d}V}_{Q_{\mathrm{tot}}}\,. \tag{17}$$

The equation states that a total net electric flux through the closed surface $\partial\boldsymbol{\Omega}$ of the volume $\boldsymbol{\Omega}$ equals the reduction of the total electric charge in the volume $\boldsymbol{\Omega}$. Equation (16) states the same for an infinitely small volume, $\boldsymbol{\Omega}$.

1.4 Conservation of energy or Poynting's theorem

In addition to the conservation of charges, Maxwell's equations comprise the conservation of energy. This results from the multiplication of Eq. (7) with the magnetic field $\mathbf{H}(\mathbf{r},t)$, the multiplication of Eq. (8) with the electric field $\mathbf{E}(\mathbf{r},t)$, and the subtraction of both equations:

$$\begin{aligned}\mathbf{H}(\mathbf{r},t)\cdot\big(\nabla\times\mathbf{E}(\mathbf{r},t)\big)-\mathbf{E}(\mathbf{r},t)\cdot\big(\nabla\times\mathbf{H}(\mathbf{r},t)\big)\\ =-\underbrace{\mathbf{H}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)}_{\partial_t w_{\mathrm{m}}(\mathbf{r},t)}-\underbrace{\mathbf{E}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)}_{\partial_t w_{\mathrm{e}}(\mathbf{r},t)}-\underbrace{\mathbf{E}(\mathbf{r},t)\cdot\mathbf{J}(\mathbf{r},t)}_{p_{\mathrm{int}}(\mathbf{r},t)}\,,\end{aligned}\tag{18}$$

where $\partial_t w_{\mathrm{e}}(\mathbf{r},t)$ and $\partial_t w_{\mathrm{m}}(\mathbf{r},t)$ describe the change of energy stored in an infinitely small volume element in the form of electric and magnetic fields, respectively. The loss of energy per volume element is denoted $p_{\mathrm{int}}(\mathbf{r},t)$. Simplifying the left-hand side of Eq. (18) by means of the vector calculus identity $\nabla\cdot\big(\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)\big)=\mathbf{H}(\mathbf{r},t)\cdot\big(\nabla\times\mathbf{E}(\mathbf{r},t)\big)-\mathbf{E}(\mathbf{r},t)\cdot\big(\nabla\times\mathbf{H}(\mathbf{r},t)\big)$ (see e.g., Ref. [3, p. 879, (I. 24)]) and shifting $p_{\mathrm{int}}(\mathbf{r},t)$ from the right-hand side to the left-hand side of the equation gives

$$p_{\mathrm{int}}(\mathbf{r},t)+\nabla\cdot\big(\underbrace{\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)}_{\mathbf{S}(\mathbf{r},t)}\big)=-\frac{\partial}{\partial t}w_{\mathrm{e}}(\mathbf{r},t)-\frac{\partial}{\partial t}w_{\mathrm{m}}(\mathbf{r},t)\,.\tag{19}$$

The cross product of the electric field with the magnetic field is often referred to as the Poynting vector, $\mathbf{S}(\mathbf{r},t)$. It specifies the flow of energy per unit area due to electromagnetic fields. Integrating Eq. (19) over the volume $\mathbf{\Omega}$ and applying Gauss's divergence theorem yields

$$\underbrace{\iiint\limits_{\mathbf{\Omega}}p_{\mathrm{int}}(\mathbf{r},t)\,\mathrm{d}V}_{P_{\mathrm{int}}(t)}+\underbrace{\oiint\limits_{\partial\mathbf{\Omega}}\mathbf{S}(\mathbf{r},t)\cdot\mathrm{d}\mathbf{A}}_{P_{\mathrm{ext}}(t)}=-\frac{\partial}{\partial t}\underbrace{\iiint\limits_{\mathbf{\Omega}}w_{\mathrm{e}}(\mathbf{r},t)\,\mathrm{d}V}_{W_{\mathrm{e}}(t)}-\frac{\partial}{\partial t}\underbrace{\iiint\limits_{\mathbf{\Omega}}w_{\mathrm{m}}(\mathbf{r},t)\,\mathrm{d}V}_{W_{\mathrm{m}}(t)}\,.\tag{20}$$

The equation balances the internal energy losses, $P_{\mathrm{int}}(t)$, in $\mathbf{\Omega}$ and the energy flowing through the boundary $\partial\mathbf{\Omega}$, often referred to as external energy losses, $P_{\mathrm{ext}}(t)$, with the change of total energy stored in electric, $W_{\mathrm{e}}(t)$, and magnetic, $W_{\mathrm{m}}(t)$, fields. The internal losses, $P_{\mathrm{int}}(t)$, can result from ohmic dissipation effects as a consequence of finite resistivity and dielectric or magnetic losses. Furthermore, the transfer of energy to charged particles residing in $\mathbf{\Omega}$ is covered in $P_{\mathrm{int}}(t)$ as well. The external losses, $P_{\mathrm{ext}}(t)$, account, e.g., for the propagation of energy through waveguide ports. Note that $P_{\mathrm{int}}(t)$ and $P_{\mathrm{ext}}(t)$ can become negative, for instance if charged particles are decelerated in $\mathbf{\Omega}$ or energy propagates into the structure through $\partial\mathbf{\Omega}$, owing to an external excitation of the waveguide ports. Consequently, $P_{\mathrm{int}}(t)$ and $P_{\mathrm{ext}}(t)$ can also model energy gains.

2 Interaction of electromagnetic fields with matter

The interaction of electromagnetic fields with matter is described by the material equations. They relate the electric flux density $\mathbf{D}(\mathbf{r},t)$ with the electric field strength $\mathbf{E}(\mathbf{r},t)$ (see Subsection 2.1), the magnetic flux density $\mathbf{B}(\mathbf{r},t)$ with the magnetic field strength $\mathbf{H}(\mathbf{r},t)$ (see Subsection 2.2), and the electric current density $\mathbf{J}(\mathbf{r},t)$ with the electric field strength $\mathbf{E}(\mathbf{r},t)$ (see Subsection 2.3). All subsequent material equations describe the interaction of electromagnetic fields with matter, based on a macroscopic level, accounting for a sufficiently large number of molecules and atoms. Typically, these interactions are transient effects, as polarization of matter does not take place instantaneously. These transient effects are not considered hereinafter. In other words, it is assumed that the polarization takes place much faster than the change of the respective field strengths. If this does not hold, the dynamic properties of polarizations are often accounted for by frequency-dependent and complex-valued material properties, as discussed for instance in Refs. [1, 2]. Furthermore, the properties of some materials depend on the direction. For these cases of anisotropy, it is not sufficient to describe the material properties by means of scalars but tensors are required. However, these cases are not discussed within this report, nor is the class of non-linear materials, such as iron.

2.1 Electric fields

The interaction of matter with electric fields is described by

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t)\,, \tag{21}$$

where $\varepsilon_0 = 8.8542 \times 10^{-12}\,\mathrm{As/Vm}$ is the permittivity of free space and $\mathbf{P}(\mathbf{r}, t)$ is the electric polarization vector. The polarization vector $\mathbf{P}(\mathbf{r}, t)$ models the electric polarization of the material as a result of an external electric field. In fact, the following three polarization effects [2] can be distinguished.

– Dipole polarization results from the fact that polar materials, such as water, have a permanent dipole moment. In the presence of an electric field, the dipoles align themselves parallel to the electric field lines.
– Molecular polarization describes the displacement of positively or negatively charged atoms in a molecule.
– Electronic polarization is referred to as the displacement of an electron cloud centre in the vicinity of an atomic nucleus.

Figure 4 depicts these mechanisms. The electric polarization depends on the electric field via

$$\mathbf{P}(\mathbf{r}, t) = \varepsilon_0 \chi_\mathrm{e} \mathbf{E}(\mathbf{r}, t)\,, \tag{22}$$

where the material property χ_e is referred to as the electric susceptibility. Combining Eq. (21) with Eq. (22) gives

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \varepsilon_0 \chi_\mathrm{e} \mathbf{E}(\mathbf{r}, t) = \varepsilon_0 \underbrace{(1 + \chi_\mathrm{e})}_{\varepsilon_\mathrm{r}} \mathbf{E}(\mathbf{r}, t)\,, \tag{23}$$

with the relative dielectric constant ε_r. Typical values for ε_r are 1.0006 for air, 3.8 for quartz, and 80 for water.

2.2 Magnetic fields

The interaction between matter and magnetic fields is characterized by

$$\mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{H}(\mathbf{r}, t) + \mathbf{M}(\mathbf{r}, t)\,, \tag{24}$$

with the permeability of free space $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{Vs/Am}$ and the magnetic polarization vector $\mathbf{M}(\mathbf{r}, t)$. The polarization vector $\mathbf{M}(\mathbf{r}, t)$ describes the magnetic polarization of the material in the presence of an external magnetic field strength. Employing a very simple atomic model, magnetic dipoles result from negatively charged electrons orbiting around positively charged nuclei or from electrons spinning around their own axes. As shown in Fig. 5, these orbiting electrons generate fields similar to those of a magnetic bar, i.e., fields of a magnetic dipole. The magnetic polarization is connected with magnetic field strength by means of

$$\mathbf{M}(\mathbf{r}, t) = \mu_0 \chi_\mathrm{m} \mathbf{H}(\mathbf{r}, t)\,, \tag{25}$$

where the material property χ_m is referred to as the magnetic susceptibility. Replacing Eq. (25) in Eq. (24) yields

$$\mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{H}(\mathbf{r}, t) + \mu_0 \chi_\mathrm{m} \mathbf{H}(\mathbf{r}, t) = \mu_0 \underbrace{(1 + \chi_\mathrm{m})}_{\mu_\mathrm{r}} \mathbf{H}(\mathbf{r}, t)\,, \tag{26}$$

where μ_r is the relative permeability. The relative permeability ranges from 0.999834 for bismuth to $1{,}000{,}000$ for supermalloy [2].

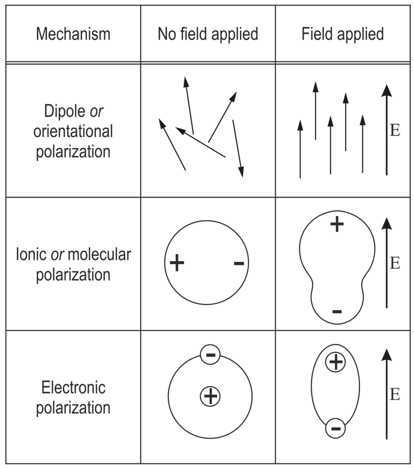


Fig. 4: Most relevant electrical polarization effects in dielectric materials. The behaviour of the dipoles is shown in the absence (middle column) and the presence (right column) of an external electric field $\mathbf{E}(\mathbf{r},t)$. Different time constants can be attributed to each effect, so that the respective polarization effects occur at different frequencies. Adapted from Ref. [2].

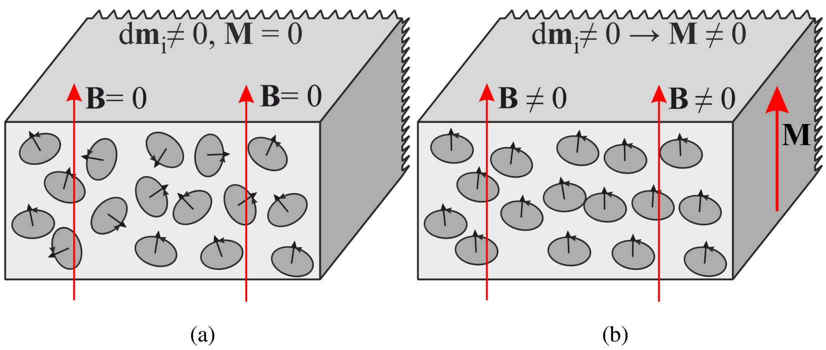


Fig. 5: Orientation of magnetic dipoles (grey circles). The arrows at the boundaries of the grey circles describe electric currents arising from negatively charged electrons orbiting around positively charged nuclei or from electrons spinning around their own axes. (a) Random orientation of magnetic dipoles in the absence of a magnetic flux density $\mathbf{B}(\mathbf{r},t)$. (b) Alignment of magnetic dipoles in the presence of a magnetic flux density $\mathbf{B}(\mathbf{r},t)$. Adapted from Ref. [2].

2.3 Ohm's law

If the electrons of a material can move from one atom to the other, they can drift as a result of an applied electric field, in consequence of Eq. (11). Owing to the drifting electron, an electric current flows. In good electric conductors, such as metals, the number of these free electrons is very large. The relationship between the electric current density due to ohmic conduction and the electric field strength is determined by

$$\mathbf{J}(\mathbf{r},t) = \sigma\,\mathbf{E}(\mathbf{r},t)\,, \tag{27}$$

with the electric conductivity σ. Depending on the electric conductivity, matter is classified as conducting, semiconducting, or dielectric (insulators). The conductivity σ varies from $1.33 \times 10^{-18}\,\mathrm{S/m}$ for fused quartz (an insulator) via $2.17\,\mathrm{S/m}$ for germanium (a semiconductor) to $6.29 \times 10^{7}\,\mathrm{S/m}$ for silver (a conductor) [6].

2.4 Boundary conditions of electromagnetic fields at material interfaces

With the help of Maxwell's equations in integral form (Eqs. (1)–(4)), boundary conditions of electromagnetic fields at the interface between different materials can be derived. They read

$$\mathbf{n} \times \Big[\mathbf{E}_2(\mathbf{r}_{\rm if}, t) - \mathbf{E}_1(\mathbf{r}_{\rm if}, t)\Big] = \mathbf{0}\,, \tag{28}$$

$$\mathbf{n} \times \Big[\mathbf{H}_2(\mathbf{r}_{\rm if}, t) - \mathbf{H}_1(\mathbf{r}_{\rm if}, t)\Big] = \mathbf{J}_{\rm s}(\mathbf{r}_{\rm if}, t)\,, \tag{29}$$

$$\mathbf{n} \cdot \Big[\mathbf{D}_2(\mathbf{r}_{\rm if}, t) - \mathbf{D}_1(\mathbf{r}_{\rm if}, t)\Big] = \rho_{\rm s}(\mathbf{r}_{\rm if}, t)\,, \tag{30}$$

$$\mathbf{n} \cdot \Big[\mathbf{B}_2(\mathbf{r}_{\rm if}, t) - \mathbf{B}_1(\mathbf{r}_{\rm if}, t)\Big] = 0\,, \tag{31}$$

where $\mathbf{n}$ denotes a unit vector normal to the interface, $\mathbf{r}_{\rm if}$ a location on the interface, $\mathbf{J}_{\rm s}(\mathbf{r}_{\rm if}, t)$ an electric surface current density, and $\rho_{\rm s}(\mathbf{r}_{\rm if}, t)$ an electric surface charge density. The latter two quantities refer to charges residing or flowing at the interface between the different materials. The subscripts of the fields indicate whether they refer to material 1 or to material 2.

Despite the fact that the statements (Eqs. (28)–(31)) look very formal, they have an intuitively accessible meaning. An explicit evaluation of the cross and the dot product reveals that Eq. (28) states that the tangential components of the electric fields are equal to each other across the interface, i.e., $E_{\rm t,2}(\mathbf{r}_{\rm if}, t) = E_{\rm t,1}(\mathbf{r}_{\rm if}, t)$. In the absence of electric surface currents ($\mathbf{J}_{\rm s}(\mathbf{r}_{\rm if}, t) = \mathbf{0}$), this also holds for tangential magnetic fields, i.e., $H_{\rm t,2}(\mathbf{r}_{\rm if}, t) = H_{\rm t,1}(\mathbf{r}_{\rm if}, t)$. In the presence of electric surface currents ($\mathbf{J}_{\rm s}(\mathbf{r}_{\rm if}, t) \neq \mathbf{0}$), the tangential fields are allowed to be discontinuous. While Eqs. (28) and (29) specify tangential components of the fields, Eqs. (30) and (31) refer to the normal components of the respective fields. Equation (30) claims that the normal component of the electric displacement fields is continuous in the absence of surface charges ($\rho_{\rm s}(\mathbf{r}_{\rm if}, t) = 0$), i.e., $D_{\rm n,2}(\mathbf{r}_{\rm if}, t) = D_{\rm n,1}(\mathbf{r}_{\rm if}, t)$. In the presence of surface charges, the normal components are related by $D_{\rm n,2}(\mathbf{r}_{\rm if}, t) = D_{\rm n,1}(\mathbf{r}_{\rm if}, t) + \rho_{\rm s}(\mathbf{r}_{\rm if}, t)$. Finally, Eq. (31) claims that the normal components of the magnetic flux density are equal to each other across the interface, i.e., $B_{\rm n,2}(\mathbf{r}_{\rm if}, t) = B_{\rm n,1}(\mathbf{r}_{\rm if}, t)$. Please refer to Fig. 6 for a graphical representation of the boundary conditions of electromagnetic fields at material interfaces. Note that the laws of refraction for the field lines can be derived directly from the boundary conditions (Eqs. (28)–(31)).

3 Electrostatics

To describe electric fields that do not depend on time, it is sufficient to restrict ourselves to electric and electric displacement fields because coupling between electric and magnetic field quantities does not take place in the static case, as the derivatives with respect to time are zero:

$$\nabla \times \mathbf{E}(\mathbf{r}) = -\underbrace{\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r})}_{\mathbf{0}}\,, \tag{32}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r})\,. \tag{33}$$

In a next step, the curl-free static electric field strength $\mathbf{E}(\mathbf{r})$ may be expressed in terms of the gradient of a scalar electrostatic potential $\phi(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})\,. \tag{34}$$

Here, $\nabla\phi(\mathbf{r})$ denotes the gradient of the electric potential. The gradient describes the change of a scalar field with respect to the spatial co-ordinates. The gradient operator acts on a scalar field and delivers a vector field. In a Cartesian system, the gradient is determined by

$$\nabla\phi(x, y, z) = \begin{pmatrix} \frac{\partial}{\partial x}\phi(x, y, z) \\ \frac{\partial}{\partial y}\phi(x, y, z) \\ \frac{\partial}{\partial z}\phi(x, y, z) \end{pmatrix}\,. \tag{35}$$

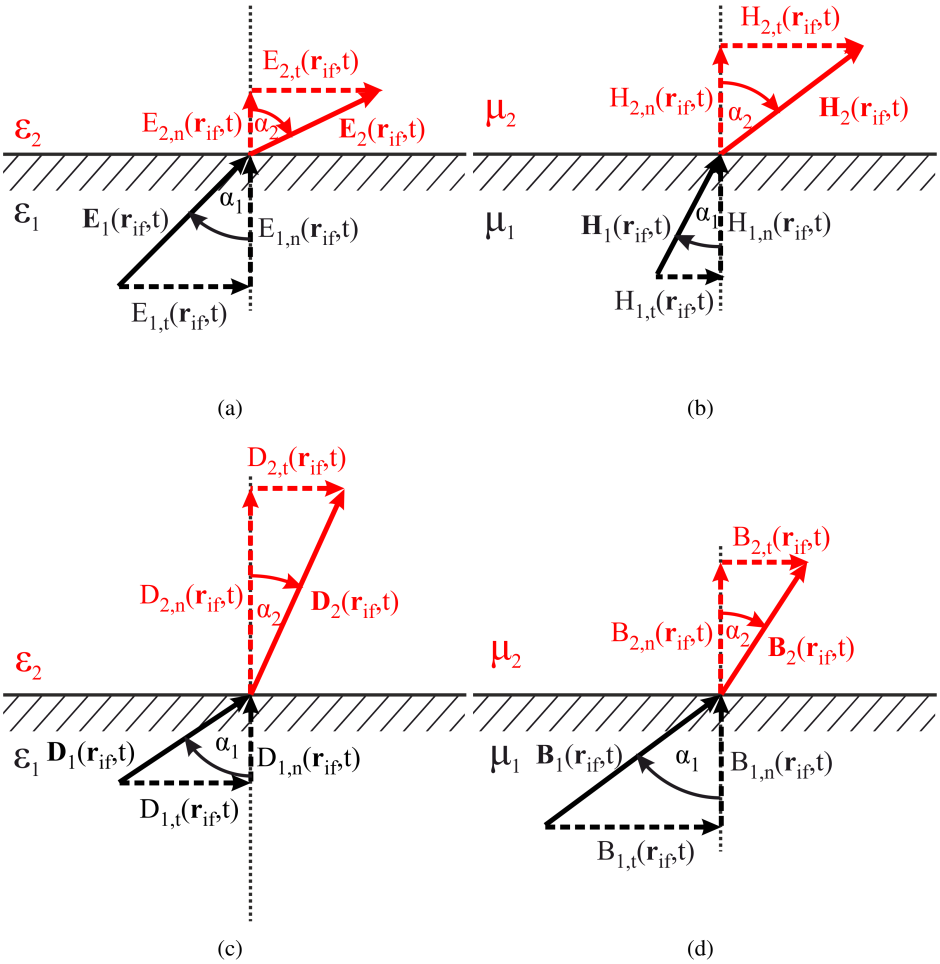


Fig. 6: Visualization of (a) electric fields, (b) magnetic fields, (c) electric displacement fields, and (d) magnetic flux densities across the interface of material 1 and material 2. At the interface, each vector field can be decomposed in terms of a normal component and a tangential component.

Expressing the static electric field $\mathbf{E}(\mathbf{r})$ in terms of the electrostatic potential $\phi(\mathbf{r})$, as in Eq. (34), has two advantages.

– The electric field $\mathbf{E}(\mathbf{r})$ is, *a priori*, free of curl, as $\nabla \times \nabla\phi(\mathbf{r}) = \mathbf{0}$, so that Eq. (32) holds in general. This can be seen by combining Eq. (35) with Eq. (10).

– Rather than searching for three unknown functions representing the components of the vector $\mathbf{E}(\mathbf{r})$, the gradient approach allows the number of sought functions to be reduced to one, i.e., the electric scalar potential $\phi(\mathbf{r})$.

Substituting the electric field strength in Eq. (33) by the ansatz (Eq. (34)) yields

$$\nabla \cdot \Big[\underbrace{-\varepsilon(\mathbf{r})\nabla\phi(\mathbf{r})}_{\mathbf{D}(\mathbf{r})}\Big] = \rho(\mathbf{r})\,. \tag{36}$$

This partial differential equation is often referred to as a potential equation. Mathematically, it is a Poisson equation. It describes electric potentials and consequently electric fields in static scenarios and must hold in the domain $\mathbf{\Omega}$. In the case of a homogeneous material distribution ($\varepsilon(\mathbf{r}) = \varepsilon = \mathrm{const.}$), the permittivity is a constant, so can be shifted in front of the divergence operator. Dividing the resulting equation on both sides by ε delivers

$$\nabla \cdot \left[\nabla \phi(\mathbf{r})\right] = -\frac{\rho(\mathbf{r})}{\varepsilon} . \tag{37}$$

The differential operator on the left-hand side corresponds to the Laplace operator:

$$\Delta \phi(\mathbf{r}) = \nabla \cdot \left[\nabla \phi(\mathbf{r})\right] . \tag{38}$$

In a Cartesian system, the Laplace operator reads as

$$\Delta \phi(x,y,z) = \frac{\partial^2}{\partial x^2}\phi(x,y,z) + \frac{\partial^2}{\partial y^2}\phi(x,y,z) + \frac{\partial^2}{\partial z^2}\phi(x,y,z) . \tag{39}$$

The solution of the Poisson equation (Eq. (37)) is given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon} \iiint\limits_{\mathbf{\Omega}} \frac{1}{|\mathbf{r}-\mathbf{r}'|}\, \rho(\mathbf{r}')\mathrm{d}\mathbf{r}' , \tag{40}$$

where the potential at infinity was set to zero, to ensure the uniqueness of the potential. Note that the static electric field strength directly results from the electrostatic potential by means of Eq. (34).

To have a unique solution for a specific field problem, boundary conditions are required. Important boundary conditions are the Dirichlet and the Neumann boundary conditions, prescribing fixed boundary values and normal derivatives, respectively. In electrostatics, the Dirichlet boundary condition is used to prescribe potentials on the boundary of the solution domain:

$$\phi(\mathbf{r}) = g_{\mathrm{D}}(\mathbf{r}) \text{ on } \partial\mathbf{\Omega}_{\mathrm{D}} , \tag{41}$$

with the function $g_{\mathrm{D}}(\mathbf{r})$ specifying the values of the potentials on the Dirichlet boundary $\partial\mathbf{\Omega}_{\mathrm{D}}$. This boundary condition is also used to model perfectly conducting boundaries. For these cases, constant potentials along the boundary are prescribed, so that the tangential component of the electric field vanishes. The very special case where $g_{\mathrm{D}}(\mathbf{r}) = 0$ is called the homogeneous Dirichlet boundary condition.

The Neumann boundary condition is employed to prescribe the change of the potential normal to the boundary of the computational domain:

$$\mathbf{n} \cdot \left[\nabla \phi(\mathbf{r})\right] = g_{\mathrm{N}}(\mathbf{r}) \text{ on } \partial\mathbf{\Omega}_{\mathrm{N}}. \tag{42}$$

Here, $\mathbf{n}$ is a unit vector normal to the boundary $\partial\mathbf{\Omega}_{\mathrm{N}}$ and $g_{\mathrm{N}}(\mathbf{r})$ is a function specifying the normal derivative of the potential on the Neumann boundary. To acquire a more intuitive understanding of Eq. (42), the gradient of the scalar electric potential is replaced by the electric field strength using Eq. (34):

$$\mathbf{n} \cdot \mathbf{E}(\mathbf{r}) = E_{\mathrm{n}}(\mathbf{r}) = -g_{\mathrm{N}}(\mathbf{r}) \text{ on } \partial\mathbf{\Omega}_{\mathrm{N}} . \tag{43}$$

The dot product of the normal vector $\mathbf{n}$ and the electric field strength $\mathbf{E}(\mathbf{r})$ delivers the normal component $E_{\mathrm{n}}(\mathbf{r})$ of the electric field strength on the Neumann boundary $\partial\mathbf{\Omega}_{\mathrm{N}}$. Following Eq. (43), the Neumann boundary condition allows the normal component of the electric field to be prescribed on the respective boundary of the computational domain. The special case where $g_{\mathrm{N}}(\mathbf{r}) = 0$ is called the homogeneous Neumann boundary condition and is often used to model perfect electric insulators. Figure 7 depicts the electric potential of a spherical charge as an example. The potential is shown for open boundary conditions (left picture) and for homogeneous Dirichlet boundary conditions (right picture). Open boundary conditions are not further discussed in this report. By employing certain approximation strategies, open

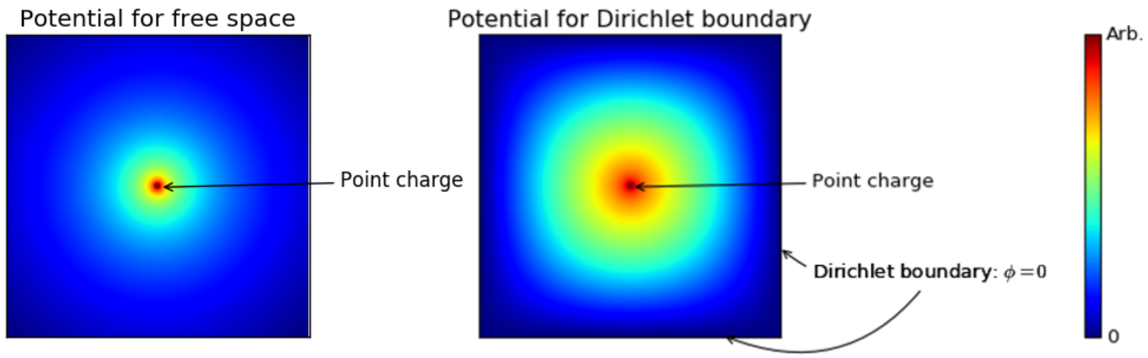


Fig. 7: Electric potential of a small spherical charge representing a single charge. The left picture shows the potential resulting from open boundaries; at the right picture, homogeneous Dirichlet conditions are enforced at the boundary of the computational domain. Both plots are created by means of numerical techniques using Ref. [11].

boundary conditions aim to mimic the computational domain to be extended to infinity. The influence of the boundary condition on the potential distribution inside the domain is clearly visible. In this example, for demonstration purposes, the boundary was set close to the source and obviously impacts the results.

In addition to the use of boundary conditions, Eqs. (41) and (42) are often used to specify symmetry conditions. The use of symmetries for symmetric field problems generally allows for a reduction of both memory requirements and computational burden.

4 Magnetostatics

As mentioned in Section 3, electric and magnetic fields are decoupled in the static case. Thus, it is sufficient to consider Eqs. (6) and (8) for magnetostatics:

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0\,, \tag{44}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \underbrace{\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r})}_{\mathbf{0}} + \mathbf{J}(\mathbf{r})\,. \tag{45}$$

Fig. 8 depicts the magnetic flux density in an iron yoke with two air gaps as an example for a field distribution obeying Eq. (44) and Eq. (45).

Rather than solving directly for the magnetic flux density, the flux density is often expressed in terms of the curl of a so-called magnetic vector potential:

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})\,. \tag{46}$$

The advantage of this approach results from the fact that Eq. (44) is then, *a priori*, fulfilled as the divergence of a field resulting from the curl operator always equalling zero. This follows, for instance, from Eqs. (9) and (10) for Cartesian co-ordinate systems. It is worth mentioning that the magnetic flux density $\mathbf{B}(\mathbf{r})$ does not uniquely determine the magnetic vector potential $\mathbf{A}(\mathbf{r})$. Adding a gradient of a scalar field, the magnetic vector potential does not influence the resulting magnetic flux density,

$$\mathbf{B}(\mathbf{r}) = \nabla \times \left[\mathbf{A}(\mathbf{r}) + \nabla\phi(\mathbf{r})\right] = \nabla \times \mathbf{A}(\mathbf{r}) + \underbrace{\nabla \times \nabla\phi(\mathbf{r})}_{\mathbf{0}} = \nabla \times \mathbf{A}(\mathbf{r})\,, \tag{47}$$

as the vector operators are linear and the curl of the gradient of scalar fields equals zero. For the static problem, Coulomb gauging is appropriate to achieve uniqueness:

$$\nabla \cdot \mathbf{A}(\mathbf{r}) = 0\,. \tag{48}$$

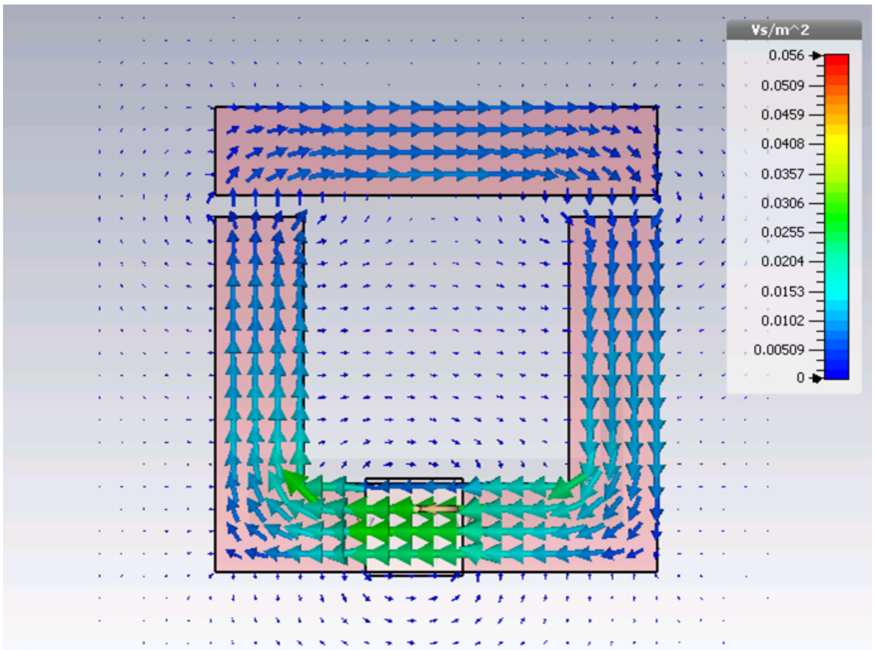


Fig. 8: Static field distribution of the magnetic flux density $\mathbf{B}(\mathbf{r})$ in an iron yoke with two air gaps. The magnetic field arises from electric currents flowing in the coil at the bottom of the yoke, where the magnetic field strength is largest. The field distribution is determined numerically using commercial software [12].

Substituting the magnetic field strength in Eq. (45) using the respective material relation and employing the ansatz (Eq. (46)) delivers

$$\nabla \times \left[\frac{1}{\mu(\mathbf{r})} \nabla \times \mathbf{A}(\mathbf{r})\right] = \mathbf{J}(\mathbf{r})\,. \tag{49}$$

Assuming the material to be homogeneous ($\mu(\mathbf{r}) = \mu = \text{const.}$), the permittivity is a constant, so it can be shifted in front of the divergence operator to obtain

$$\nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \mu \mathbf{J}(\mathbf{r})\,. \tag{50}$$

Note that the material distribution of the example structure depicted in Fig. 8 is inhomogeneous as vaccum and iron are present in the domain. The statement Eq. (50) can be modified by means of the vector calculus identity,

$$\nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \nabla\left(\nabla \cdot \mathbf{A}(\mathbf{r})\right) - \Delta \mathbf{A}(\mathbf{r})\,, \tag{51}$$

to

$$\nabla\left(\nabla \cdot \mathbf{A}(\mathbf{r})\right) - \Delta \mathbf{A}(\mathbf{r}) = \mu \mathbf{J}(\mathbf{r})\,. \tag{52}$$

This can be simplified to the vector Poisson equation,

$$\Delta \mathbf{A}(\mathbf{r}) = -\mu \mathbf{J}(\mathbf{r})\,, \tag{53}$$

by enforcing the Coulomb gauge (Eq. (48)) for the magnetostatic potential. In a Cartesian co-ordinate system, the vector Laplace operator reads as

$$\Delta \mathbf{A}(x,y,z) = \begin{pmatrix} \frac{\partial^2}{\partial x^2} A_x(x,y,z) + \frac{\partial^2}{\partial y^2} A_x(x,y,z) + \frac{\partial^2}{\partial z^2} A_x(x,y,z) \\ \frac{\partial^2}{\partial x^2} A_y(x,y,z) + \frac{\partial^2}{\partial y^2} A_y(x,y,z) + \frac{\partial^2}{\partial z^2} A_y(x,y,z) \\ \frac{\partial^2}{\partial x^2} A_z(x,y,z) + \frac{\partial^2}{\partial y^2} A_z(x,y,z) + \frac{\partial^2}{\partial z^2} A_z(x,y,z) \end{pmatrix}, \tag{54}$$

where $A_x(x, y, z)$, $A_y(x, y, z)$, and $A_z(x, y, z)$ are the components of the magnetic vector potential $\mathbf{A}(x, y, z)$. It follows from Eq. (54) that the vector Laplace operator, in fact, works as a Laplace operator on each individual component of the vector field. Note that the vector Laplace operator acts on a vector and delivers a vector.

In close analogy to Eq. (40), the solution of the vector Poisson equation (Eq. (53)) is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \iiint\limits_{\boldsymbol{\Omega}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \mathbf{J}(\mathbf{r}')\mathrm{d}\mathbf{r}' \,. \tag{55}$$

The magnetic field strength is uniquely determined by the magnetic vector potential by the employment of the Coulomb gauge (Eq. (48)). For a specific field problem, boundary conditions similar to Eqs. (41) and (42) can be formulated.

5 Electromagnetic waves

A closer inspection of Maxwell's equations reveals that these equations are mutually coupled. Before quantifying the coupling in a formal way, a qualitative description of the interplay between the field quantities is presented.

1. A time-dependent electric displacement field $\mathbf{D}(\mathbf{r}, t)$ generates a time-dependent curled magnetic field $\mathbf{H}(\mathbf{r}, t)$ according to Eq. (4).
2. This time-dependent magnetic field $\mathbf{H}(\mathbf{r}, t)$ corresponds to a time-dependent magnetic flux density $\mathbf{B}(\mathbf{r}, t)$, owing to the material relation (Eq. (26)).
3. A time-dependent magnetic flux density $\mathbf{B}(\mathbf{r}, t)$ creates a curled electric field strength $\mathbf{E}(\mathbf{r}, t)$ as stated in Eq. (3).
4. Owing to Eq. (23), this electric field strength $\mathbf{E}(\mathbf{r}, t)$ is proportional to the electric displacement field $\mathbf{D}(\mathbf{r}, t)$. In other words, a time-dependent electric displacement field $\mathbf{D}(\mathbf{r}, t)$ is generated and the process can be started again from 1.

For the formal description of electromagnetic problems, Maxwell's equations are often not employed directly, since each equation has more than one unknown field quantity to be specified. Instead, Maxwell's equations are inserted into each other so that they describe one field quantity exclusively. The resulting equations are referred to as wave equations and can be deduced for electric and for magnetic fields. The described approach, however, increases the order of the time derivative from one to two. It is worth highlighting that, despite the fact that the wave equation describes only one field quantity, the remaining quantities are generally unequal to zero and can be computed by means of Maxwell's equations. Hereinafter, the wave equations for electric and magnetic fields are derived. The derived wave equations allow for a multitude of different solutions, as discussed in Refs. [1–8] (See Fig. 9). In Subsection 5.3, the most simple frequency-domain solution of these wave equations is presented in closer detail as an example.

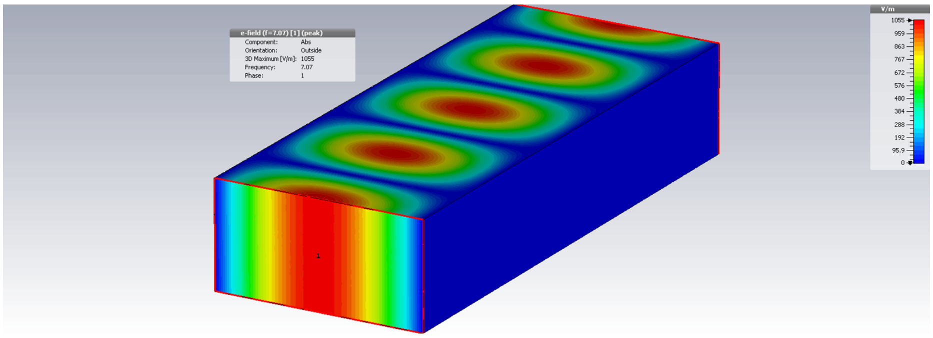
5.1 Wave equation for electric fields

In a first step, the curl of Faraday's law of induction (Eq. (7)) is evaluated:

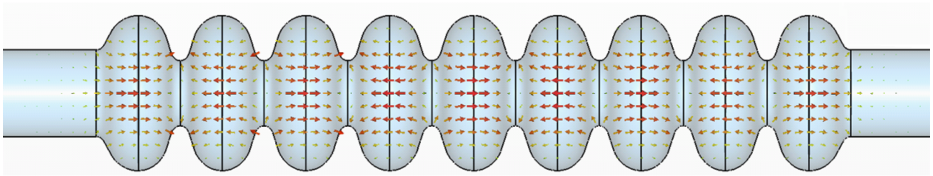
$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) = \nabla \times \left[-\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t) \right] . \tag{56}$$

Note that the minus sign can be shifted in front of the curl operator as it corresponds to the constant (-1) and that the order of derivatives can be changed as a consequence of Schwarz's theorem:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \nabla \times \mathbf{B}(\mathbf{r}, t) \,. \tag{57}$$



(a)



(b)

Fig. 9: Arbitrarily chosen examples for electromagnetic waves computed using commercial software [12]. (a) Absolute value of the electric field of a propagating TE_{10} mode in a rectangular waveguide. (b) Electric field of a standing wave in a third-harmonic TESLA cavity [13, 14] with a phase shift of π from cell to cell. This mode type is typically used for particle acceleration.

Next, the magnetic flux density is replaced by the magnetic field strength using Eq. (26):

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \nabla \times [\mu \mathbf{H}(\mathbf{r}, t)] \,. \tag{58}$$

Provided that the material distribution is homogeneous, the permeability μ does not depend on spatial co-ordinates, so it can be shifted in front of the operators:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H}(\mathbf{r}, t) \,. \tag{59}$$

The curl of the magnetic field on the right-hand side is now replaced by employing Ampère's law with Maxwell's extension (Eq. (8)) to obtain

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu \frac{\partial^2}{\partial t^2} \mathbf{D}(\mathbf{r}, t) - \mu \frac{\partial}{\partial t} \mathbf{J}(\mathbf{r}, t) \,. \tag{60}$$

Replacing the electric flux density by the electric field strength by using Eq. (23) yields the wave equation for electric fields in homogeneous media, often also denoted the curl-curl equation

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}, t) + \varepsilon \mu \frac{\partial^2}{\partial t^2} \mathbf{E}(\mathbf{r}, t) = -\mu \frac{\partial}{\partial t} \mathbf{J}(\mathbf{r}, t) \,. \tag{61}$$

By employing the vector calculus identity (Eq. (51)), the statement can be transferred to

$$\nabla (\nabla \cdot \mathbf{E}(\mathbf{r}, t)) - \Delta \mathbf{E}(\mathbf{r}, t) + \varepsilon \mu \frac{\partial^2}{\partial t^2} \mathbf{E}(\mathbf{r}, t) = -\mu \frac{\partial}{\partial t} \mathbf{J}(\mathbf{r}, t) \,. \tag{62}$$

As a result of Eqs. (5) and (23), the divergence of the electric field in homogeneous media is the charge density over the permittivity. Employing this property finally delivers the wave equation for electric fields,

$$\Delta \mathbf{E}(\mathbf{r}, t) - \varepsilon \mu \frac{\partial^2}{\partial t^2} \mathbf{E}(\mathbf{r}, t) = \mu \frac{\partial}{\partial t} \mathbf{J}(\mathbf{r}, t) + \frac{1}{\varepsilon} \nabla \rho(\mathbf{r}, t) \,. \tag{63}$$

5.2 Wave equation for magnetic fields

To derive the wave equation for magnetic fields, the curl of Ampère's law with Maxwell's extension (Eq. (8)) is initially evaluated to obtain

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}, t) = \nabla \times \left[\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t)\right] = \nabla \times \left[\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r}, t)\right] + \nabla \times \mathbf{J}(\mathbf{r}, t)\,. \tag{64}$$

The order of spatial and time derivatives is exchanged by the application of Schwarz's theorem:

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial}{\partial t}\nabla \times \mathbf{D}(\mathbf{r}, t) + \nabla \times \mathbf{J}(\mathbf{r}, t)\,. \tag{65}$$

Subsequently, the electric flux density is substituted by the electric field strength using Eq. (23):

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial}{\partial t}\nabla \times [\varepsilon\mathbf{E}(\mathbf{r}, t)] + \nabla \times \mathbf{J}(\mathbf{r}, t)\,. \tag{66}$$

Assuming the material distribution to be homogeneous, the permittivity ε does not depend on spatial co-ordinates, so this material property can be moved in front of the operators:

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}, t) = \varepsilon\frac{\partial}{\partial t}\nabla \times \mathbf{E}(\mathbf{r}, t) + \nabla \times \mathbf{J}(\mathbf{r}, t)\,. \tag{67}$$

The curl of the electric field on the right-hand side is now substituted by using Faraday's law of induction (Eq. (7)):

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}, t) = -\varepsilon\frac{\partial^2}{\partial t^2}\mathbf{B}(\mathbf{r}, t) + \nabla \times \mathbf{J}(\mathbf{r}, t)\,. \tag{68}$$

Expressing the magnetic flux density by means of the magnetic field strength according to Eq. (26) delivers the wave equation or curl-curl equation for magnetic fields in homogeneous media:

$$\nabla \times \nabla \times \mathbf{H}(\mathbf{r}, t) = -\varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{H}(\mathbf{r}, t) + \nabla \times \mathbf{J}(\mathbf{r}, t)\,. \tag{69}$$

The vector calculus identity (Eq. (51)) allows this statement to be transferred to:

$$\nabla\left(\nabla \cdot \mathbf{H}(\mathbf{r}, t)\right) - \Delta\mathbf{H}(\mathbf{r}, t) = -\varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{H}(\mathbf{r}, t) + \nabla \times \mathbf{J}(\mathbf{r}, t)\,. \tag{70}$$

Combining Eqs. (6) and (26) shows that divergence of the magnetic field in homogeneous media is equal to zero. Accounting for this property finally yields the wave equation for magnetic fields:

$$\Delta\mathbf{H}(\mathbf{r}, t) - \varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{H}(\mathbf{r}, t) = -\nabla \times \mathbf{J}(\mathbf{r}, t)\,. \tag{71}$$

5.3 Plane waves

The simplest solution of the wave equations is a harmonic plane wave in free space. For this case, the excitation current density ($\mathbf{J}(\mathrm{r}, t) = \mathbf{0}$) is assumed to be zero, as is the charge density ($\rho(\mathrm{r}, t) = 0$). The electric field of a plane wave polarized in the x-direction and propagating parallel to the z-axis is given by

$$\underline{\mathbf{E}}(\mathbf{r}, t) = \underbrace{\mathbf{n}_x\left(E_\mathrm{p}\mathrm{e}^{-\mathrm{j}kz} + E_\mathrm{m}\mathrm{e}^{\mathrm{j}kz}\right)}_{\underline{\mathbf{E}}(\mathbf{r})}\mathrm{e}^{\mathrm{j}\omega t}\,, \tag{72}$$

where $\mathbf{n}_x$ is the normal component in the x-direction, E_p the amplitude of a wave propagating in the positive z-direction, and E_m the amplitude of a wave propagating in the negative z-direction. It is a crucial property of the wave equation that it allows for propagation in both positive and negative directions. The parameter k is referred to as the wavenumber. Substituting the equation for the plane wave in Eq. (63) and using the fact that the amplitudes E_p and E_m can be freely chosen delivers

$$k = \frac{\omega}{c}\,, \tag{73}$$

with the speed of light $c = 1/\sqrt{\varepsilon\mu}$.

Despite the fact that magnetic fields neither show up in Eq. (63) nor in Eq. (72), they are present, owing to the mutual coupling of electric and magnetic fields for frequencies greater than zero. To determine the magnetic field strength, which accompanies the electric field of a plane wave, Eq. (72) is inserted into Faraday's law of induction (Eq. (7)). Subsequently, the magnetic flux density is replaced by the magnetic field strength using the material relation (Eq. (26)). These two steps yield the magnetic field strength,

$$\underline{\mathbf{H}}(\mathbf{r},t) = \underbrace{\mathbf{n}_y \underbrace{\frac{\mathrm{j}k}{\mathrm{j}\omega\mu}}_{Z} \left(E_\mathrm{p}\mathrm{e}^{-\mathrm{j}kz} - E_\mathrm{m}\mathrm{e}^{\mathrm{j}kz)}\right)}_{\underline{\mathbf{H}}(\mathbf{r})} \mathrm{e}^{\mathrm{j}\omega t}\,, \tag{74}$$

of the plane wave, where $\mathbf{n}_y$ is the normal component in the y direction and $Z = \sqrt{\mu/\varepsilon}$ is the so-called wave impedance. The wave impedance of the plane wave is a quantity depending on the material in which the wave is propagating. This impedance results from the ratio of the amplitude of the electric field and the amplitude of the magnetic field of a propagating wave.

The power per unit area transported by the plane wave is determined by the frequency-domain representation of the Poynting vector (Eq. (19)) and reads as

$$\underline{\mathbf{S}}(\mathbf{r}) = \frac{1}{2}\underline{\mathbf{E}}(\mathbf{r}) \times \underline{\mathbf{H}}^*(\mathbf{r})\,, \tag{75}$$

where $\underline{\mathbf{H}}^*(\mathbf{r})$ denotes the complex conjugate of $\underline{\mathbf{H}}(\mathbf{r})$. According to Eqs. (72), (74), and (75), a wave travelling in the positive z-direction with the amplitude E_p transports the following power per unit area:

$$\underline{\mathbf{S}}(\mathbf{r}) = \mathbf{n}_z \frac{1}{2Z} E_\mathrm{p}^2\,. \tag{76}$$

6 Field attenuation in conductors

To discuss the attenuation of fields in conductors, a plane wave (Eq. (72)) with amplitude E_p is considered in lossy material. For this purpose, the wavenumber k is assumed to be complex-valued, i.e., $\underline{k} = k' - \mathrm{j}k''$. Here, k' and k'' denote the real and the negative imaginary part of $\underline{k}$, respectively. The plane wave (Eq. (72)) with the complex-valued wavenumber must satisfy Eq. (63). However, as the material under consideration is a conductor, ohmic currents can flow, so that Eq. (63) becomes

$$\Delta\mathbf{E}(\mathbf{r},t) - \varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = \mu\sigma\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t)\,, \tag{77}$$

by employment of Eq. (27). Note that the conductor under consideration is assumed to be free of free charges, so that $\rho(\mathbf{r},t) = 0$ holds. Plugging the complex-valued electric field strength of the plane wave into Eq. (77) determines the complex-valued wavenumber:

$$\underline{k}^2 = \varepsilon\mu\omega^2 - \mathrm{j}\omega\mu\sigma\,. \tag{78}$$

Taking the square root of this equation gives

$$\underline{k} = \underbrace{\frac{\mu\sigma\omega}{\sqrt{2}\sqrt{\sqrt{\mu^2\sigma^2\omega^2 + \varepsilon^2\mu^2\omega^4} - \varepsilon\mu\omega^2}}}_{k'} - \mathrm{j}\underbrace{\frac{1}{\sqrt{2}}\sqrt{\sqrt{\mu^2\sigma^2\omega^2 + \varepsilon^2\mu^2\omega^4} - \varepsilon\mu\omega^2}}_{k''}\,. \tag{79}$$

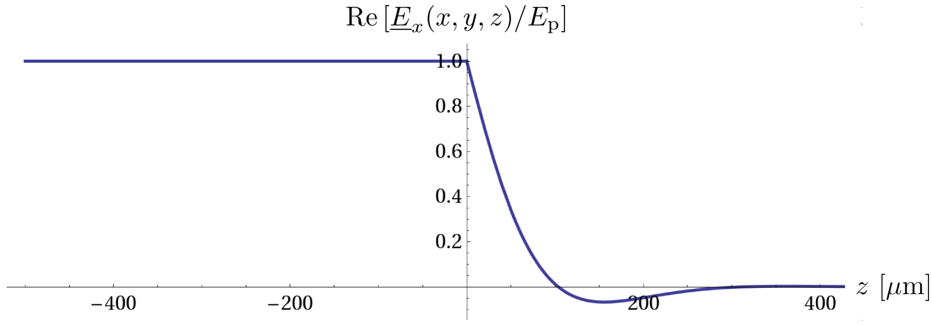


Fig. 10: Amplitude of a plane wave with frequency $f = 1\,\mathrm{MHz}$ polarized in the x-direction and propagating in the z-direction. For $z < 0$, vacuum with the material properties ε_0, μ_0, and $\sigma = 0\,\mathrm{S/m}$ is assumed. For $z > 0$, copper with the material properties ε_0, μ_0, and $\sigma = 58 \times 10^6\,\mathrm{S/m}$ is assumed. In other words, there is an interface between vacuum and copper at $z = 0$. The exponential decay of the field strength is clearly visible in copper, i.e., for $z > 0$. In this scenario, the penetration depth, $\delta = 66.085$ µm, is much smaller than the wavelength $\lambda = 1/(f\sqrt{\varepsilon_0\mu_0}) = 299.863\,\mathrm{m}$ in vacuum, so that the field strength appears to be constant for $z < 0$ on the given scale. Note that the electric field resulting from the reflection of the wave at the copper surface (i.e., a wave propagating in negative z-direction) is not depicted.

(Recall that a quadratic equation always has two solutions. Here, the solution referring to a wave travelling in the positive z-direction is selected.) Substituting the complex-valued wave number into the formula of the electric field of the plane wave (Eq. (72)) and exclusively assuming a wave propagating in the positive z-direction yields

$$\underline{\mathbf{E}}(\mathbf{r}) = \mathbf{n}_x E_\mathrm{p} \mathrm{e}^{-\mathrm{j}\underline{k}z} = \mathbf{n}_x E_\mathrm{p} \mathrm{e}^{-\mathrm{j}k'z} \mathrm{e}^{-k''z} \,. \tag{80}$$

The statement shows that the constant k'' describes the exponential decay of the field strength owing to the conductivity of the material. In this context, the inverse of k'',

$$\delta = \frac{1}{k''} = \frac{\sqrt{2}}{\sqrt{\sqrt{\mu^2\omega^2\left(\sigma^2+\varepsilon^2\omega^2\right)} - \varepsilon\mu\omega^2}} \approx \sqrt{\frac{2}{\mu\omega\sigma}} \,, \tag{81}$$

is of special relevance. The so-called penetration depth δ is the length that is required for the wave amplitude to decay by a factor of $\mathrm{e}^{-1} \approx 0.3678$ in the direction of propagation. Typical values for the penetration depth are of the order of $10^{-5}\,\mathrm{m}$ for copper at $1\,\mathrm{MHz}$. Figure 10 shows the field strength of a plane wave with a frequency of $1\,\mathrm{MHz}$ at an interface from vacuum ($z < 0$) to copper ($z > 0$). Note that, for the described scenario, the wavelength $\lambda = 1/(f\sqrt{\varepsilon_0\mu_0})$ in free space is much larger than the penetration depth δ, so that the field strength appears to be constant in vacuum in the diagram.

7 Summary and conclusion

This report shows that profound knowledge of mathematical tools, such as vector analysis, is required for the formal description of electromagnetic phenomena. However, once one is able to apply and to work with these mathematical tools, one realizes the beauty of the theory of electromagnetic fields: only four differential equations plus three constitutive equations are required to describe electromagnetic phenomena completely in a classical and macroscopic way. Moreover, it is very satisfactory to see that these equations inherently conserve energy. Recall that conservation of energy can be considered as one of the 'holy grails' of physics.

After providing a short overview with this report, readers are invited to continue their studies in the exciting field of the theory of electromagnetic fields by reading the literature listed in the reference section. The authors would like to point out that in addition to the analytical techniques for solving electromagnetic field problems, numerical methods for field problems are a very active research area. In contrast with analytical approaches, numerical methods can tackle field problems that are by far much more complex. Unfortunately, there exist a large variety of pitfalls when numerically solving field problems. To avoid these pitfalls, a profound knowledge of numerical mathematics and good command of analytical methods of the theory of electromagnetic fields is mandatory.

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