

Quantum field theory and the Electroweak Standard Model

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Abstract

These lecture notes cover the basics of Quantum Field Theory (QFT) and peculiarities in the construction of the Electroweak (EW) sector of the Standard Model (SM). In addition, the present status, issues, and prospects of the SM are discussed.

Keywords

Quantum field theory; standard model; electroweak; lectures.

1 Introduction

The Standard Model (SM) [1–3] was established in the mid-1970s. Its success is incredible: even after almost half a century, no significant deviations from the SM predictions have been found.

But what is the SM?

After the discovery [4, 5] of the Higgs boson at the LHC, it is fair to give the following *short* answer [6]:

The Absolutely Amazing Theory of Almost Everything.

There are many excellent lectures (e.g., [7–10]) and textbooks (e.g., [11, 12]) that can provide a lot of convincing arguments for such a fancy name. In this course we are not able to cover all the aspects of the SM, but just review some basic facts and underlying principles of the model emphasizing salient features of the latter.

Let us start with a brief overview of the SM particle content (see Fig. 1). One usually distinguishes fermions (half-integer spin) from bosons (integer spin). Traditionally, fermions are associated with “matter”, while bosons take the role of “force carriers” that mediate interactions between spin-1/2 particles. In the SM, there are three *generations* involving two types of fermions - *quarks* and *leptons*. In total, we have

- 6 quarks of different flavour ($q = u, d, c, s, t, b$),
- 3 charged ($l = e, \mu, \tau$) and 3 neutral ($\nu_l = \nu_e, \nu_\mu, \nu_\tau$) leptons.

All of them participate in the weak interactions. Both quarks q and charged leptons l take part in the electromagnetic interactions. In addition, quarks carry a *colour* charge and are influenced by the strong force. In the SM the above-mentioned interactions are mediated by the exchange of spin-1 (or vector) bosons:

- 8 gluons are responsible for the strong force between quarks;
- 4 electroweak bosons mediate the electromagnetic (photon - γ) and weak (Z, W^\pm) interactions.

There is also a famous spin-0 Higgs boson h , which plays an important role in the construction of the SM. It turns out that only gluons and photons (γ) are assumed to be massless.¹ All other *elementary* particles are massive due to the Higgs mechanism.

¹Initially neutrinos ν_l were assumed to be massless in the SM but experiments show that it is not the case.

Standard Model of Elementary Particles

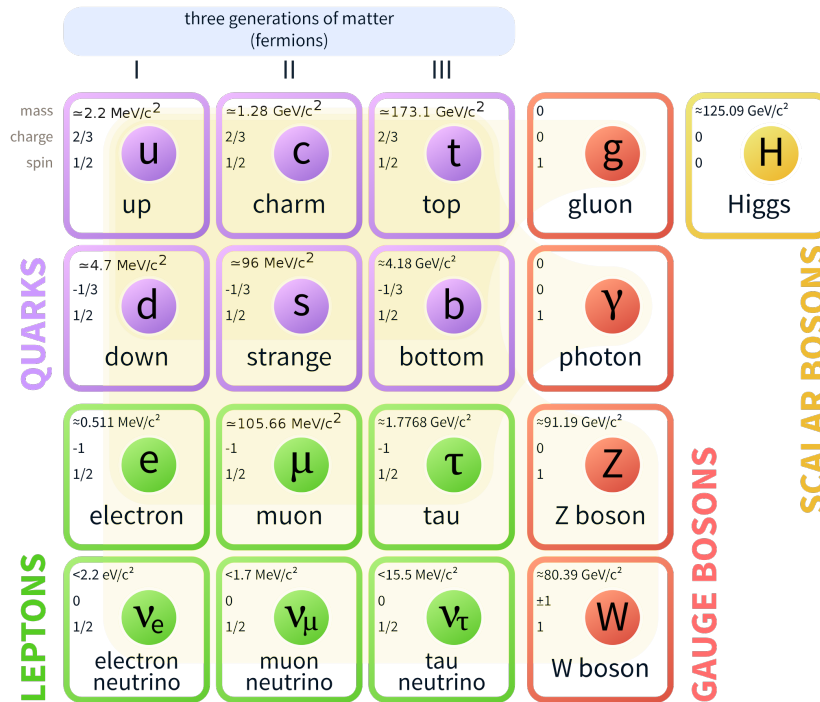


Fig. 1: Particle content of the Standard Model. Courtesy to Wikipedia.

In the SM the properties of the particle interactions can be read off the SM *Lagrangian* \mathcal{L}_{SM} . One can find its compact version on the famous CERN T-shirt. However, there is a lot of structure behind the short expression and it is *Quantum Field Theory* or QFT (see, e.g., textbooks [12–16]) that allows us to derive the full Lagrangian and understand why the T-shirt Lagrangian is unique in a sense.

The form of \mathcal{L}_{SM} is *restricted* by various kinds of (postulated) *symmetries*. Moreover, the SM is a *renormalizable* model. The latter fact allows us to use *perturbation theory* (PT) to provide high-precision predictions for thousands and thousands observables and verify the model experimentally. All these peculiarities will be discussed during the lectures, which have the following structure.

We begin by introducing quantum fields in Sec. 2 as the key objects of the relativistic quantum theory of particles. Then we discuss (global) symmetries in Sec. 3 and emphasize the relation between symmetries and particle properties. We switch from free to interacting fields in Sec. 4 and give a brief overview of techniques used to perform calculations in QFT models. We introduce gauge (or local) symmetries in Sec. 5 and discuss how they are realized in the SM (Sec. 6). The experimental status of the SM can be found in Sec. 7. Final remarks and conclusions are provided in Sec. 8.

2 From particles to quantum fields

Before we begin our discussion of quantum fields let us set up our notation. We work in natural units with the speed of light $c = 1$ and the (reduced) Planck constant $\hbar = 1$. In this way, all the quantities in particle physics are expressed in powers of electron-Volts (eV). To recover ordinary units, the following conversion constants can be used:

$$\left[\text{smiley face} \right] \quad \hbar \simeq 6.58 \cdot 10^{-22} \text{ MeV} \cdot \text{s}, \quad \hbar c \simeq 1.97 \cdot 10^{-14} \text{ GeV} \cdot \text{cm} \quad \left[\text{smiley face} \right]. \quad (1)$$

In High-Energy Physics (HEP) we routinely deal with particles traveling at speed $v \lesssim c$. As a consequence, we require that our theory should respect Lorentz *symmetry* that leaves a scalar product²

$$px \equiv p_\mu x_\mu = g_{\mu\nu} p_\mu x_\nu = p_0 x_0 - \mathbf{p} \cdot \mathbf{x}, \quad g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \quad (2)$$

of any four-vectors, e.g., space-time coordinates x_μ and energy-momenta p_μ

$$\begin{aligned} x_\mu &= \{x_0, \mathbf{x}\}, \text{ with time } t \equiv x_0, \\ p_\mu &= \{p_0, \mathbf{p}\}, \text{ with energy } E \equiv p_0, \end{aligned}$$

invariant under rotations and boosts parametrized by $\Lambda_{\mu\nu}$:

$$x_\mu \rightarrow x'_\mu = \Lambda_{\mu\nu} x_\nu, \quad x_\mu x_\mu = x'_\mu x'_\mu \Rightarrow \Lambda_{\mu\alpha} \Lambda_{\mu\beta} = g_{\alpha\beta} \quad (3)$$

It is this requirement that forces us to use QFT as a theory of *relativistic* particles. Relativistic quantum mechanics (RQM) describing a fixed number of particles turns out to be inconsistent. Indeed, from the energy-momentum relation for a free relativistic particle

$$E^2 = \mathbf{p}^2 + m^2 \quad (\text{instead of } E = \frac{\mathbf{p}^2}{2m} \text{ in the non-relativistic case}),$$

and the *correspondence* principle

$$E \rightarrow i \frac{\partial}{\partial t}, \quad \mathbf{p} \rightarrow -i \nabla$$

one obtains a relativistic analog of the Schrödinger equation - the Klein-Gordon (KG) equation

$$(\partial_t^2 - \nabla^2 + m^2) \phi(t, \mathbf{x}) = 0 \quad (\text{instead of } i \partial_t \psi = -\frac{\nabla^2}{2m} \psi) \quad (4)$$

for a wave-function $\phi(t, \mathbf{x}) \equiv \langle \mathbf{x} | \phi(t) \rangle$. It has two plane-wave solutions for any three-dimensional \mathbf{p} :

$$\phi_{\mathbf{p}}(t, \mathbf{x}) = e^{-iEt + i\mathbf{p}\mathbf{x}}, \quad \text{with } E = \pm \omega_p, \quad \omega_p = +\sqrt{\mathbf{p}^2 + m^2}. \quad (5)$$

One can see that the spectrum (5) is not bounded from below. Another manifestation of this problem is the fact that for a *general* wave-packet solution

$$\phi(t, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} [a(\mathbf{p}) e^{-i\omega_p t + i\mathbf{p}\mathbf{x}} + b(\mathbf{p}) e^{+i\omega_p t - i\mathbf{p}\mathbf{x}}] \quad (6)$$

we are not able to introduce a *positive-definite* probability density ρ

$$\rho \equiv j_0 = i(\phi^* \partial_t \phi - \phi \partial_t \phi^*) \Rightarrow 2E \text{ for } \phi \propto e^{-iEt}, \quad (7)$$

required to interpret ϕ as a wave-function of a single particle. Of course, one can try to impose the positive-energy condition [$b(\mathbf{p}) \equiv 0$] but it is not stable under *interactions*. A single-particle interpretation fails to account for the appearance of negative-energy modes and we need a new formalism to deal with such situations. Moreover, in RQM space coordinates play a role of dynamical variables and are represented by operators, while time is an evolution parameter. Obviously, a *consistent* relativistic theory should treat space and time on equal footing.

In order to circumvent these difficulties, one can re-interpret $\phi(\mathbf{x}, t)$ satisfying (4) as a *quantum* field, i.e., an *operator*³ $\hat{\phi}(\mathbf{x}, t)$. The space coordinates \mathbf{x} can be treated as a *label* for infinitely many

²Summation over repeated indices is implied.

³We use the Heisenberg picture, in which operators $\mathcal{O}_H(t)$ depend on time, while in the Schrödinger picture it is the states that evolve: $\langle \psi(t) | \mathcal{O}_S | \psi(t) \rangle = \langle \psi | \mathcal{O}_H(t) | \psi \rangle$ with $\mathcal{O}_S = \mathcal{O}_H(t=0)$, $|\psi\rangle = |\psi(t=0)\rangle$.

dynamical variables and we are free to choose a system of reference, in which we evolve these variables. As a consequence, a single field can account for an infinite number of particles, which are treated as field *excitations*. In the QFT notation the solution of the KG equation ($p_0 = \omega_p$) can be rewritten

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} [a_{\mathbf{p}}^- e^{-ipx} + b_{\mathbf{p}}^+ e^{+ipx}] \quad (8)$$

as a linear combination of *operators* $a_{\mathbf{p}}^\pm$ and $b_{\mathbf{p}}^\pm$ obeying

$$a_{\mathbf{p}}^- a_{\mathbf{p}'}^+ - a_{\mathbf{p}'}^+ a_{\mathbf{p}}^- \equiv [a_{\mathbf{p}}^-, a_{\mathbf{p}'}^+] = \delta^3(\mathbf{p} - \mathbf{p}'), \quad [b_{\mathbf{p}}^-, b_{\mathbf{p}'}^+] = \delta^3(\mathbf{p} - \mathbf{p}'). \quad (9)$$

All other commutators are zero, e.g., $[a_{\mathbf{p}}^\pm, a_{\mathbf{p}'}^\pm] = 0$. The operators satisfy $a_{\mathbf{p}}^\pm = (a_{\mathbf{p}}^\mp)^\dagger$ and $b_{\mathbf{p}}^\pm = (b_{\mathbf{p}}^\mp)^\dagger$, and for $a_{\mathbf{p}}^\pm \equiv b_{\mathbf{p}}^\pm$ the field is hermitian $\phi^\dagger(x) = \phi(x)$.

The operator (8) needs some space to act on and in QFT we consider the *Fock* space. It consists of a *vacuum* $|0\rangle$, which is *annihilated* by $a_{\mathbf{p}}^-$ (and $b_{\mathbf{p}}^-$) for every \mathbf{p}

$$\langle 0|0\rangle = 1, \quad a_{\mathbf{p}}^- |0\rangle = 0, \quad \langle 0|a_{\mathbf{p}}^+ = (a_{\mathbf{p}}^- |0\rangle)^\dagger = 0,$$

and field excitations. The latter are *created* from the vacuum by acting with $a_{\mathbf{k}}^+$ (and/or $b_{\mathbf{k}}^+$), e.g.,

$$|f_1\rangle = \int d\mathbf{k} \cdot f_1(\mathbf{k}) a_{\mathbf{k}}^+ |0\rangle, \quad \text{1-particle state;} \quad (10)$$

$$|f_2\rangle = \int d\mathbf{k}_1 d\mathbf{k}_2 \cdot f_2(\mathbf{k}_1, \mathbf{k}_2) a_{\mathbf{k}_1}^+ a_{\mathbf{k}_2}^+ |0\rangle \quad \text{2-particle state,} \quad (11)$$

...

where $f_i(\mathbf{k}, \dots)$ are supposed to be square-integrable, so that, e.g., $\langle f_1|f_1\rangle = \int |f_1(\mathbf{k})|^2 d\mathbf{k} < \infty$. In spite of the fact that it is more appropriate to deal with such normalizable states, in QFT we usually consider (basis) states that have definite momentum \mathbf{p} , i.e., we assume that $f_1(\mathbf{k}) = \delta(\mathbf{k} - \mathbf{p})$.

The two set of operators a^\pm and b^\pm correspond to particles and antiparticles. From the commutation relations we deduce that $a_{\mathbf{p}}^+ a_{\mathbf{k}}^+ = a_{\mathbf{k}}^+ a_{\mathbf{p}}^+$, so particles are *not distinguishable* by construction.

The commutation relations (10) should remind us about a bunch of independent *quantum harmonic oscillators*. Indeed, the corresponding Hamiltonian

$$\hat{\mathcal{H}}_{osc} = \sum_j \frac{1}{2} (\hat{p}_j^2 + \omega_j^2 \hat{x}_j^2) = \sum_j \frac{\omega_j}{2} (a_j^+ a_j^- + a_j^- a_j^+) = \sum_j \omega_j \left(\hat{n}_j + \frac{1}{2} \right) \quad (12)$$

can be expressed in terms of *ladder operators* $\sqrt{2\omega_j} a_j^\pm = (\omega_j \hat{x}_j \mp i \hat{p}_j)$ (no summation), which satisfy $[a_j^-, a_k^+] = \delta_{jk}$ similar to Eq.(10). For convenience we re-order operators entering into $\hat{\mathcal{H}}_{osc}$ and introduce $\hat{n}_j = a_j^+ a_j^-$ that counts energy quanta $\hat{n}_j |n_j\rangle = n_j |n_j\rangle$. A direct consequence of the re-ordering is the fact that the lowest possible state (vacuum $|0\rangle$) has non-zero energy, which is equal to the sum of zero-point energies $\sum_j \omega_j/2$ of all oscillators.

We can make the analogy between a (free) field and harmonic oscillators more pronounced if we put our field in a box of size L . In this case, the energy ω_p and momentum \mathbf{p} are quantized

$$\mathbf{p} \rightarrow \mathbf{p}_j = (2\pi/L)\mathbf{j}, \quad \omega_p \rightarrow \omega_j = \sqrt{(2\pi/L)^2 \mathbf{j}^2 + m^2}, \quad \mathbf{j} = (j_1, j_2, j_3), \quad j_i \in \mathbb{Z}.$$

The corresponding $\hat{\mathcal{H}}_{osc}$ (12) can be used to deduce the (QFT) Hamiltonian (by taking the limit $L \rightarrow \infty$):

$$\hat{\mathcal{H}}_{part} = \lim_{L \rightarrow \infty} \underbrace{\left[\left(\frac{2\pi}{L} \right)^3 \sum_{\mathbf{j}} \right]}_{\int d\mathbf{p}} \omega_{\mathbf{j}} \left[\underbrace{\left(\frac{L}{2\pi} \right)^{\frac{3}{2}} a_{\mathbf{j}}^+}_{a_{\mathbf{p}}^+} \underbrace{\left(\frac{L}{2\pi} \right)^{\frac{3}{2}} a_{\mathbf{j}}^-}_{a_{\mathbf{p}}^-} + \frac{1}{2} \underbrace{\left(\frac{L}{2\pi} \right)^3}_{\delta(0)} \right].$$

Since our field (8) involves two kinds of ladder operators, we have

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{part} + \hat{\mathcal{H}}_{antipart} = \int d\mathbf{p} \omega_p [n_{\mathbf{p}} + \bar{n}_{\mathbf{p}}] + \int d\mathbf{p} \omega_p \delta(\mathbf{0}) \quad (13)$$

with $\bar{n}_{\mathbf{p}} \equiv b_{\mathbf{p}}^+ b_{\mathbf{p}}^-$ and $n_{\mathbf{p}} \equiv a_{\mathbf{p}}^+ a_{\mathbf{p}}^-$. The interpretation of the first term is straightforward: ($\bar{n}_{\mathbf{p}}$) $n_{\mathbf{p}}$ counts (anti-)particles with definite momentum \mathbf{p} and there is a sum over the corresponding energies. The second term in Eq.(13) looks disturbing. It is associated with *infinite* vacuum (no particles) energy:

$$E_0 = \langle 0 | \hat{\mathcal{H}} | 0 \rangle = \int d\mathbf{p} \omega_p \delta(\mathbf{0}).$$

Actually, there are two kinds of infinities in E_0 :

- InfraRed (large distances, $L \rightarrow \infty$) due to $L^3 \rightarrow (2\pi)^3 \delta(\mathbf{0})$;
- UltraViolet (small distances, $\mathbf{p}, \omega_p \rightarrow \infty$).

One usually “solves” this problem by introducing *normal-ordered* Hamiltonians, e.g.,

$$:\hat{\mathcal{H}}_{osc}:= \frac{\omega_j}{2} \left(:a_j^+ a_j^- + a_j^- a_j^+ : \right) = \omega_j :a_j^+ a_j^- := \omega_j a_j^+ a_j^-.$$

With $:\hat{\mathcal{H}}:$ we measure all energies with respect to the vacuum $\hat{\mathcal{H}} \rightarrow : \hat{\mathcal{H}} := \hat{\mathcal{H}} - \langle 0 | \hat{\mathcal{H}} | 0 \rangle$ and ignore (non-trivial) dynamics of the latter. In what follows we assume that operators are normal-ordered by default.

It is easy to check that $[\hat{\mathcal{H}}, a_{\mathbf{p}}^{\pm}] = \pm \omega_{\mathbf{p}} a_{\mathbf{p}}^{\pm}$ and $[\hat{\mathcal{H}}, b_{\mathbf{p}}^{\pm}] = \pm \omega_{\mathbf{p}} b_{\mathbf{p}}^{\pm}$. As a consequence, single-particle states with definite momentum \mathbf{p}

$$|\mathbf{p}\rangle = a_{\mathbf{p}}^+ |0\rangle, \quad \hat{\mathcal{H}}|\mathbf{p}\rangle = \omega_p |\mathbf{p}\rangle, \quad |\bar{\mathbf{p}}\rangle = b_{\mathbf{p}}^+ |0\rangle, \quad \hat{\mathcal{H}}|\bar{\mathbf{p}}\rangle = \omega_p |\bar{\mathbf{p}}\rangle \quad (14)$$

are eigenvectors of the Hamiltonian with *positive* energies and we avoid introduction of negative energies in our formalism from the very beginning. One can generalize Eq. (12) and “construct” the momentum $\hat{\mathbf{P}}$ and charge \hat{Q} operators⁴:

$$\hat{\mathbf{P}} = \int d\mathbf{p} \mathbf{p} [n_{\mathbf{p}} + \bar{n}_{\mathbf{p}}], \quad \hat{\mathbf{P}}|0\rangle = 0|0\rangle, \quad \hat{\mathbf{P}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle \quad \hat{\mathbf{P}}|\bar{\mathbf{p}}\rangle = \mathbf{p}|\bar{\mathbf{p}}\rangle, \quad (15)$$

$$\hat{Q} = \int d\mathbf{p} [n_{\mathbf{p}} - \bar{n}_{\mathbf{p}}], \quad \hat{Q}|0\rangle = 0|0\rangle, \quad \hat{Q}|\mathbf{p}\rangle = +|\mathbf{p}\rangle \quad \hat{Q}|\bar{\mathbf{p}}\rangle = -|\bar{\mathbf{p}}\rangle. \quad (16)$$

The charge operator \hat{Q} distinguishes particles from anti-particles. One can show that the field ϕ^\dagger (ϕ) increases (decreases) the charge of a state

$$\left[\hat{Q}, \phi^\dagger(x) \right] = +\phi^\dagger(x), \quad \left[\hat{Q}, \phi(x) \right] = -\phi(x)$$

and consider the following amplitudes:

$$\begin{array}{ll} t_2 > t_1 : & \langle 0 | \underbrace{\phi(x_2)}_{a^-} \underbrace{\phi^\dagger(x_1)}_{a^+} | 0 \rangle \\ & \text{Particle (charge +1)} \\ & \text{propagates from } x_1 \text{ to } x_2 \end{array} \quad \begin{array}{ll} t_1 > t_2 : & \langle 0 | \underbrace{\phi^\dagger(x_1)}_{b^-} \underbrace{\phi(x_2)}_{b^+} | 0 \rangle \\ & \text{Antiparticle (charge -1)} \\ & \text{propagates from } x_2 \text{ to } x_1 \end{array}$$

⁴It is worth pointing here that by construction both \hat{Q} and $\hat{\mathbf{P}}$ do not depend on time and commute. In the next section, we look at this fact from a different perspective and connect it to various symmetries.

Both possibilities can be taken into account in one function:

$$\underbrace{\langle 0|T[\phi(x_2)\phi^\dagger(x_1)]|0\rangle}_{-iD_c(x-y)} \equiv \theta(t_2 - t_1)\langle 0|\phi(x_2)\phi^\dagger(x_1)|0\rangle + \theta(t_1 - t_2)\langle 0|\phi^\dagger(x_1)\phi(x_2)|0\rangle, \quad (17)$$

with T being the *time-ordering* operation ($\theta(t) = 1$ for $t \geq 0$ and zero otherwise).

Equation (17) is nothing else but the famous Feynman propagator, which has the following momentum representation:

$$D_c(x - y) = \frac{-1}{(2\pi)^4} \int d^4p \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}. \quad (18)$$

The $i\epsilon$ -prescription ($\epsilon \rightarrow 0$) picks up certain poles in the complex p_0 plane (see Fig. 2) and gives rise to the time-ordered expression (17). The propagator plays a key role in the construction of perturbation theory for interacting fields (see Sec. 4.1).

For the moment, let us mention a couple of facts about $D_c(x)$. It is a Green-function for the KG equation, i.e.,

$$(\partial_x^2 + m^2) D_c(x - y) = \delta(x - y). \quad (19)$$

This gives us an alternative way to find the expression (18). One can also see that $D_c(x - y)$ is a Lorentz and translational invariant function.

The propagator of particles can be connected to the force between two classical static sources $J_i(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_i)$ located at $\mathbf{x}_i = (\mathbf{x}_1, \mathbf{x}_2)$. The sources disturb the vacuum $|0\rangle \rightarrow |\Omega\rangle$, since the Hamiltonian of the system is modified $\mathcal{H} \rightarrow \mathcal{H}_0 + J \cdot \phi$. Assuming for simplicity that $\phi = \phi^\dagger$, we can find the energy of the disturbed vacuum from

$$\begin{aligned} \langle \Omega | e^{-i\mathcal{H}T} | \Omega \rangle &\equiv e^{-iE_0(J)T} \Rightarrow \text{in the limit } T \rightarrow \infty \\ &= e^{\frac{i^2}{2T} \int dx dy J(x) \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle J(y)} = e^{+\frac{i}{2} \int dx dy J(x) D_c(x-y) J(y)} \end{aligned}$$

Evaluating the integral for $J(x) = J_1(x) + J_2(x)$ and neglecting ‘‘self-interactions’’, we get the contribution δE_0 to $E_0(J)$ due to interactions between two sources

$$\begin{aligned} \lim_{T \rightarrow \infty} \delta E_0 T &= - \int dx dy J_1(x) D_c(x-y) J_2(y) \\ \delta E_0 &= - \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{e^{+i\mathbf{p}(\mathbf{x}_1 - \mathbf{x}_2)}}{\mathbf{p}^2 + m^2} = - \frac{1}{4\pi r} e^{-mr}, \quad r = |\mathbf{x}_1 - \mathbf{x}_2| \end{aligned}$$

This is nothing else but the *Yukawa* potential. It is *attractive* and *falls off* exponentially over the distance scale $1/m$. Obviously, for $m = 0$ we get a Coulomb-like potential.

3 Symmetries and fields

Let us switch to the discussion of symmetries and their role in QFT. A convenient way to deal with quantum fields and the symmetries of the corresponding physical systems is to consider the following

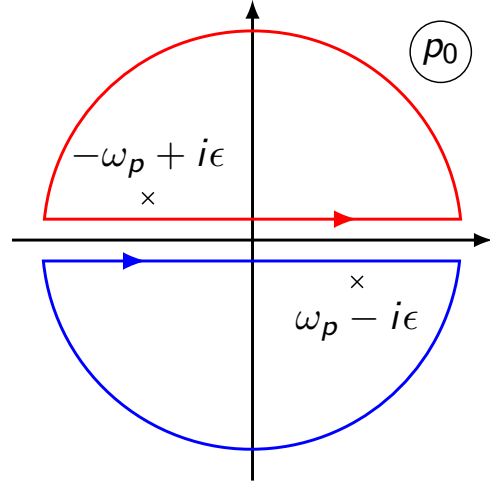


Fig. 2: Integration contours in p_0 plane.

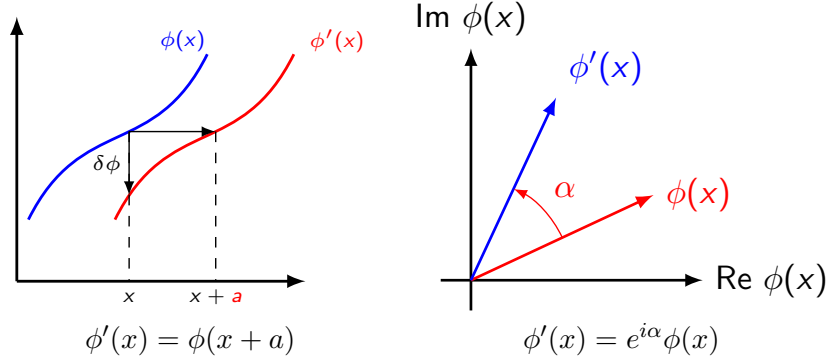


Fig. 3: Translations (left) and phase transformations (right).

Action functional⁵

$$\mathcal{A}[\phi(x)] = \int d^4x \underbrace{\mathcal{L}(\phi(x), \partial_\mu \phi)}_{\text{Lagrangian (density)}} = \int d^4x \underbrace{\left(\partial_\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi \right)}_{\phi^\dagger \cdot K \cdot \phi}. \quad (20)$$

To have an analogy with a mechanical system, one can rewrite $\mathcal{A}[\phi]$ as

$$\begin{aligned} \mathcal{A}[\phi(x)] &= \int dt L(t), & L &= T - U, & H &= T + U \\ T &= \int d\mathbf{x} |\partial_t \phi|^2, & U &= \int d\mathbf{x} (|\partial_{\mathbf{x}} \phi|^2 + m^2 |\phi|^2) \end{aligned}$$

with T and U being kinetic and potential energy of a system of *coupled* oscillators (a “mattress”).

Given a Lagrangian \mathcal{L} , one can derive the *equations of motions* (EOM) via the *Action Principle*. For this we consider variation of the action

$$\underbrace{\mathcal{A}[\phi'(x)] - \mathcal{A}[\phi(x)]}_{\delta \mathcal{A}[\phi(x)] = 0} = \int d^4x \left[\underbrace{\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} \right)}_{(\partial_\mu^2 + m^2)\phi = 0} \delta \phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta \phi \right)}_{\text{surface term} = 0} \right]. \quad (21)$$

due to tiny (infinitesimal) shifts in the field $\phi'(x) = \phi(x) + \delta\phi(x)$. If we require that $\delta\mathcal{A}[\phi(x)] = 0$ for *any* variation $\delta\phi(x)$ of *some* $\phi(x)$, we will immediately deduce that this can be achieved only for *specific* $\phi(x)$ that satisfy EOM. These *particular* fields are usually called “on-mass-shell”. From the Lagrangian for our free scalar field (20) we derive the KG equation. It is related in a straightforward way to the quadratic form K in Eq.(20). Having in mind Eq.(19), one can see that the (Feynman) propagator can also be obtained by inverting K . This statement is easily generalized to the case of other fields.

The Action functional for a physical system allows one to study *Symmetries*. The latter are intimately connected with *transformations*, which leave something *invariant*. The transformations can be *discrete*, such as

$$\begin{aligned} \text{Parity} &: \phi'(\mathbf{x}, t) = P\phi(\mathbf{x}, t) = \phi(-\mathbf{x}, t), \\ \text{Time-reversal} &: \phi'(\mathbf{x}, t) = T\phi(\mathbf{x}, t) = \phi(\mathbf{x}, -t), \\ \text{Charge-conjugation} &: \phi'(\mathbf{x}, t) = C\phi(\mathbf{x}, t) = \phi^\dagger(\mathbf{x}, t), \end{aligned}$$

⁵Contrary to ordinary functions that produce numbers from numbers, a *functional* takes a function and produces a number.

or depend on *continuous* parameters. One distinguishes *space-time* from *internal* transformations. Lorentz boosts, rotations, and translations are typical examples of the former, while phase transformations belong to the latter (see Fig. 3). At the moment, we only consider *global* symmetries with parameters independent of space-time coordinates and postpone the discussion of *x*-dependent or *local* (*gauge*) transformations to Sec. 5.

Given $\mathcal{A}[\phi]$, one can *find* its *symmetries*, which can be defined as *particular* infinitesimal variations $\delta\phi(x)$ that for *any* $\phi(x)$ leave $\mathcal{A}[\phi]$ invariant up to a surface term (*cf.* the Action Principle)

$$\mathcal{A}[\phi'(x)] - \mathcal{A}[\phi(x)] = \int d^4x \partial_\mu \mathcal{K}_\mu, \quad \phi'(x) \equiv \phi(x) + \delta\phi(x).$$

If we compare this with the general expression

$$\mathcal{A}[\phi'(x)] - \mathcal{A}[\phi(x)] = \int d^4x \left[\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \right) \right].$$

and require in addition that ϕ satisfy EOM⁶, we get a *local conservation law*

$$\partial_\mu J_\mu = 0, \quad J_\mu \equiv \mathcal{K}_\mu - \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi. \quad (22)$$

The integration of Eq. (22) over *space* leads to *conserved charge*:

$$\frac{d}{dt} Q = 0, \quad Q = \int d\mathbf{x} J_0. \quad (23)$$

If $\delta\phi = \rho_i \delta_i \phi$ depends on parameters ρ_i , we have a conservation law for every ρ_i . This is the essence of the *first Noether theorem* [17].

A careful reader might notice that we somehow forgot about the quantum nature of our fields and in our discussion of symmetries treat them as classical objects. Let us comment on this fact. In Classical Physics *symmetries* allow one to find

- new solutions to EOM from the given one, keeping some features of the solutions (*invariants*) intact;
- how a solution in one coordinate system (as seen by one observer) looks in other coordinates (as seen by another observer).

In a quantum world a symmetry \mathcal{S} guarantees that transition *probabilities* \mathcal{P} between *states* do not change upon transformation:

$$|A_i\rangle \xrightarrow{\mathcal{S}} |A'_i\rangle, \quad \mathcal{P}(A_i \rightarrow A_j) = \mathcal{P}(A'_i \rightarrow A'_j), \quad |\langle A_i | A_j \rangle|^2 = |\langle A'_i | A'_j \rangle|^2. \quad (24)$$

One can see that symmetries can be *represented* by *unitary*⁷ operators U :

$$|A'_i\rangle = U|A_j\rangle, \quad \langle A'_i | A'_j \rangle = \langle A_i | \underbrace{U^\dagger U}_1 | A_j \rangle. \quad (25)$$

In QFT one usually reformulates a symmetry transformation of *states* as a change of *operators* \mathcal{O}_k via

$$\langle A_i | \mathcal{O}_k(x) | A_j \rangle \xrightarrow{\mathcal{S}} \langle A'_i | \mathcal{O}_k(x) | A'_j \rangle = \langle A_i | \mathcal{O}'_k(x) | A_j \rangle, \quad \mathcal{O}'_k(x) \equiv U^\dagger \mathcal{O}_k(x) U. \quad (26)$$

⁶This requirement is crucial.

⁷or anti-unitary (e.g., in the case of time reversal).

For example, translational invariance leads to a relation between matrix elements of *quantum* fields, e.g.,

$$\langle A_i | \phi(x) | A_j \rangle = \langle A_i | \phi'(x+a) | A_j \rangle = \langle A_i | U^\dagger(a) \phi(x+a) U(a) | A_j \rangle \quad (27)$$

for any states. As a consequence, any *quantum* field in a translational invariant theory should satisfy

$$\phi(x+a) = U(a)\phi(x)U^\dagger(a). \quad (28)$$

One can also find similar *constraints* on quantum operators due to other symmetries.

By means of the Noether theorem we can get almost at no cost the expressions for energy-momentum $P_\mu = (\mathcal{H}, \mathbf{P})$ and charge Q , which we used in Sec. 2. For example, P_μ is nothing else but the conserved “charges”, which correspond to space-time translations. Indeed, the Noether current in this case is just the energy-momentum tensor $T_{\mu\nu}$

$$\phi'(x+a) = \phi(x), \quad \text{expand in } a \Rightarrow \delta\phi(x) = -a_\nu \partial_\nu \phi(x), \quad (29)$$

$$\delta\mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \partial_\nu (-a_\nu \mathcal{L}) \Rightarrow J_\mu = -a_\mu \mathcal{L} + a_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi = a_\nu T_{\mu\nu}. \quad (30)$$

According to Eq. (23), for every a_μ we have $P_\nu = \int d\mathbf{x} T_{0\nu}$, i.e., conserved total energy-momentum. In the same way, we can apply the Noether theorem to phase transformations of our *complex* field and get

$$\phi'(x) = e^{i\alpha} \phi(x), \quad \delta\phi(x) = i\alpha \phi(x), \quad J_\mu = i(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger), \quad Q = \int d\mathbf{x} J_0. \quad (31)$$

The corresponding quantum operators, i.e., $\hat{\mathcal{H}}$ (12) or \hat{Q} (16), are obtained⁸ (modulo ordering issues) from these (classical) expressions by plugging in quantum field $\hat{\phi}$ from Eq.(8). It turns out that the charges act as *generators* of symmetries, e.g., for space-time translations the unitary operator from Eq.(28) is given by

$$U(a) = \exp\left(i\hat{P}_\mu a_\mu\right), \quad \hat{\phi}(x+a) = U(a)\hat{\phi}(x)U^\dagger(a). \quad (32)$$

In addition, *conserved* quantities can be used to define a convenient *basis* of states, e.g., we characterize our particle states by eigenvalues of P_μ , and Q :

$$|\mathbf{p}\rangle \equiv |\mathbf{p}, +1\rangle, \quad |\bar{\mathbf{p}}\rangle \equiv |\mathbf{p}, -1\rangle \Rightarrow \hat{Q}|\mathbf{p}, q\rangle = q|\mathbf{p}, q\rangle, \quad \hat{\mathbf{P}}|\mathbf{p}, q\rangle = \mathbf{p}|\mathbf{p}, q\rangle. \quad (33)$$

It is worth mentioning that some symmetries can mix fields, e.g.,

$$\phi'_i(x') = S_{ij}(a)\phi_j(x) \Rightarrow \phi_i(x') = S_{ij}(a)U(a)\phi_j(x)U^\dagger(a), \quad x' = x'(x, a). \quad (34)$$

Typical examples are fields with non-zero spin: they have several components, which also change under coordinate rotations (more generally, under Lorentz transformations). Moreover, it is the Lorentz symmetry that allows us to classify fields as different *representations* of the corresponding *group*.

Let us discuss this in more detail. We can describe fields involving several degrees of freedom (per space point) by adding more (and more) *indices* $\phi(x) \rightarrow \Phi_\alpha^i(x)$. One can split the indices into two groups: *space-time* (α) and *internal* (i). The former are associated with space-time transformations, while the latter with transformations in the “internal” space:

$$\text{Lorentz transform } \Lambda : \Phi_\alpha^i(\Lambda x) = S_{\alpha\beta}(\Lambda)\Phi_\beta^i(x), \quad (35)$$

$$\text{Internal transform } a : \Phi_\alpha^i(x) = U^{ij}(a)\Phi_\alpha^j(x). \quad (36)$$

⁸We leave this as an exercise.

A quantum field in this case can be represented as

$$\Phi_\alpha^i(x) = \frac{1}{(2\pi)^{3/2}} \sum_s \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[u_\alpha^s(\mathbf{p}) (a_{\mathbf{p}}^-)^i e^{-ipx} + v_\alpha^{*s}(\mathbf{p}) (b_{\mathbf{p}}^+)^i e^{+ipx} \right]. \quad (37)$$

Here the factors $e^{\pm ipx}$ with $p_0 = \omega_{\mathbf{p}}$ (plane waves) guarantee that every component of Φ_α^i satisfies the KG equation. The sum in Eq.(37) is over all polarization states, which are characterized by polarization ‘‘vectors’’ for particles $u_\alpha^s(\mathbf{p})$ annihilated by $(a_{\mathbf{p}}^-)^i$, and anti-particles $v_\alpha^{*s}(\mathbf{p})$ created by $(b_{\mathbf{p}}^+)^i$. The conjugated field $(\Phi_\alpha^i)^\dagger$ involves (conjugated) polarization vectors for (anti) particles that are (annihilated) created. Let us give a couple of examples:

- Quarks are *coloured fermions* Ψ_α^i and, e.g., $(a_{\mathbf{p}}^-)^b$ annihilates the ‘‘blue’’ quark in a spin state s . The latter is characterized by a spinor $u_\alpha^s(\mathbf{p})$;
- There are *eight vector* gluons G_μ^a . So $(a_{\mathbf{p}}^-)^a$ annihilates a gluon a in spin state s having polarization $u_\alpha^s(\mathbf{p}) \rightarrow \epsilon_\mu^s(\mathbf{p})$.

Since Lorentz symmetry plays a key role in QFT, we elaborate on some of its non-trivial representations and consider vector and fermion fields in more detail.

3.1 Massive vector fields

A charged Vector Field (e.g., a W -boson) can be written as

$$W_\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=1}^3 \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[\left(\epsilon_\mu^\lambda(\mathbf{p}) a_{\mathbf{p}}^-(\mathbf{p}) e^{-ipx} + \epsilon_\mu^{*\lambda}(\mathbf{p}) b_{\mathbf{p}}^+(\mathbf{p}) e^{+ipx} \right) \right]. \quad (38)$$

A massive spin-1 particle has 3 independent polarization vectors, which satisfy

$$p_\mu \epsilon_\mu^\lambda(\mathbf{p}) = 0, \quad \epsilon_\mu^\lambda(\mathbf{p}) \epsilon_\mu^{*\lambda'}(\mathbf{p}) = -\delta^{\lambda\lambda'}, \quad \sum_{\lambda=1}^3 \epsilon_\mu^\lambda \epsilon_\nu^{*\lambda} = -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \right) \quad [p_0 = \omega_{\mathbf{p}}].$$

The Feynman propagator can be found by considering time-ordered product of two fields

$$\langle 0|T(W_\mu(x)W_\nu^\dagger(y))|0\rangle = \frac{1}{(2\pi)^4} \int d^4p e^{-ip(x-y)} \left[\frac{-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \right)}{p^2 - m^2 + i\epsilon} \right] \quad [p_0 - \text{arbitrary}] \quad (39)$$

or by inverting the quadratic form of the (free) Lagrangian

$$\mathcal{L} = -\frac{1}{2} W_{\mu\nu}^\dagger W_{\mu\nu} + m^2 W_\mu^\dagger W_\mu, \quad W_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu.$$

One can show that one of the polarization vectors $\epsilon_\mu^L \simeq p_\mu/m + \mathcal{O}(m)$ and *diverges* in the limit $p_\mu \rightarrow \infty$ ($m \rightarrow 0$). This indicates that one should be careful when constructing models with massive vector fields. We will return to this issue later.

3.2 Massless vector fields

Massless (say photon) vectors are usually represented by

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=0}^3 \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left[\epsilon_\mu^\lambda(\mathbf{p}) a_{\mathbf{p}}^-(\mathbf{p}) e^{-ipx} + \text{h.c.} \right]. \quad (40)$$

with

$$\epsilon_\mu^\lambda(\mathbf{p})\epsilon_\mu^{*\lambda'}(\mathbf{p}) = g^{\lambda\lambda'}, \quad \epsilon_\mu^\lambda(\mathbf{p})\epsilon_\nu^{*\lambda}(\mathbf{p}) = g_{\mu\nu}, \quad [a_\lambda^-(\mathbf{p}), a_{\lambda'}^+(\mathbf{p}')] = -g_{\lambda\lambda'}\delta_{\mathbf{p},\mathbf{p}'}.$$

The corresponding Feynman propagator can be given by

$$\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle = \frac{1}{(2\pi)^4} \int d^4p e^{-ip(x-y)} \left[\frac{-ig_{\mu\nu}}{p^2 + i\epsilon} \right]$$

In spite of the fact that we sum over four polarizations in Eq.(40) only *two* of them are *physical*! This reflects the fact that the vector-field Lagrangian in the massless case $m = 0$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

is invariant under $A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x)$ for arbitrary $\alpha(x)$ (*gauge symmetry*). Additional *conditions* (gauge-fixing) are needed to get rid of unphysical states.

3.3 Fermion fields

Spin-1/2 fermion fields (e.g., leptons) are given by⁹

$$\psi^\alpha(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \sum_{s=1,2} [u_s^\alpha(\mathbf{p})a_s^-(\mathbf{p})e^{-ipx} + v_s^\alpha(\mathbf{p})b_s^+(\mathbf{p})e^{+ipx}],$$

where we explicitly write the *spinor* (Dirac) index α for u_s, v_s and the quantum operator ψ . The former satisfy the 4×4 matrix (Dirac) equations

$$(\hat{p} - m)u_s(\mathbf{p}) = 0, \quad (\hat{p} + m)v_s(\mathbf{p}) = 0, \quad \hat{p} \equiv \gamma_\mu p_\mu, \quad p_0 \equiv \omega_{\mathbf{p}} \quad (41)$$

and correspond to particles (u_s) or antiparticles (v_s). In Eq.(41) we use gamma-matrices

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu \equiv [\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu} \mathbf{1} \quad \Rightarrow \quad \gamma_0^2 = \mathbf{1}, \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -\mathbf{1}$$

to account for *two* spin states ($s = 1, 2$) of particles and antiparticles. Fermion fields transform under the Lorentz group $x' = \Lambda x$ as (cf. Eq.(35))

$$\psi'(x') = \mathcal{S}_\Lambda \psi(x), \quad \psi'(x')^\dagger = \psi(x) \mathcal{S}_\Lambda^\dagger. \quad (42)$$

It turns out that the 4×4 matrix $\mathcal{S}_\Lambda^\dagger \neq \mathcal{S}_\Lambda^{-1}$ but $\mathcal{S}^{-1} = \gamma_0 \mathcal{S}^\dagger \gamma_0$. Due to this, it is convenient to introduce a *Dirac-conjugated* spinor $\bar{\psi}(x) \equiv \psi^\dagger \gamma_0$. The latter enters into

$$\begin{aligned} \bar{\psi}'(x')\psi'(x') &= \bar{\psi}(x)\psi(x), & \text{Lorentz scalar;} \\ \bar{\psi}'(x')\gamma_\mu\psi'(x') &= \Lambda_{\mu\nu}\bar{\psi}(x)\gamma_\nu\psi(x), & \text{Lorentz vector.} \end{aligned}$$

This allows us to convince ourselves that the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi} \left(i\hat{\partial} - m \right) \psi$$

is also a Lorentz scalar, i.e., respects Lorentz symmetry. Dirac-conjugated spinors can be used to impose Lorentz-invariant normalization on u and v :

$$\bar{u}_s(\mathbf{p})u_r(\mathbf{p}) = 2m\delta_{rs}, \quad \bar{v}_s(\mathbf{p})v_r(\mathbf{p}) = -2m\delta_{rs},$$

⁹There exists a charge-conjugation matrix $C = i\gamma_2$, which relates spinors for particles u and antiparticles v , e.g., $v = Cu^*$.

An important fact about quantum fermion fields is that, contrary to the case of scalar or vector (*boson*) fields, the creation/annihilation operators for fermions $a_{s,\mathbf{p}}^\pm$ and antifermions $b_{s,\mathbf{p}}^\pm$ *anticommute*:

$$\begin{aligned} \left[a_{r,\mathbf{p}}^-, a_{s,\mathbf{p}'}^+ \right]_+ &= \left[b_{r,\mathbf{p}}^-, b_{s,\mathbf{p}'}^+ \right]_+ = \delta_{sr} \delta(\mathbf{p} - \mathbf{p}') \\ \left[a_{r,\mathbf{p}}^\pm, a_{s,\mathbf{p}'}^\pm \right]_+ &= \left[b_{r,\mathbf{p}}^\pm, b_{s,\mathbf{p}'}^\pm \right]_+ = \left[a_{r,\mathbf{p}}^\pm, b_{s,\mathbf{p}'}^\pm \right]_+ = 0. \end{aligned}$$

Due to this, fermions obey the *Pauli principle*, e.g., $a_{r,\mathbf{p}}^+ a_{r,\mathbf{p}}^+ = 0$. Moreover, one can explicitly show that quantization of bosons (integer spin) with anticommutators or fermions (half-integer spin) with commutators leads to inconsistencies (violates the *Spin-Statistics* theorem).

Let us continue our discussion of free fermions by emphasizing the difference between the notions of *Chirality* and *Helicity*. Two independent solutions for *massive* fermions ($u_{1,2}$) can be chosen to correspond to two different *helicities* — projections of spin vector \mathbf{s} onto direction of \mathbf{p} :

$$\mathcal{H} = \mathbf{s} \cdot \mathbf{n}, \quad \mathbf{n} = \mathbf{p}/|\mathbf{p}|. \quad \begin{array}{cc} \text{Left-Handed} & \text{Right-Handed} \\ \begin{array}{c} \text{p} \\ \rightarrow \\ \text{s} \\ \leftarrow \end{array} & \begin{array}{c} \text{p} \\ \rightarrow \\ \text{s} \\ \rightarrow \end{array} \end{array} \quad (43)$$

In *free* motion it is *conserved* and serves as a good quantum number. However, it is not a Lorentz-invariant quantity. Indeed, we can flip the sign of particle momentum by moving with speed faster than $v = |\mathbf{p}|/p_0$. As a consequence, $\mathbf{n} \rightarrow -\mathbf{n}$ and $\mathcal{H} \rightarrow -\mathcal{H}$. However, *helicity* for a *massless* particle is the same for all inertial observers and coincides with *chirality*, which is a *Lorentz-invariant* concept.

By definition Left (ψ_L) and Right (ψ_R) *chiral* spinors are eigenvectors of

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \Rightarrow [\gamma_\mu, \gamma_5]_+ = 0, \quad \gamma_5^2 = 1, \quad \gamma_5^\dagger = \gamma_5, \quad (44)$$

where

$$\gamma_5 \psi_L = -\psi_L, \quad \gamma_5 \psi_R = +\psi_R. \quad (45)$$

Any spinor ψ can be decomposed as

$$\psi = \psi_L + \psi_R, \quad \psi_{L/R} = P_{L/R} \psi, \quad P_{L/R} = \frac{1 \mp \gamma_5}{2}. \quad (46)$$

Rewriting the Dirac Lagrangian in terms of chiral components

$$\mathcal{L} = i \underbrace{(\bar{\psi}_L \hat{\partial} \psi_L + \bar{\psi}_R \hat{\partial} \psi_R)}_{\text{conserve chirality}} - m \underbrace{(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)}_{\text{break chirality}}, \quad (47)$$

we see that, indeed, it is the mass term that mixes two chiralities. Due to this, it violates *chiral* symmetry corresponding to the independent rotation of left and right components

$$\psi \rightarrow e^{i\gamma_5 \alpha} \psi. \quad (48)$$

Consequently, if we drop the mass term, the symmetry of the Lagrangian is enhanced.

Up to now we were discussing the so-called Dirac mass term. For *neutral* fermions (e.g., neutrino) there is another possibility — a *Majorana* mass. Since charge-conjugation applied to fermion fields, $\psi \rightarrow \psi^c$, *flips* chirality, we can use ψ_L^c in place of ψ_R to write

$$\mathcal{L} = \frac{1}{2} (i\bar{\psi}_L \hat{\partial} \psi_L - m\bar{\psi}_L \psi_L^c). \quad (49)$$

As a consequence, to describe Majorana particles, we need only two components instead of four since antiparticles coincide with particles in this case. At the moment, the nature of neutrinos is unclear and we refer to [18] for more details.

4 From free to interacting fields

Let us summarize what we have learned so far. If we have a Lagrangian \mathcal{L} at hand, we can

- Derive *EOM* (via the *Action Principle*);
- Find the *Symmetries* of the Action $\mathcal{A} = \int d^4x \mathcal{L}$;
- Find *Conserved* quantities (via the *Noether Theorem*).

However, we usually *start* building our models by *postulating* symmetries. Indeed, we assume that a general QFT Lagrangian \mathcal{L} is

- a Lorentz (Poincare) invariant (i.e., a sum of Lorentz scalars),
- Local (involves a finite number of partial derivatives),
- Real (hermitian) (respects unitarity=conservation of probability)

In addition, one can impose other symmetries and get further restrictions on the model. Having all this in mind, we can proceed further and discuss particle *interactions*.

In HEP, a typical collision/scattering experiment deals with “*free*” initial and final states and considers *transitions* between these states. To account for this in a quantum theory, one introduces the *S-matrix* with matrix elements

$$\mathcal{M} = \langle \beta | S | \alpha \rangle, \quad \mathcal{M} = \delta_{\alpha\beta} + (2\pi)^4 \delta^4(p_\alpha - p_\beta) i M_{\alpha\beta} \quad (50)$$

giving amplitudes for possible transitions between *in* $|\alpha\rangle$ and *out* $|\beta\rangle$ states:

$$|\alpha\rangle = \tilde{a}_{\mathbf{p}_1}^+ \dots \tilde{a}_{\mathbf{p}_r}^+ |0\rangle, \quad |\beta\rangle = \tilde{a}_{\mathbf{k}_1}^+ \dots \tilde{a}_{\mathbf{k}_s}^+ |0\rangle, \quad \tilde{a}_{\mathbf{p}}^+ = (2\pi)^{3/2} \sqrt{2\omega_{\mathbf{p}}} a_{\mathbf{p}}^+, \quad (51)$$

where for convenience¹⁰ (see also Eq.(62)) we rescale our creation/annihilation operators. Given the matrix element $M_{\alpha\beta}$, one can calculate the differential probability (per unit volume per unit time) to evolve from $|\alpha\rangle$ to $|\beta\rangle$:

$$dw = \frac{n_1 \dots n_r}{(2\omega_{p_1}) \dots (2\omega_{p_r})} |M_{\alpha\beta}|^2 d\Phi_s, \quad (52)$$

where n_i correspond to initial-state particle densities, and an element of phase space is given by

$$d\Phi_s = (2\pi)^4 \delta^4(p_{in} - k_{out}) \frac{d\mathbf{k}_1}{(2\pi)^3 (2\omega_{k_1})} \dots \frac{d\mathbf{k}_i}{(2\pi)^3 (2\omega_{k_i})} \quad (53)$$

with $p_{in} = \sum p_i$ and $k_{out} = \sum k_i$. Since we are usually interested in processes involving one or two particles in the initial state, it is more convenient to consider the differential decay width $d\Gamma$ in the rest frame of a particle with mass m , or cross-section $d\sigma$ of a process $2 \rightarrow s$:

$$d\Gamma = \Phi_\Gamma |M_{1 \rightarrow s}|^2 d\Phi_s, \quad \Phi_\Gamma = \frac{1}{2m}, \quad (54)$$

$$d\sigma = \Phi_\sigma |M|^2 d\Phi_s, \quad \Phi_\sigma = \frac{1}{4\sqrt{(p_1 p_2)^2 - p_1^2 p_2^2}}. \quad (55)$$

In Eq.(55) the factor Φ_σ is *Lorentz-invariant* and is expressed in terms of four-momenta of initial particles p_1 and p_2 . The total width Γ and total cross-section σ can be obtained by integration over the momenta of final particles restricted by energy-momentum conservation due to the four-dimensional δ -function in Eq.(53).

In QFT, the S-matrix is given by the time-ordered exponent

$$S = T e^{-i \int d^4x \mathcal{H}_I(x)} = T e^{i \int d^4x \mathcal{L}_I(x)}. \quad (56)$$

¹⁰The states created by \tilde{a}^+ are normalized in the *relativistic-invariant* way.

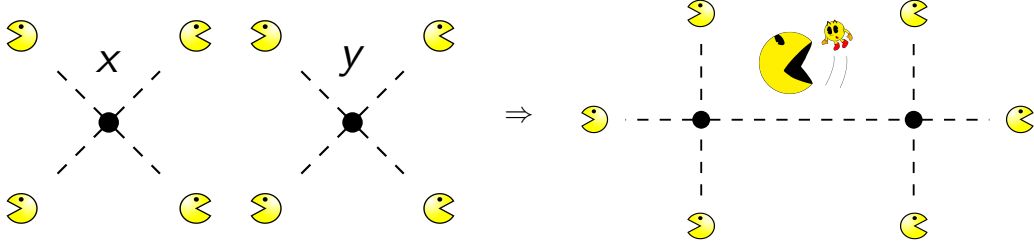


Fig. 4: The Wick theorem at work: one of the contributions.

involving the interaction Hamiltonian \mathcal{H}_I (Lagrangian \mathcal{L}_I).

The interaction Lagrangian $\mathcal{L}_I = \mathcal{L}_{full} - \mathcal{L}_0$ is a sum of *Lorentz-invariant* terms having more than *two* fields and more ∂_μ than in the quadratic part \mathcal{L}_0 , which corresponds to free particles. It is worth noting that in Eq.(56) we treat \mathcal{L}_I (\mathcal{H}_I) as an operator built from *free*¹¹ quantum fields (i.e., certain combinations of a^\pm and b^\pm).

The *time-ordering* operation, which was used to define particle propagators, is generalized in Eq.(56) to account for more than two fields originating from \mathcal{L}_I

$$T\Phi_1(x_1)\dots\Phi_n(x_n) = (-1)^k \Phi_{i_1}(x_{i_1})\dots\Phi_{i_n}(x_{i_n}), \quad x_{i_1}^0 > \dots > x_{i_n}^0. \quad (57)$$

Here the factor $(-1)^k$ appears due to k possible permutations of *fermion* fields.

As it was mentioned earlier, (interaction) Lagrangians should be hermitian. Any scalar combination of quantum fields can, in principle, be included in \mathcal{L}_I , e.g.,

$$\begin{aligned} \mathcal{L}_I : \quad & g\phi^3(x), \quad \lambda\phi^4(x), \quad y\bar{\psi}(x)\psi(x)\phi(x) \\ & e\bar{\psi}(x)\gamma_\mu\psi(x)A_\mu(x), \quad G[(\bar{\psi}_1\gamma_\mu\psi_2)(\bar{\psi}_3\gamma_\mu\psi_4) + \text{h.c.}] \end{aligned}$$

The parameters (couplings) g , λ , e , y , and G set the strength of the interactions. An important characteristic of any coupling in the QFT model is its *dimension*, which can be deduced from the fact that Lagrangian has dimension $[\mathcal{L}] = 4$. One can notice that all the couplings (hidden) in the T-shirt Lagrangian are *dimensionless*. This fact has crucial consequences for the self-consistency of the SM model.

4.1 Perturbation theory

In an interacting theory it is very hard, if not impossible, to calculate the S-matrix (56) exactly. Usually, we assume that the couplings in \mathcal{L}_I are small allowing us to treat the terms in \mathcal{L}_I as *perturbations* to \mathcal{L}_0 . As a consequence, we expand the T-exponent and restrict ourselves to a finite number of terms. In the simplest case of $\mathcal{L}_I = -\lambda\phi^4/4!$ we have at the n th order

$$\frac{i^n}{n!} \left[\frac{\lambda}{4!} \right]^n \langle 0 | \tilde{a}_{\mathbf{k}_1}^- \dots \tilde{a}_{\mathbf{k}_s}^- \int dx_1 \dots dx_n T [\phi(x_1)^4 \dots \phi(x_n)^4] \tilde{a}_{\mathbf{p}_1}^+ \dots \tilde{a}_{\mathbf{p}_r}^+ | 0 \rangle. \quad (58)$$

To proceed, one uses the *Wick theorem*:









$$T\Phi_1\dots\Phi_n = \sum (-1)^\sigma \langle 0 | T(\Phi_{i_1} \Phi_{i_2}) | 0 \rangle \dots \langle 0 | T(\Phi_{i_{m-1}} \Phi_{i_m}) | 0 \rangle : \Phi_{i_{m+1}} \dots \Phi_{i_n} :, \quad (59)$$

where the sum goes over all possible ways to pair the fields. The Wick theorem (59) expresses *time-ordered* products of fields in terms of *normal-ordered* ones and propagators. The normal-ordered operation puts *all* annihilation operators originating from different Φ s to the right. It also cares about fermions, e.g.,

$$: a_1^- a_2^+ a_3^- a_4^- a_5^+ a_6^- := (-1)^\sigma a_2^+ a_5^+ a_1^- a_3^- a_4^- a_6^-, \quad (60)$$

¹¹More precisely, operators in the *interaction* picture.

Table 1: Feynman rules for external states.

incoming scalar	1		incoming fermion	$u_s(\mathbf{p})$	
outgoing scalar	1		outgoing fermion	$\bar{u}_s(\mathbf{p})$	
incoming vector	$\epsilon_\mu^\lambda(\mathbf{p})$		incoming antifermion	$\bar{v}_s(\mathbf{p})$	
outgoing vector	$\epsilon_\mu^{*\lambda}(\mathbf{p})$		outgoing antifermion	$v_s(\mathbf{p})$	

where σ correspond to the number of fermion permutations (cf. Eq.(57)). In Fig. 4 a cartoon, which illustrates Eq.(59) for one of the contributions to $T[\mathcal{L}_I(x)\mathcal{L}_I(y)]$, is provided.

After application of the Wick theorem we have to calculate

$$\langle 0 | \tilde{a}_{\mathbf{k}_1}^- \dots \tilde{a}_{\mathbf{k}_s}^- : \Phi_{i_{m+1}} \dots \Phi_{i_n} : \tilde{a}_{\mathbf{p}_1}^+ \dots \tilde{a}_{\mathbf{p}_r}^+ | 0 \rangle. \quad (61)$$

To get a *non-zero* matrix element, all a^- (a^+) in the normal product of fields from the Lagrangian have to be “killed” by (commuted with) a^+ (a^-) from the initial (final) states.

For our *generalized* field (37) we have

$$\begin{aligned} [\Phi_\alpha^i(x), (a_{\mathbf{p}}^+)_s^i] &= \underbrace{\frac{e^{-ipx}}{(2\pi)^{3/2} \sqrt{2\omega_p}}}_{\text{common to all fields}} u_\alpha^s(\mathbf{p}), & \text{initial state polarization (particle);} \\ [(b_{\mathbf{p}}^-)_s^i, \Phi_\alpha^i(x)] &= \frac{e^{+ipx}}{(2\pi)^{3/2} \sqrt{2\omega_p}} v_\alpha^{*s}(\mathbf{p}), & \text{final state polarization (antiparticle).} \end{aligned} \quad (62)$$

and one clearly sees that the factors in the denominators Eq.(62) are avoided when the rescaled \tilde{a}^\pm (or \tilde{b}^\pm) operators (51) are used.

All this machinery can be implemented in a set of *Feynman rules*, which are used to draw (and evaluate) *Feynman diagrams*. Every Feynman diagram involves interaction *vertices*, *external* and *internal* lines. Internal lines connect two vertices and correspond to propagators. The expression for propagators can be derived from \mathcal{L}_0 , e.g.,

$$\left. \begin{aligned} \langle 0 | T(\phi(x)\phi^\dagger(y)) | 0 \rangle \\ \langle 0 | T(\psi(x)\bar{\psi}(y)) | 0 \rangle \\ \langle 0 | T(W_\mu(x)W_\nu^\dagger(y)) | 0 \rangle \end{aligned} \right\} = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} \left\{ \begin{array}{ll} 1 & \begin{array}{l} \text{---} \xrightarrow{p} \text{---} \\ \phi; \end{array} \\ \hat{p} + m & \begin{array}{l} \text{---} \xrightarrow{p} \text{---} \\ \psi; \end{array} \\ -g_{\mu\nu} + p_\mu p_\nu / m^2 & \begin{array}{l} \text{---} \xrightarrow{p} \text{---} \\ \mu \quad \nu \\ W_\mu. \end{array} \end{array} \right. \quad (63)$$

One can notice that all the dependence on x_i of the integrand in Eq.(58) comes from either Eq.(62) or Eq.(63). As a consequence, it is possible to carry out the integration for *every* x_i

$$\int d^4x_i e^{-ix_i(p_1 + \dots + p_n)} = (2\pi)^4 \delta^4(p_1 + \dots + p_n) \quad (64)$$

and obtain a δ -function reflecting energy-momentum conservation at the corresponding vertex.

Depending on the direction of momenta, the external lines represent incoming or outgoing particles (see Table 1). Again, the corresponding factors (=polarization vectors) are derived from \mathcal{L}_0 . Notice that we explicitly write the Lorentz indices for vector particles and suppress the Dirac indices for

fermions. To keep track of the index contractions in the latter case, one uses *arrows* on the fermion lines.¹²

Let us turn to interaction vertices. The corresponding Feynman rules can be derived from $\mathcal{A}_I = \int d^4\mathcal{L}_I$. It is convenient to do this by carrying out a Fourier transform to “convert” coordinate derivatives to momenta and considering variations of the action. In the case of $\mathcal{L}_I = -\lambda\phi^4/4!$ we have (all momenta are assumed to be incoming)

$$i \frac{\delta^4 \mathcal{A}_I[\phi]}{\delta\phi(p_1)\delta\phi(p_2)\delta\phi(p_3)\delta\phi(p_4)} \Big|_{\phi=0} \Rightarrow \underbrace{(2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)}_{\text{conservation of energy-momentum}} \times [-i\lambda]. \quad (65)$$

In a typical diagram all $(2\pi)^4\delta(\dots)$ factors (but *one*¹³) reflecting the energy-momentum conservation at each vertex, are removed by the momentum integration originating from propagators, Eq.(63). Due to this, we also omit these factors (see, Table 2 for examples).

Given Feynman rules, we can draw all possible diagrams that contribute to a process and evaluate the amplitude. We do not provide the precise prescription here (see textbooks [12–16] for details) but just mention the fact that one should keep in mind various *symmetry* factors and relative *signs* that can appear in real calculations.

In order to get probabilities, we have to *square* matrix elements, e.g.,

$$|M|^2 = MM^\dagger \Rightarrow \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \quad (66)$$

Sometimes we do not care about polarization states of initial or final particles, so we have to *sum* over *final* polarization and *average* over *initial* ones. That is where *spin-sum* formulas, e.g.,

$$\sum_s u_s(\mathbf{p}_1)\bar{u}_s(\mathbf{p}_1) = \hat{p}_1 + m, \quad \sum_s v_s(\mathbf{p}_2)\bar{v}_s(\mathbf{p}_2) = \hat{p}_2 - m \quad (67)$$

become handy

$$MM^\dagger \rightarrow \sum_{s,r} (\bar{u}_s A v_r)(\bar{v}_r A^\dagger u_s) = \text{Tr} \left[(\hat{p}_1 + m) A (\hat{p}_2 - m) A^\dagger \right]. \quad (68)$$

As a consequence, one can utilize the well-known machinery for gamma-matrix traces to evaluate probabilities in an efficient way.

Let us continue by mentioning that only in *tree* graphs, such as

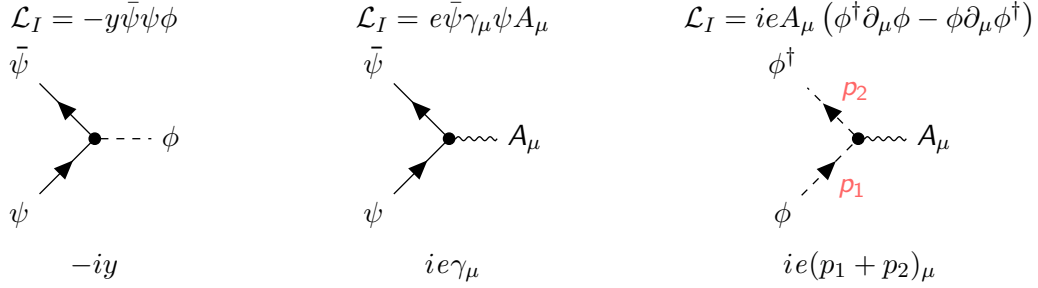
$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} \rightarrow \begin{array}{c} p_4 \\ p_5 \\ p_6 \end{array} \quad \Rightarrow \quad (2\pi)^4 \delta^4 \left(\sum_{i=1}^3 p_i - \sum_{i=4}^6 p_i \right) [-i\lambda]^2 \frac{i}{q^2 - m^2 + i\epsilon},$$

all the integrations (due to propagators) are “killed” by vertex δ -functions. However, nothing forbids us from forming *loops*. In this case, we have *integrals* over unconstrained momenta, e.g., in the ϕ^4 -theory

$$\begin{array}{c} q \\ \text{---} \text{---} \text{---} \text{---} \\ k - q \end{array} : \quad I_2(k) \equiv \int \frac{d^4q}{[q^2 + i\epsilon][(k-q)^2 + i\epsilon]} \sim \int^\infty \frac{|q|^3 d|q|}{|q|^4} \sim \ln \infty,$$

¹²There are subtleties when interactions involve Majorana fermions.

¹³We factor it out in the definition of $M_{\alpha\beta}$, see Eq.(50).

Table 2: Vertex Feynman rules. Derivatives in \mathcal{L}_I correspond to particle momenta.


which can lead to *divergent* (meaningless?) results. This is again a manifestation of *UV* divergences due to *large* momenta (“small distances”).

A natural question arises: Do we have to abandon QFT? Since we still use it, there are reasons *not* to do this. Indeed, we actually do not know physics up to infinitely small scales and our extrapolation can not be adequate in this case. To make sense of the integrals, we can *regularize* them, e.g., introduce a “cut-off” $|q| < \Lambda$,

$$I_2^\Lambda(k) = i\pi^2 \left[\ln \frac{\Lambda^2}{k^2} + 1 \right] + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right) = i\pi^2 \left[\ln \frac{\Lambda^2}{\mu^2} - \ln \frac{k^2}{\mu^2} + 1 \right] + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right) \quad (69)$$

or use another convenient possibility — *dimensional* regularization, when $d = 4$ space-time is formally continued to $d = 4 - 2\varepsilon$ dimensions:

$$I_2^{4-2\varepsilon}(k) = \mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon}q}{q^2(k-q)^2} = i\pi^2 \left(\frac{1}{\varepsilon} - \ln \frac{k^2}{\mu^2} + 2 \right) + \mathcal{O}(\varepsilon). \quad (70)$$

Both the regularized integrals are now convergent¹⁴ and share the same logarithmic dependence on external momentum k . One can also notice a (one-to-one) correspondence between a *logarithmically* divergent contribution $\log \Lambda^2/\mu^2$ in Eq.(69) and the pole term $1/\varepsilon$ in Eq.(70). However, the constant terms are *different*. How do we make sense of this ambiguity?

The crucial observation here is that the divergent pieces, which blow up when we try to remove the regulators ($\Lambda \rightarrow \infty$ or $\varepsilon \rightarrow 0$), are *local*, i.e., depend polynomially on external kinematical parameters. This fact allows us to *cancel* them by the so-called counterterm (CT) vertices. The latter can be interpreted as new terms in \mathcal{L}_I . Moreover, in a *renormalizable* QFT model additional (divergent) contributions have the same form as the initial Lagrangian and thus can be “absorbed” into redefinition of fields and parameters.

One can revert the reasoning and assume that the initial Lagrangian is written in terms of the so-called *bare* (unobservable) quantities. The predictions of the model are finite since the explicit dependence of Feynman integrals on the cut-off Λ (or ε) is actually compensated by the implicit dependence of bare fields and parameters. In some sense these quantities represent our ignorance of dynamics at tiny scales. Physical fields and parameters are always “dressed” by clouds of virtual particles.

It is obvious that working with *bare* quantities is not very convenient. One usually makes the dependence on Λ (or ε) explicit by introduction of divergent Z -factors for *bare* fields (ϕ_B), masses (m_B^2), and couplings (λ_B), e.g.,

$$\mathcal{L}_{full} = \frac{1}{2}(\partial\phi_B)^2 - \frac{m_B^2}{2}\phi_B^2 + \frac{\lambda_B\phi_B^4}{4!} = \frac{Z_2}{2}(\partial\phi)^2 - \frac{Z_m m^2}{2}Z_2\phi^2 + \frac{Z_\lambda\lambda}{4!}(Z_2\phi^2)^2 \quad (71)$$

¹⁴We do not discuss the issue of possible IR divergences here.

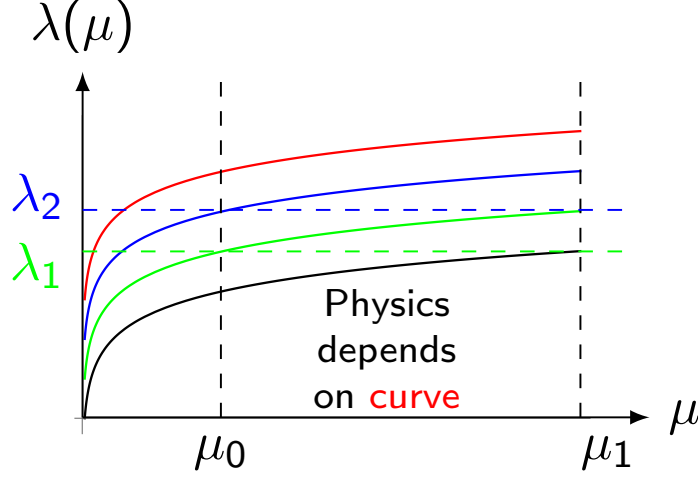


Fig. 5: Solutions of RGE for different boundary conditions.

$$= \frac{(\partial\phi)^2}{2} - \frac{m^2\phi^2}{2} + \frac{\lambda\phi^4}{4!} + \underbrace{\frac{(Z_2 - 1)}{2}(\partial\phi)^2 - \frac{(Z_m Z_2 - 1)m^2}{2}\phi^2 + (Z_4 Z_2^2 - 1)\frac{\lambda\phi^4}{4!}}_{\text{counterterms}}. \quad (72)$$

Here ϕ , m and λ denote *renormalized* (finite) quantities. Since we can always subtract something finite from infinity, there is a certain freedom¹⁵ in this procedure. So we have to impose additional *conditions* on Z s, i.e., define a *renormalization* scheme. For example, in the minimal (MS) schemes we subtract only the divergent terms, e.g., only poles in ε , while in the so-called momentum-subtraction (MOM) schemes we require amplitudes (more generally *Green functions*) to have a certain value at some fixed kinematics.

As an illustration, let us consider a scattering amplitude $2 \rightarrow 2$ in the ϕ^4 model calculated in perturbation theory:

$$\text{Diagram} = \text{Tree} + \text{One-loop} + \text{permutations} + \text{more loops} \quad (73)$$

$$= \lambda_B(\Lambda) - \frac{\lambda_B(\Lambda)^2}{2(16\pi^2)} \left(\ln \frac{\Lambda^2}{\mu^2} - \ln \frac{k^2}{\mu^2} + \dots \right) + \dots \quad (74)$$

$$= \left[\lambda(\mu) + \frac{3}{2} \frac{\lambda^2(\mu)}{16\pi^2} \ln \frac{\Lambda^2}{\mu^2} \right] - \frac{\lambda(\mu)^2}{2(16\pi^2)} \left(\ln \frac{\Lambda^2}{\mu^2} - \ln \frac{k^2}{\mu^2} + \dots \right) + \dots \quad (75)$$

$$= \lambda(\mu) + \frac{\lambda(\mu)^2}{2(16\pi^2)} \left(\ln \frac{k^2}{\mu^2} + \dots \right) + \dots \quad (76)$$

In Eq.(73) the tree-level and one-loop diagrams contributing to the matrix element are shown. The corresponding expression in terms of the bare coupling $\lambda_B(\Lambda)$ that implicitly depends on the regularization parameter Λ is given in Eq.(74). We introduce a renormalized¹⁶ coupling $\lambda(\mu)$ in Eq.(75) to make the dependence explicit:

$$\lambda_B(\Lambda) = \lambda(\mu)Z_\lambda = \lambda(\mu) \left(1 + \frac{3}{2} \frac{\lambda(\mu)}{16\pi^2} \ln \frac{\Lambda^2}{\mu^2} + \dots \right). \quad (77)$$

¹⁵Different constant terms in Eq.(69) and Eq.(70) are one manifestation of this fact.

¹⁶We use minimal subtractions here and the factor of three comes from the fact that all three one-loop graphs (s , t and u) give rise to the same *divergent* term.

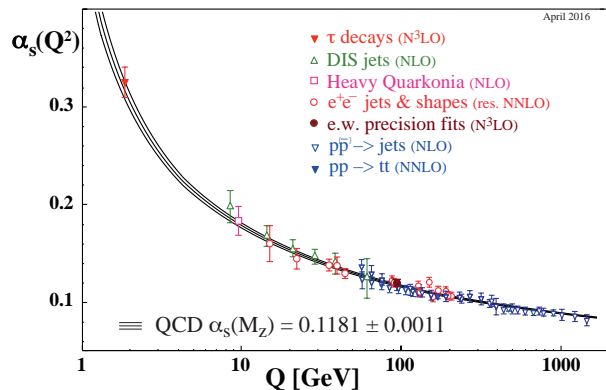


Fig. 6: The scale dependence of the strong coupling α_s .

The final result (76) is finite (when $\Lambda \rightarrow \infty$) and can be confronted with experiment. It seems to depend on an auxiliary scale μ , which inevitably appears in any renormalization scheme. The crucial point here is that *observables* (if all orders of PT are taken into account) actually do *not* depend on μ . Changing μ corresponds to a certain reshuffling of the PT series: some terms from loop corrections are absorbed into the rescaled (*running*) couplings. This allows one to improve the “convergence”¹⁷ of the series.

The scale-dependence of the *running* couplings is governed by renormalization-group equations (RGE). In the considered case we have

$$\lambda(\mu_0) \rightarrow \lambda(\mu), \quad \frac{d}{d \ln \mu} \lambda = \beta_\lambda(\lambda), \quad \beta_\lambda = \frac{3}{2} \frac{\lambda^2}{16\pi^2} + \dots \quad (78)$$

The *beta-function* β_λ can be calculated order-by-order in PT. However, the (initial) value $\lambda(\mu_0)$ needed to solve Eq.(78) is *not predicted* and has to be extracted from experiment.

It is worth pointing out here that two different numerical values of the *renormalized* self-coupling, λ_1 and λ_2 , do not necessarily correspond to different Physics. Indeed, if they are fitted from measurements at different scales, e.g., μ_0 and μ , and are related by means of RGE, they represent the *same* Physics (see Fig. 5). A prominent example is the running of the strong coupling in Quantum chromodynamics (QCD) described by (see [19])

$$\beta_{\alpha_s} = -\frac{\alpha_s^2}{4\pi} \left(11 - \frac{2}{3} n_f \right) + \dots + \mathcal{O}(\alpha_s^7), \quad n_f - \text{number of flavours.} \quad (79)$$

In Fig. 6 one can see a remarkable consistency between different measurements of $\alpha_s(\mu)$ and the scale dependence predicted by perturbative QCD.

4.2 Renormalizable or non-renormalizable?

Let us stress again that the model is called *renormalizable* if *all* the divergences that appear in loop integrals can be canceled by local counterterms due to renormalization of bare parameters and couplings from \mathcal{L}_{full} . But what happens if there is a divergent amplitude but the structure of the required subtraction does not have a counter-part in our initial Lagrangian, i.e., we do not have a coupling to absorb the infinity? Obviously, we can modify \mathcal{L}_{full} and *add* the required term (and the coupling).

¹⁷Actually, the PT series are *asymptotic* (divergent) and we speak about the behavior of a limited number of first terms here.

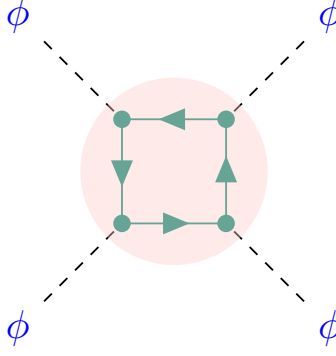


Fig. 7: One-loop correction to higgs self-interaction.

An example of such a situation can be found in the model with a scalar ϕ (e.g., Higgs) coupled to a fermion ψ (e.g., top quark) via the Yukawa interaction characterized by the coupling y

$$\mathcal{L}_I \ni \delta\mathcal{L}_Y = -y \cdot \bar{\psi}\psi\phi. \quad (80)$$

Let us assume for the moment that we set the self-coupling to zero $\lambda = 0$ and want to calculate the Higgs-scattering amplitude due to top quarks (see, Fig. 7). We immediately realize that the contribution is divergent and without $\delta\mathcal{L}_4 = -\lambda\phi^4/4!$ we are not able to cancel it. Due to this, we are forced to consider the ϕ^4 term in a consistent theory.

Since we modified \mathcal{L}_{full} , we have to re-calculate all the amplitudes. In principle, new terms in \mathcal{L}_I will generate new diagrams, which can require new interactions to be added to \mathcal{L}_I . Will this process terminate? In the case of *renormalizable* models the answer is positive. We just need to make sure that \mathcal{L}_I include *all* possible terms with *dimensionless* couplings¹⁸, or, *equivalently*, local dimension-4 operators built from quantum fields and their derivatives.

On the contrary, if one has to add more and more terms to \mathcal{L}_I , this is a signal of a *non-renormalizable* model. It looks like that we have to abandon such models since we need to measure an infinite number of couplings to predict something in this situation! However, it should be stressed that non-renormalizable models, contrary to renormalizable ones, involve couplings G_i with *negative* mass dimension $[G_i] < 0!$ Due to this, not all of them are important at *low* energies, such as

$$G_i E^{-[G_i]} \ll 1. \quad (81)$$

This explains the success of the *Fermi model* involving the dimension-6 four-fermion operator

$$- \mathcal{L}_I = G \bar{\Psi}_p \gamma_\rho \Psi_n \cdot \bar{\Psi}_e \gamma_\rho \Psi_\nu + \text{h.c.} \quad (82)$$

in the description of the β -decay $n \rightarrow p + e^- + \bar{\nu}_e$. Since the model turns out to be a harbinger of the modern electroweak theory, let us consider it in more detail and discuss its features, which eventually lead to the construction of the SM.

In 1957 R. Marshak and G.Sudarshan, R. Feynman and M. Gell-Mann modified the original Fermi theory of beta-decay to incorporate 100 % violation of Parity discovered by C.S. Wu in 1956:

$$- \mathcal{L}_{\text{Fermi}} = \frac{G_F}{2\sqrt{2}} (J_\mu^+ J_\mu^- + \text{h.c.}). \quad (83)$$

Here the current

$$J_\rho^- = (V - A)_\rho^{\text{nucleons}} + \bar{\Psi}_e \gamma_\rho (1 - \gamma_5) \Psi_{\nu_e} + \bar{\Psi}_\mu \gamma_\rho (1 - \gamma_5) \Psi_{\nu_\mu} + \dots \quad (84)$$

¹⁸Remember the T-shirt Lagrangian?

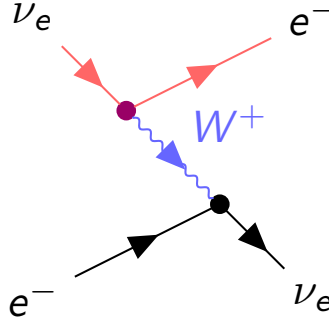


Fig. 8: A contribution to $\nu_e e$ -scattering due to charged W -boson

is the difference between Vector (V) and Axial (A) parts. This kind of *current-current* interactions can describe not only the proton beta-decay but also the muon decay $\mu \rightarrow e \nu_\mu \bar{\nu}_e$ or the process of $\nu_e e$ -scattering. Since the *Fermi* constant $G_F \simeq 10^{-5} \text{ GeV}^{-1}$, from simple *dimensional* grounds we have

$$\sigma(\nu_e e \rightarrow \nu_e e) \propto G_F^2 s, \quad s = (p_e + p_\nu)^2. \quad (85)$$

With such a dependence on energy we eventually *violate unitarity*. This is another manifestation of the fact that non-renormalizable interactions are not self-consistent.

However, a modern view on the Fermi model treats it as an *effective* field theory [20] with certain *limits of applicability*. It perfectly describes low-energy experiments and one can fit the value of G_F very precisely (see [21]). The *magnitude* of G_F tells us something about a *more fundamental* theory (the SM in our case): around $G_F^{-1/2} \sim 10^2 - 10^3 \text{ GeV}$ there should be some “New Physics” (NP) to cure the above-mentioned shortcomings. Indeed, by analogy with (renormalizable) QED we can introduce *mediators* of the weak interactions – electrically charged *vector* fields W_μ^\pm (see, e.g., Fig. 8):

$$\mathcal{L}_{\text{Fermi}} = -\frac{G_F}{2\sqrt{2}}(J_\mu^+ J_\mu^- + \text{h.c.}) \rightarrow \mathcal{L}_I = \frac{g}{2\sqrt{2}}(W_\mu^+ J_\mu^- + \text{h.c.}) \quad (86)$$

with a *dimensionless* coupling g . Since we know that weak interactions are *short-range*, the W -bosons should be *massive*. Given \mathcal{L}_I we can calculate the tree-level scattering amplitude due to the exchange of W^\pm between two fermionic currents:

$$T = i(2\pi)^4 \frac{g^2}{8} J_\alpha^+ \left[\frac{g_{\alpha\beta} - p_\alpha p_\beta / M_W^2}{p^2 - M_W^2} \right] J_\beta^-. \quad (87)$$

In the limit $|p| \ll M_W$, Eq.(87) reproduces the prediction of the effective theory (Fermi model) if we identify (“match”)

$$\text{(effective theory)} \quad \frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \quad \text{(more fundamental theory)}. \quad (88)$$

However, one can see that in the UV region ($|p| \gg M_W$) the amplitude (87) still has bad behavior, leading to all the above-mentioned problems. To deal with the issue, we utilize *gauge* symmetry, which will be discussed in the next section.

5 Gauge symmetries

We are seeking for a model of weak interactions that has good UV-properties. Let us revise how the *gauge principle* is implemented in QED. First of all, consider

$$\mathcal{L}_0 = \bar{\psi} (i\hat{\partial} - m) \psi \quad (89)$$

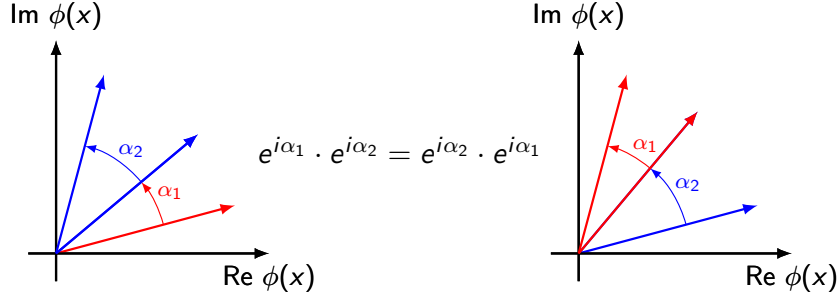


Fig. 9: $U(1)$ transformations commute with each other.

and make the *global* $U(1)$ -symmetry of \mathcal{L}_0

$$\psi \rightarrow \psi' = e^{ie\omega} \psi \quad (90)$$

local, i.e., $\omega \rightarrow \omega(x)$. In this case, the Lagrangian ceases to be invariant¹⁹:

$$\delta\mathcal{L}_0 = \partial_\mu\omega \cdot J_\mu, \quad J_\mu = -e\bar{\psi}\gamma_\mu\psi, \quad (91)$$

To compensate this term, we add the interaction of the current J_μ with the photon field A_μ :

$$\mathcal{L}_0 \rightarrow \mathcal{L} = \mathcal{L}_0 + A_\mu J_\mu = \bar{\psi} \left[i(\hat{\partial} + ie\hat{A}) - m \right] \psi, \quad A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu\omega. \quad (92)$$

The photon A_μ is an example of *gauge* field. To get the full QED Lagrangian, we should also add a kinetic term for the photon:

$$\mathcal{L}_{QED} = \bar{\psi} \left(i\hat{D} - m \right) \psi - \frac{1}{4} F_{\mu\nu}^2 \quad (93)$$

$$D_\mu = \partial_\mu + ieA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (94)$$

Here we introduce a *covariant* derivative D_μ and a *field-strength* tensor $F_{\mu\nu}$. One can check that Eq.(93) is invariant under

$$\begin{aligned} \psi &\rightarrow \psi' = e^{ie\omega(x)} \psi \\ A_\mu &\rightarrow A'_\mu = A_\mu - \partial_\mu\omega \\ D_\mu\psi &\rightarrow D'_\mu\psi' = e^{ie\omega(x)} D_\mu\psi. \end{aligned}$$

The *second* Noether theorem [17] states that theories possessing *gauge* symmetries are *redundant*, i.e., some degrees of freedom are not physical. This makes quantization non-trivial. To deal with this problem in QED, one adds a *gauge-fixing term* to the free vector-field Lagrangian:

$$\mathcal{L}_0(A) = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2\xi} (\partial_\mu A_\mu)^2 \equiv -\frac{1}{2} A_\mu K_{\mu\nu} A_\nu. \quad (95)$$

This term allows one to obtain the photon propagator by inverting²⁰ $K_{\mu\nu}$:

$$\langle 0|T A_\mu(x) A_\nu(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i [g_{\mu\nu} - (1-\xi)p_\mu p_\nu/p^2]}{p^2 + i\epsilon} e^{-ip(x-y)} \quad (96)$$

¹⁹Note that one can use this fact to get an expression for the Noether current J_μ .

²⁰If we omit the gauge-fixing term, we will not be able to invert the quadratic form.

The propagator now involves an auxiliary parameter ξ . It controls the propagation of *unphysical* longitudinal polarization $\epsilon_\mu^L \propto p_\mu$. The polarization turns out to be harmless in QED since the corresponding terms *drop out* of physical quantities, e.g., due to current conservation

$$e_\mu^L J_\mu \propto p_\mu J_\mu = 0 \quad [\text{we have no source for unphysical } \gamma]. \quad (97)$$

One can see that the propagator has good UV behaviour and falls down as $1/p^2$ for large p . The gauge symmetry of QED is $U(1)$. It is *Abelian* since the order of two transformations is irrelevant (see Fig. 9). However, if we want to apply the *gauge principle* to the case of EW interactions, we have to generalize $U(1)$ to the *Non-Abelian* case. Let us consider the $SU(n)$ group, i.e., unitary $n \times n$ matrices U_{ij} depending on $n^2 - 1$ parameters ω^a and having $\det U = 1$:

$$\psi_i \rightarrow \psi'_i = U_{ij}(\omega)\psi_j, \quad U(\omega) = e^{igt^a\omega^a}. \quad (98)$$

In general, different transformations do not commute in the non-Abelian case. This fact is reflected in commutation relations for the group *generators* t^a , which obey the $su(n)$ -algebra:

$$[t^a, t^b] = if^{abc}t^c, \quad f^{abc} - \text{structure constants}. \quad (99)$$

For *constant* ω^a the transformation (98) is a symmetry of the Lagrangian

$$\mathcal{L}_0 = \bar{\psi}_i \left(i\hat{\partial} - m \right) \psi_i, \quad i = 1, \dots, n \quad (100)$$

describing n free fermions in the *fundamental* representation of $SU(n)$.

In order to make the symmetry local, we introduce a (matrix) *covariant derivative* depending on $n^2 - 1$ gauge fields W_μ^a :

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} - igt_{ij}^a W_\mu^a. \quad (101)$$

The transformation properties of W_μ^a should guarantee that for space-time dependent $\omega^a(x)$ the covariant derivative of ψ transforms in the same way as the field itself:

$$D'_\mu \psi' = U(\omega)(D_\mu \psi), \quad U(\omega) = e^{igt^a\omega^a}. \quad (102)$$

One can find that

$$W_\mu^a \rightarrow W'^a_\mu = W_\mu^a + \partial_\mu \omega^a + gf^{abc}W_\mu^b \omega^c \quad (103)$$

$$= W_\mu^a + (D_\mu)^{ab} \omega_b, \quad (D_\mu)^{ab} \equiv \partial_\mu \delta^{ab} - ig(-if^{abc})W_\mu^c, \quad (104)$$

where we introduce the covariant derivative (101) D_μ^{ab} with generators $(t^c)^{ab} = -if^{cab}$ in the *adjoint* representation. The field-strength tensor for each component of W_μ^a is given by the commutator

$$[D_\mu, D_\nu] = -igt^a \mathcal{F}_{\mu\nu}^a, \quad \mathcal{F}_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + gf^{abc}W_\mu^a W_\nu^b. \quad (105)$$

Contrary to the $U(1)$ case, $\mathcal{F}_{\mu\nu}^a$ contains an additional term quadratic in W_μ^a . Due to this, the gauge symmetry predicts not only interactions between fermions ψ (or fields in the fundamental representation of the gauge group) and W_μ^a but also *self-interactions* of the latter (the gauge fields are “charged” under the group).

Combining all the ingredients, we can write down the following Lagrangian for an $SU(n)$ gauge (Yang-Mills) theory :

$$\mathcal{L} = \bar{\psi} \left(i\hat{D} - m \right) \psi - \frac{1}{4} \mathcal{F}_{\mu\nu}^a \mathcal{F}_{\mu\nu}^a = \mathcal{L}_0 + \mathcal{L}_I, \quad (106)$$

$$\mathcal{L}_0 = \bar{\psi} \left(i\hat{\partial} - m \right) \psi - \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a, \quad F_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a, \quad (107)$$

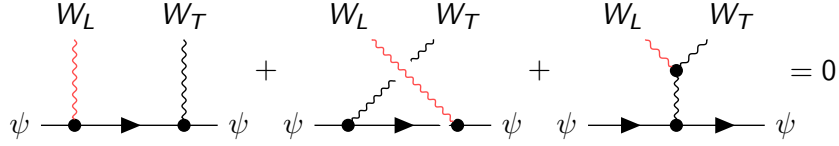


Fig. 10: Gauge symmetry at work: tree-level amplitudes with unphysical polarization (L) vanish.

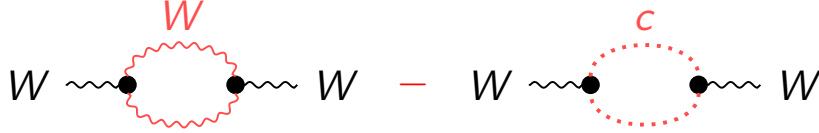


Fig. 11: Ghosts cancel contributions due to virtual unphysical states.

$$\mathcal{L}_I = g\bar{\psi}_\alpha^i \gamma_\mu^\alpha \psi_\beta^j t_{ij}^a W_\mu^a - \frac{g}{2} f^{abc} W_\mu^b W_\nu^c F_{\mu\nu}^a - \frac{g^2}{4} f^{abc} f^{ade} W_\mu^a W_\nu^b W_\mu^d W_\nu^e. \quad (108)$$

For illustration purposes we explicitly specify all the indices in the first term of interaction Lagrangian \mathcal{L}_I : the Greek ones correspond to Dirac (α, β) and Lorentz (μ) indices, while the Latin ones belong to different representations of $SU(n)$: i, j – fundamental, a – adjoint. One can also see that the strength of all interactions in \mathcal{L}_I is governed by the single dimensionless coupling g .

To quantize a Yang-Mills theory, we generalize the QED gauge-fixing term and write, e.g.,

$$\mathcal{L}_{gf} = -\frac{1}{2\xi} (F^a)^2, \quad F^a = \partial_\mu W_\mu^a \quad (109)$$

with F^a being a gauge-fixing function. This again introduces unphysical states in the W_μ^a propagator. However, contrary to the case of QED, the *fermionic* current $J_\mu^a = g\bar{\psi} t^a \gamma_\mu \psi$ is not conserved and can produce longitudinal W_μ^a . Nevertheless, the *structure* of vector-boson self-interactions guarantees that at *tree* level amplitudes involving unphysical polarizations for external W_μ^a *vanish* (see, e.g., Fig. 10).

Unfortunately, this is not sufficient to get rid of unphysical states in loops. To deal with the problem in a *covariant* way, one introduces the so-called *Faddeev-Popov ghosts* \bar{c}_a and c_a . They are *anticommuting* “scalars” and precisely cancel the annoying contribution. The Lagrangian for the fictitious particles is related to the gauge-fixing function $F_a(W_\mu) = \partial_\mu W_\mu^a$ via

$$\begin{aligned} \mathcal{L}_{ghosts} &= -\bar{c}^a \frac{\partial F_a(W^\omega)}{\partial \omega_b} c^b = -\bar{c}^a \partial_\mu D_\mu^{ab} c^b \\ &= -\bar{c}^a \partial^2 c^a - g f^{abc} (\partial_\mu \bar{c}^a) c^b A_\mu^b. \end{aligned} \quad (110)$$

The ghosts are charged under $SU(n)$ and interact with gauge fields in the same way as the unphysical modes. However, there is an additional minus sign for the loops involving anticommuting ghosts (see, e.g., Fig. 11) that leads to the above-mentioned cancellations.

6 Gauge theory of electroweak interactions

6.1 Fermion couplings to gauge bosons

In the SM we use the gauge principle to introduce EW interactions. Indeed, we utilize

$$SU(2)_L \otimes U(1)_Y \quad (111)$$

gauge group that has four generators or, equivalently, four gauge bosons. Three of them, W_μ , belong to *weak-isospin* $SU(2)_L$, while the photon-like B_μ mediates *weak-hypercharge* $U(1)_Y$ interactions. The

SM fermions are charged under the group (111). To account for the $(V - A)$ pattern only *left* fermions interact with W_μ and form $SU(2)_L$ doublets:

$$L = \begin{pmatrix} \nu_l \\ l^- \end{pmatrix}_L, Q = \begin{pmatrix} q_u \\ q_d \end{pmatrix}_L, \quad q_u = u, c, t; \quad q_d = d, s, b; \quad l = e, \mu, \tau. \quad (112)$$

Since the generators of $SU(2)$ are just the Pauli matrices, we immediately write the following expression for the corresponding covariant derivative

$$D_\mu^L = \begin{pmatrix} \partial_\mu - \frac{i}{2} \left(gW_\mu^3 + g'Y_L^f B_\mu \right) & -i\frac{g}{\sqrt{2}}W_\mu^+ \\ -i\frac{g}{\sqrt{2}}W_\mu^- & \partial_\mu + \frac{i}{2} \left(gW_\mu^3 - g'Y_L^f B_\mu \right) \end{pmatrix}. \quad (113)$$

The *right* fermions²¹ are $SU(2)_L$ singlets and do not couple to W_μ :

$$D_\mu^R = \partial_\mu - ig' \frac{Y_R^f}{2} B_\mu. \quad (114)$$

The covariant derivatives involve two gauge couplings g, g' corresponding to $SU(2)_L$ and $U(1)_Y$, respectively. Different $Y_{L/R}^f$ denote weak hypercharges of the fermions and up to now the values are not fixed. Let us put some constraints on $Y_{L/R}^f$. The first restriction comes from the $SU(2)_L$ symmetry, i.e., $Y_L^u = Y_L^d \equiv Y_L^Q$, and $Y_L^\nu = Y_L^e \equiv Y_L^L$.

One can see that the EW interaction Lagrangian

$$\mathcal{L}_W = \mathcal{L}_{NC} + \mathcal{L}_{CC}, \quad (115)$$

in addition to the *charged-current* interactions of the form

$$\mathcal{L}_{CC}^l = \frac{g}{\sqrt{2}} \bar{\nu}_L^e \gamma_\mu W_\mu^+ e_L + \text{h.c.} = \frac{g}{2\sqrt{2}} \bar{\nu}_e \gamma_\mu W_\mu^+ (1 - \gamma_5) e + \text{h.c.} \quad (116)$$

also involves *neutral-current* interactions

$$\mathcal{L}_{NC}^l = \bar{\nu}_L^e \gamma_\mu \left(\frac{1}{2} g W_\mu^3 + \frac{Y_L^l}{2} g' B_\mu \right) \nu_L^e + \bar{e}_L \gamma_\mu \left(-\frac{1}{2} g W_\mu^3 + \frac{Y_L^l}{2} g' B_\mu \right) e_L + g' \bar{e}_R \gamma_\mu \frac{Y_R^e}{2} B_\mu e_R. \quad (117)$$

It is obvious that we have to account for QED in the SM and should predict a photon field that couples to fermions with the correct values of the electric charges. Since both W_μ^3 and B_μ are *electrically neutral*, they can mix

$$\begin{aligned} W_\mu^3 &= Z_\mu \cos \theta_W + A_\mu \sin \theta_W \\ B_\mu &= -Z_\mu \sin \theta_W + A_\mu \cos \theta_W. \end{aligned} \quad (118)$$

Here we introduce the *Weinberg* angle θ_W . One can try to fix $\sin \theta_W$ and various $Y_{L/R}^f$ from the requirement that, e.g., A_μ has the same interactions as the photon in QED. Indeed, given fermion *electric* charges Q_f (see Fig. 1) in the units of the elementary charge e , one can derive the following relations:

$$\begin{aligned} g \sin \theta_W &= e(Q_\nu - Q_e) = e(Q_u - Q_d), \\ g' Y_L^l \cos \theta_W &= e(Q_\nu + Q_e) = -e, \\ g' Y_L^Q \cos \theta_W &= e(Q_u + Q_d) = \frac{1}{3}e, \end{aligned}$$

²¹In what follows we do not consider right-handed neutrino and refer again to Ref. [18].

$$g'Y_R^f \cos \theta_W = 2eQ_f, \quad f = e, u, d. \quad (119)$$

As a consequence, $e = g \sin \theta_W$ and, e.g., $e = 3g'Y_L^Q \cos \theta_W$, so that

$$Y_L^l = -3Y_L^Q, \quad Y_R^e = -6Y_L^Q, \quad Y_R^u = 4Y_L^Q, \quad Y_R^d = -2Y_L^Q \quad (120)$$

are fixed in terms of one (arbitrary chosen) Y_L^Q . It is convenient to normalize the $U(1)_Y$ coupling g' so that $e = g' \cos \theta_W$, so $Y_L^Q = 1/3$. As a consequence, the photon field couples to the electric charge Q_f of a fermion f . The latter is related to the weak hypercharge and the third component of weak isospin T_3^f via the Gell-Mann–Nishijima formula:

$$\mathcal{L}_{NC} \ni \bar{f} \left[\left(gT_3^f \sin \theta_W + g' \frac{Y_f^L}{2} \cos \theta_W \right) P_L + \left(g' \frac{Y_f^R}{2} \cos \theta_W \right) P_R \right] \gamma_\mu f A_\mu \quad (121)$$

$$= e \bar{f} \left(T_3 + \frac{Y}{2} \right) \gamma_\mu f A_\mu = e Q_f \bar{f} \gamma_\mu f A_\mu, \quad (122)$$

where in Eq.(122) we assume that T_3 and Y are operators, which give T_3^f and Y_L^f , when acting on left components, and $T_3 = 0$ and $Y_R^f = 2Q_f$ for right fermions.

The relations (120) allow one to rewrite the neutral-current Lagrangian as

$$\mathcal{L}_{NC} = e J_\mu^A A^\mu + \frac{g}{\cos \theta_W} J_\mu^Z Z_\mu, \quad (123)$$

where the photon A_μ and a new Z -boson couple to the currents of the form

$$J_\mu^A = \sum_f Q_f \bar{f} \gamma_\mu f, \quad J_\mu^Z = \frac{1}{4} \sum_f \bar{f} \gamma_\mu (v_f - a_f \gamma_5) f, \quad (124)$$

$$v_f = 2T_3^f - 4Q_f \sin^2 \theta_W, \quad a_f = 2T_3^f, \quad (125)$$

where $T_3^f = \pm \frac{1}{2}$ for left up-type/down-type fermions. For example, in the case of u -quarks, $Q_u = 2/3$, $T_3^u = 1/2$, so

$$v_u = 1 - \frac{8}{3} \sin^2 \theta_W, \quad a_u = 1. \quad (126)$$

For completeness, let us give the expression for the charged-current interactions in the EW model

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} (J_\mu^+ W^{+\mu} + J_\mu^- W^{-\mu}), \quad J_\mu^+ = \frac{1}{2} \sum_f \bar{f}_u \gamma_\mu (1 - \gamma_5) f_d, \quad (127)$$

where $f_u(f_d)$ is the up-type (down-type) component of an $SU(2)_L$ doublet f . The corresponding interaction vertices are given in Fig. 12. It is worth emphasizing that in the SM the couplings between fermions and gauge bosons exhibit *Universality*.

It turns out that it was a *prediction* of the electroweak SM that there should be an additional neutral gauge boson Z_μ . Contrary to the photon, the Z -boson also interacts with neutrinos. This crucial property was used in the experiment called *Gargamelle* at CERN, where in 1973 the discovery was presented (Fig. 8). About ten years later both W and Z were directly produced at Super Proton Synchrotron (SPS) at CERN. Finally, in the early 90s a comprehensive analysis of the $e^+e^- \rightarrow f\bar{f}$ process, which was carried out at the Large Electron Proton (LEP) Collider (CERN) and at the Stanford Linear Collider (SLAC) confirmed the SM predictions for the Z couplings to fermions (125).

It is also worth mentioning the fact that the (hyper)-charge assignment (120) satisfies very non-trivial constraints related to cancellation of *gauge anomalies*. Anomalies correspond to situations when

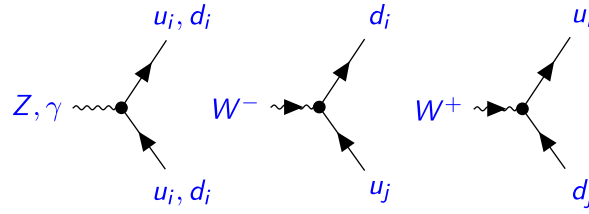


Fig. 12: Gauge-boson–quark vertices. Leptons interact with the EW bosons in the same way.

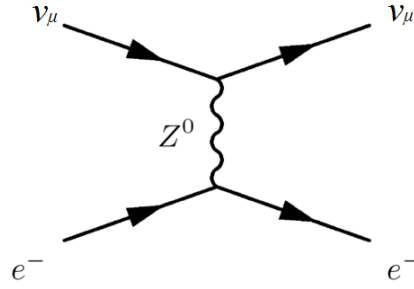


Fig. 13: The chamber of Gargamelle at CERN (left), ν_μ scattering due to Z -boson (right). From Wikipedia.

a symmetry of the classical Lagrangian is violated at the quantum level. A well-known example is *Axial or Chiral or Adler–Bell–Jackiw(ABJ) anomaly* when the classical conservation law for the axial current J_μ^A is modified due to quantum effects:

$$J_\mu^A = \bar{\Psi}\gamma_\mu\gamma_5\Psi, \quad \partial_\mu J_\mu^A = 2im\Psi\gamma_5\Psi + \underbrace{\frac{\alpha}{2\pi}F_{\mu\nu}\tilde{F}_{\mu\nu}}_{\text{anomaly}}, \quad \tilde{F}_{\mu\nu} = 1/2\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}. \quad (128)$$

The $F\tilde{F}$ -term appears due to loop diagrams presented in Fig. 14.

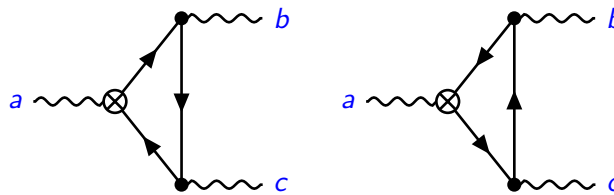


Fig. 14: Diagrams contributing to the anomaly of an axial current (crossed vertex).

There is nothing wrong when the anomalous current J_μ^A corresponds to a global symmetry and does not enter into \mathcal{L} . It just implies that a classically forbidden processes may actually occur in the quantum theory. For example, it is the anomaly in the *global axial flavour* symmetry that is responsible for the decay $\pi \rightarrow \gamma\gamma$. On the contrary, if an axial current couples to a gauge field, anomalies break gauge invariance, thus rendering the corresponding QFT inconsistent. In the SM left and right fermions (eigenvectors of γ_5) have different $SU(2)_L \times U(1)_Y$ quantum numbers, leaving space for potential anomalies. However, since we have to take into account all fermions which couple to a gauge field, there is a possibility that contributions from different species cancel each other due to a special assignment of

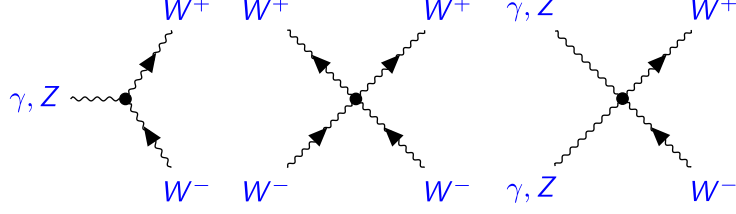


Fig. 15: Gauge-boson self-interaction vertices.

fermion charges. Indeed, in the case of chiral theories²², anomalies are proportional to ($\gamma_5 = P_R - P_L$)

$$\text{Anom} \propto \text{Tr}[t^a, \{t^b, t^c\}]_L - \text{Tr}[t^a, \{t^b, t^c\}]_R, \quad (129)$$

where t^a are generators of the considered symmetries and the traces are over left (L) or right (R) fields. In the SM the requirement that all anomalies should be zero imposes the following conditions on fermion hypercharges:

$$0 = 2Y_L^Q - Y_R^u - Y_R^d, \quad U(1)_Y - SU(3)_c - SU(3)_c, \quad (130a)$$

$$0 = N_c Y_L^Q + Y_L^l, \quad U(1)_Y - SU(2)_L - SU(2)_L, \quad (130b)$$

$$0 = N_c \left[2(Y_L^Q)^3 - (Y_R^u)^3 - (Y_R^d)^3 \right] + \left[2(Y_L^l)^3 - (Y_R^e)^3 \right], \quad U(1)_Y - U(1)_Y - U(1)_Y, \quad (130c)$$

$$0 = N_c \left[2Y_L^Q - Y_R^u - Y_R^d \right] + \left[2Y_L^l - Y_R^e \right], \quad U(1)_Y - \text{grav.} - \text{grav.}, \quad (130d)$$

where, in addition to the EW gauge group, we also consider strong interactions of quarks that have $N_c = 3$ colours²³. While the first three conditions come from the SM interactions, the last one (130d) is due to the coupling to gravity. Other anomalies are trivially zero. One can see that the hypercharges introduced in Eq.(120) do satisfy the equations. It is interesting to note that contributions due to colour quarks miraculously cancel those of leptons and the cancellation works within a single generation. This put a rather strong restriction on possible new fermions that can couple to the SM gauge bosons: new particles should appear in a complete generation (quarks + leptons) in order not to spoil anomaly cancellation within the SM. Moreover, the anomaly cancellation condition can select viable models that go beyond the SM (BSM).

6.2 Properties of the EW gauge bosons

Due to the non-Abelian nature of the $SU(2)_L$ group, the gauge fields W_i have triple and quartic self-interactions (see Eq.(108)). Since W_3 is a linear combination of the Z -boson and photon, the same is true for Z and γ . In Fig. 15, self-interaction vertices for the EW gauge bosons are depicted.

The triple vertices $WW\gamma$ and WWZ predicted by the SM were tested at LEP2 in the $e^+e^- \rightarrow W^+W^-$ process (Fig. 16) and agreement with the SM predictions was found. Subsequent studies at hadron colliders (Tevatron and LHC) aimed at both quartic and triple gauge couplings (QGC and TGC, respectively) also show consistency with the SM and put limits on possible deviations (so-called anomalous TGC and QGC).

Since we do not observe Z -bosons flying around like photons, Z_μ should have a non-zero mass M_Z and similar to W^\pm give rise to Fermi-like interactions between *neutral* currents J_Z^μ at low energies. The relative strength of the *charged* and *neutral* current-current interactions $(J_\mu^Z J_\mu^Z)/(J^{+\mu} J_\mu^+)$ can be

²²Chiral theories distinguish left and right fermions.

²³In the SM coloured quarks belong to the fundamental representation of the corresponding gauge group $SU(3)_c$.

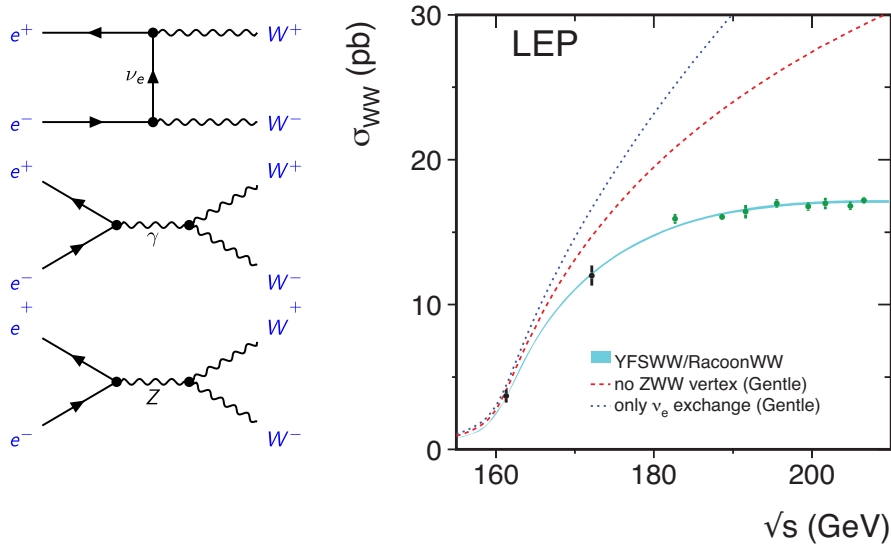


Fig. 16: $e^+e^- \rightarrow W^+W^-$.

measured by the parameter ρ :

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2 \theta_W}. \quad (131)$$

Up to now, we do not specify any relations between M_Z and M_W . Due to this, the value of ρ can, in principle, be arbitrary. However, it is a prediction of the full SM that $\rho \simeq 1$ (see below).

The fact that both W and Z should be massive poses a serious problem for theoretical description of the EW interactions. First of all, the naive introduction of the corresponding mass terms breaks the *gauge* symmetry (111). For example, $m_W^2 W_\mu^+ W_\mu^-$ is forbidden due to $W_\mu \rightarrow W_\mu + \partial_\mu \omega + \dots$. One can also mention an issue with unitarity, which arises in the scattering of longitudinal EW bosons due to gauge self-interactions in Fig. 15.

In addition, the symmetry also forbids *explicit* mass terms for fermions, since e.g., $m_\mu(\bar{\mu}_L \mu_R + \text{h.c.})$, which accounts for muon mass, mixes left and right fields that transform differently under the electroweak group (111). In the next section, we discuss how these problems can be solved by coupling the SM fermions and gauge bosons to the scalar (Higgs) sector (see also [22]).

6.3 Spontaneous symmetry breaking and gauge-boson masses

We need to *generate* masses for W_μ^\pm and Z_μ (but not for A_μ) without *explicit* breaking of the gauge symmetry. Let us consider for simplicity *scalar* electrodynamics:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial_\mu \phi - V(\phi^\dagger \phi) - \frac{1}{4} F_{\mu\nu}^2 + ie \left(\phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger \right) A_\mu + e^2 A_\mu A_\mu \phi^\dagger \phi \equiv \mathcal{L}_1, \quad (132)$$

which is invariant under $U(1)$

$$\phi \rightarrow e^{ie\omega(x)} \phi, \quad A_\mu \rightarrow A_\mu + \partial_\mu \omega. \quad (133)$$

In Eq.(132) a *complex* scalar ϕ interacts with the photon A_μ . We can use *polar* coordinates to rewrite the Lagrangian in terms of new variables

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \rho)^2 + \frac{e^2 \rho^2}{2} \left(A_\mu - \frac{1}{e} \partial_\mu \theta \right) \left(A_\mu - \frac{1}{e} \partial_\mu \theta \right) - V(\rho^2/2) - \frac{1}{4} F_{\mu\nu}^2, \quad (134)$$

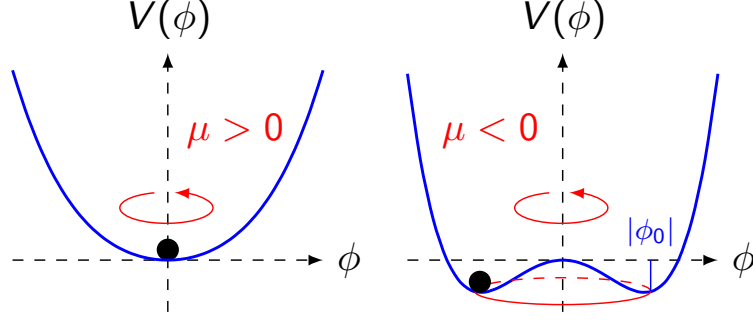


Fig. 17: A symmetric vacuum (left) and degenerate vacua (right).

$$= \frac{1}{2}(\partial_\mu \rho)^2 + \frac{e^2 \rho^2}{2} B_\mu B_\mu - V(\rho^2/2) - \frac{1}{4} F_{\mu\nu}^2(B), \quad (135)$$

where ρ is gauge invariant, while the $U(1)$ transformation (133) gives rise to a *shift* in θ :

$$\phi = \frac{1}{\sqrt{2}} \rho(x) e^{i\theta(x)}, \quad \rho \rightarrow \rho, \quad \theta \rightarrow \theta + e\omega. \quad (136)$$

One can also notice that $B_\mu \equiv A_\mu - \frac{1}{e} \partial_\mu \theta$ is also invariant! Moreover, since $F_{\mu\nu}(A) = F_{\mu\nu}(B)$, we can completely get rid of θ . As a consequence, the gauge symmetry becomes “hidden” when the system is described by the variables $B_\mu(x)$ and $\rho(x)$.

If in Eq.(132) we replace our *dynamical* field $\rho(x)$ by a constant $\rho \rightarrow v = \text{const}$, we get the mass term for B_μ . This can be achieved by considering the potential $V(\phi)$ of the form (written in terms of initial variables)

$$V = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2. \quad (137)$$

One can distinguish two different situations (see Fig. 17):

- $\mu^2 > 0$ — a *single* minimum with $\phi = 0$;
- $\mu^2 < 0$ — a valley of *degenerate* minima with $\phi \neq 0$.

In both cases we solve EOM for the homogeneous (in space and time) field. When $\mu^2 > 0$ the *solution* is unique and symmetric, i.e., it does not transform under $U(1)$. In the second case, in which we are interested here, the potential has non-trivial minima

$$\left. \frac{\partial V}{\partial \phi^\dagger} \right|_{\phi=\phi_0} = 0 \Rightarrow \phi_0^\dagger \phi_0 = -\frac{\mu^2}{2\lambda} = \frac{v^2}{2} > 0 \Rightarrow \phi_0 = \frac{v}{\sqrt{2}} e^{i\beta}, \quad (138)$$

which are *related* by *global* $U(1)$ transformations (133) that change $\beta \rightarrow \beta + e\omega$. So, in spite of the fact that we do not break the symmetry *explicitly*, it is *spontaneously broken* (SSB) due to a particular choice of our solution (β).

In QFT we interpret ϕ_0 as a characteristic of our *vacuum* state, i.e., as a *vacuum expectation value* (vev) or *condensate* of the quantum field:

$$\phi_0 = \langle 0 | \phi(x) | 0 \rangle \stackrel{\beta=0}{=} \frac{v}{\sqrt{2}}. \quad (139)$$

Since we want to introduce particles as *excitations* above the vacuum, we have to shift the field:

$$\phi(x) = \frac{v + h(x)}{\sqrt{2}} e^{i\zeta(x)/v}, \quad \langle 0 | h(x) | 0 \rangle = 0, \quad \langle 0 | \zeta(x) | 0 \rangle = 0. \quad (140)$$

As a consequence, Eq.(135) can be rewritten as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 + \frac{e^2 v^2}{2} \left(1 + \frac{h}{v}\right)^2 B_\mu B_\mu - V(h) - \frac{1}{4} F_{\mu\nu}^2(B) \equiv \mathcal{L}_2, \quad (141)$$

$$V(h) = -\frac{|\mu|^2}{2} (v+h)^2 + \frac{\lambda}{4} (v+h)^4 = \frac{2\lambda v^2}{2} h^2 + \lambda v h^3 + \frac{\lambda}{4} h^4 - \frac{\lambda}{4} v^4. \quad (142)$$

One can see that the Lagrangian (142) describes a massive vector field B_μ with $m_B^2 = e^2 v^2$ and a massive scalar h with $m_h^2 = 2\lambda v^2$. We do not break the symmetry explicitly. It is again *hidden* in the relations between couplings and masses. This is the essence of the *Brout-Englert-Higgs-Hagen-Guralnik-Kibble* mechanism [23–25].

The Lagrangians \mathcal{L}_1 (132) and \mathcal{L}_2 (142) describe the same Physics but written in terms of different quantities (variables). Expression (132) involves a *complex* scalar ϕ with 2 (real) degrees of freedom (DOFs) and a *massless* gauge field (A_μ) also having 2 DOFs. It is manifestly gauge invariant but not suitable for perturbative expansion (ϕ has imaginary mass).

On the contrary, in \mathcal{L}_2 the gauge symmetry is hidden²⁴ and it is written in terms of *physical* DOFs, i.e., a *real* scalar h (1 DOF) and a *massive* vector B_μ (3 DOFs). In a sense, one *scalar* DOF (ζ) is “eaten” by the gauge field to become massive. It is important to note that the postulated *gauge* symmetry allows us to avoid the consequences of the *Goldstone* theorem, which states that if the vacuum breaks a *global* continuous symmetry there is a *massless* boson (Nambu-Goldstone) in the spectrum²⁵. This boson is associated with ‘oscillations’ along the valley, i.e., in the *broken* direction (see Fig. 17). However, due to the local character of symmetry, χ is not physical anymore, its disappearance (or appearance, see below) reflects the *redundancy*, which was mentioned above.

In Sec. 4.2, we demonstrated that the massive-vector propagator has rather bad UV behavior and is not very convenient for doing calculations in PT. It looks like we gain nothing from the gauge principle. But it is not true. We can write the model Lagrangian in the *Cartesian* coordinates $\phi = \frac{1}{\sqrt{2}}(v + \eta + i\chi)$:

$$\mathcal{L}_3 = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{e^2 v^2}{2} A_\mu A_\mu + \frac{1}{2} \partial_\mu \chi \partial_\mu \chi - \underline{ev A_\mu \partial_\mu \chi} + \frac{1}{2} \partial_\mu \eta \partial_\mu \eta - \frac{2v^2 \lambda}{2} \eta^2 + \frac{v^4 \lambda}{4} \quad (143)$$

$$+ e A_\mu \chi \partial_\mu \eta - e A_\mu \eta \partial_\mu \chi - v \lambda \eta (\eta^2 + \chi^2) - \frac{\lambda}{4} (\eta^2 + \chi^2)^2 + \frac{e^2}{2} A_\mu A_\mu (2v\eta + \eta^2 + \chi^2). \quad (144)$$

The “free” part (143) of \mathcal{L}_3 seems to describe 5 real DOFs: a massive scalar η , a *massless* (would-be *Nambu-Goldstone*) boson χ and a massive A_μ . However, there is a mixing between the *longitudinal* component of A_μ and χ that spoils this naive counting (unphysical χ is “partially eaten” by A_μ).

In spite of this subtlety, \mathcal{L}_3 is more convenient for calculations in PT. To quantize the model, one can utilize the gauge-fixing freedom and add the following expression to \mathcal{L}_3

$$\delta \mathcal{L}_{g.f.} = -\frac{1}{2\xi} (\partial_\mu A_\mu + ev\xi\chi)^2 = -\frac{1}{2\xi} (\partial_\mu A_\mu)^2 - \underline{ev\chi\partial_\mu A_\mu} - \frac{e^2 v^2 \xi}{2} \chi^2. \quad (145)$$

It removes the mixing from Eq.(143) and introduces a mass for χ , $m_\chi^2 = (e^2 v^2)\xi$. In addition, the vector-boson propagator in this case looks like

$$\langle 0|T A_\mu(x) A_\nu(y)|0\rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{-i \left[g_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2 - \xi m_A^2} \right]}{p^2 - m_A^2 + i\epsilon} e^{-ip(x-y)}, \quad m_A = ev. \quad (146)$$

One can see that for $\xi \rightarrow \infty$ we reproduce Eq.(39), while for finite ξ the propagator behaves like $1/p^2$ as $p \rightarrow \infty$, thus making it convenient for PT calculations.

²⁴One can also say that \mathcal{L}_2 corresponds to the *unitary* gauge, i.e., no unphysical “states” in the particle spectrum.

²⁵Any non-derivative interactions violate the shift symmetry $\zeta \rightarrow \zeta + ev\omega$ for $\omega = \text{const}$.

It should be mentioned that contrary to \mathcal{L}_2 the full Lagrangian corresponding to \mathcal{L}_3 involves also unphysical *ghosts*, which do not decouple in the considered case. Nevertheless, it is a relatively small price to pay for the ability to perform high-order calculations required to obtain high-precision predictions.

Let us switch back to the SM. We have three gauge bosons that should become massive. As a consequence, three symmetries should be broken by the SM vacuum to feed hungry W_μ^\pm and Z_μ with (would-be) Goldstone bosons

$$SU(2)_L \times U(1)_Y \rightarrow U(1)_{em}. \quad (147)$$

The photon should remain massless and correspond to the unbroken electromagnetic $U(1)_{em}$. This can be achieved by considering an $SU(2)_L$ doublet of scalar fields:

$$\Phi = \frac{1}{\sqrt{2}} \exp\left(i \frac{\zeta_j(x) \sigma^j}{2v}\right) \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \quad \Phi_0 \equiv \langle 0 | \Phi | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad (148)$$

where we decompose $\Phi(x)$ in terms of three (would-be) Goldstone bosons ζ_j and a Higgs h . The Pauli matrices σ_j represent broken generators of $SU(2)_L$. Let Φ also be charged under $U(1)_Y$:

$$\Phi \rightarrow \exp\left(ig \frac{\sigma^i}{2} \omega_a + ig' \frac{Y_H}{2} \omega'\right) \Phi. \quad (149)$$

We do not want to break $U(1)_{em}$ spontaneously so the vacuum characterized by the vev Φ_0 should be invariant under $U(1)_{em}$, i.e., has no electric charge Q

$$e^{ieQ\theta} \Phi_0 = \Phi_0 \rightarrow Q\Phi_0 = 0. \quad (150)$$

The operator Q is a linear combination of diagonal generators of $SU(2)_L \times U(1)_Y$, $T_3 = \sigma_3/2$ and $Y/2$:

$$Q\Phi_0 = \left(T_3 + \frac{Y}{2}\right) \Phi_0 = \frac{1}{2} \begin{pmatrix} 1 + Y_H & 0 \\ 0 & -1 + Y_H \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \stackrel{?}{=} 0. \quad (151)$$

As a consequence, to keep $U(1)_{em}$ unbroken, we should set $Y_H = 1$. Since Φ transforms under the EW group, we have to introduce gauge interactions for the Higgs doublet to make sure that the scalar sector respects the corresponding local symmetry:

$$\mathcal{L}_\Phi = (D_\mu \Phi)^\dagger (D_\mu \Phi) - V(\Phi), \quad \text{with} \quad V(\Phi) = m_\Phi^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2. \quad (152)$$

For $m_\Phi^2 < 0$ the symmetry is spontaneously broken. In the *unitary* gauge (Goldstone bosons are gauged away: in Eq.(148) we put $\zeta_j = 0$) the first term in Eq.(152) can be cast into

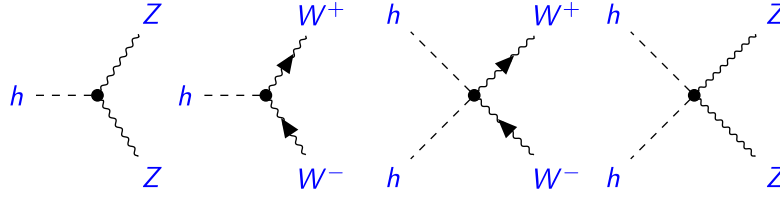
$$|D_\mu \Phi|^2 = \frac{1}{2} (\partial_\mu h)^2 + \frac{g^2}{8} (v+h)^2 |W_\mu^1 + iW_\mu^2|^2 + \frac{1}{8} (v+h)^2 (gW_\mu^3 - g'Y_H B_\mu)^2 \quad (153)$$

$$= \frac{1}{2} (\partial_\mu h)^2 + \frac{g^2}{4} (v+h)^2 W^+ W^- \quad \left[\sqrt{2} W^\pm = W_\mu^1 \mp iW_\mu^2 \right] \\ + \frac{1}{8} (v+h)^2 \left[Z_\mu (g \cos \theta_W + g' \sin \theta_W) + A_\mu (g \sin \theta_W - g' \cos \theta_W) \right]^2 \quad (154)$$

$$= \frac{1}{2} (\partial_\mu h)^2 + M_W^2 \left(1 + \frac{h}{v}\right)^2 W^+ W^- + \frac{M_Z^2}{2} \left(1 + \frac{h}{v}\right)^2 Z_\mu Z_\mu, \quad (155)$$

where we *require* the photon to be massless after SSB, i.e.,

$$g \sin \theta_W - g' \cos \theta_W = 0 \quad \Rightarrow \quad \sin \theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \quad \cos \theta_W = \frac{g}{\sqrt{g^2 + g'^2}} \quad (156)$$


Fig. 18: Gauge-boson–Higgs interactions.

and, consequently,

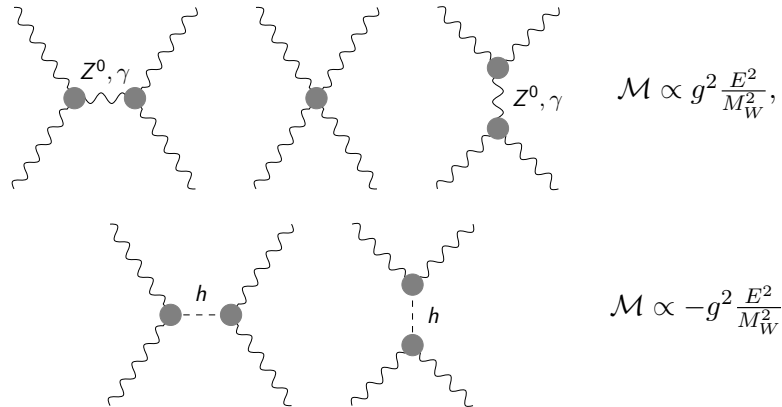
$$g \cos \theta_W + g' \sin \theta_W = \sqrt{g^2 + g'^2}, \quad e = g \sin \theta_W = g' \cos \theta_W = \frac{gg'}{\sqrt{g^2 + g'^2}}. \quad (157)$$

The masses of the Z and W -bosons are proportional to the EW gauge couplings

$$M_W^2 = \frac{g^2 v^2}{4}, \quad M_Z^2 = \frac{(g^2 + g'^2) v^2}{4}. \quad (158)$$

One can see that the Higgs-gauge boson vertices (Fig. 18) are related to the masses M_W and M_Z .

An important consequence of the SM gauge symmetry and the existence of the Higgs boson is the *unitarization* of massive vector-boson scattering. By means of simple power counting, one can easily convince oneself that the amplitude for (longitudinal) W -boson scattering originating from the quartic vertex in Fig. 15 scales with energy as E^4/M_W^4 . This kind of dependence will eventually violate unitarity for $E \gg M_W$. However, in the SM, thanks to gauge symmetry, QGC and TGC are related. This results in E^2/M_W^2 behavior when Z/γ exchange is taken into account. Moreover, since the gauge bosons couple also to Higgs, we need to include the corresponding contribution to the total amplitude. It turns out that it is this contribution that cancels the E^2 terms and saves unitarity in the WW -scattering. Obviously, this pattern is a consequence of the EW symmetry breaking in the SM and can be modified by the presence of New Physics. Due to this, experimental studies of vector boson scattering (VBS) play a role in proving overall consistency of the SM.


Fig. 19: WW -scattering and Unitarity.

Having in mind Eq.(88), one can derive the relation

$$G_F = \frac{1}{\sqrt{2}v^2} \Rightarrow v \simeq 246 \text{ GeV}, \quad (159)$$

which gives a numerical estimate of v . One can also see that due to (158) we have (at the tree level)

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} = 1. \quad (160)$$

Let us emphasize that it is a consequence of the fact that the SM Higgs is a weak *doublet* with *unit* hypercharge. Due to this, $\rho \simeq 1$ imposes important constraints on possible extensions of the SM Higgs sector. For example, we can generalize expression (160) to account for n scalar $(2I_i + 1)$ -plets ($i = 1, \dots, n$) that transform under $SU(2)_L$ and have hypercharges Y_i . In case they acquire vevs v_i , which break the EW group, we have

$$\rho = \frac{\sum_i (I_i(I_i + 1) - Y_i^2) v_i^2}{\sum_i 2Y_i^2 v_i^2}. \quad (161)$$

Consequently, any non-doublet (with total weak isospin $I_i \neq 1/2$) vev leads to a deviation from $\rho = 1$.

6.4 Fermion-higgs interactions and masses of quarks and leptons

Since we fixed all the gauge quantum numbers of the SM fields, it is possible to construct the following *gauge-invariant* Lagrangian:

$$\mathcal{L}_Y = -y_e \begin{pmatrix} \bar{L} & \Phi \end{pmatrix} \begin{pmatrix} e_R \\ -2 \end{pmatrix} - y_d \begin{pmatrix} \bar{Q} & \Phi \end{pmatrix} \begin{pmatrix} d_R \\ -\frac{2}{3} \end{pmatrix} - y_u \begin{pmatrix} \bar{Q} & \Phi^c \end{pmatrix} \begin{pmatrix} u_R \\ -\frac{4}{3} \end{pmatrix} + \text{h.c.}, \quad (162)$$

which involves *dimensionless* Yukawa couplings y_f . It describes interactions between the Higgs field Φ , left fermion doublets (112) and right singlets. In Eq.(162) we also indicate weak hypercharges of the corresponding fields. One can see that combinations of two doublets, $(\bar{Q}\Phi)$ etc., are invariant under $SU_L(2)$ but have a non-zero charge under $U(1)_Y$. The latter is compensated by hypercharges of right fermions. In addition, $U(1)_Y$ symmetry forces us to use a charge-conjugated Higgs doublet $\Phi^c = i\sigma_2\Phi^*$ with $Y = -1$ to account for Yukawa interactions involving u_R .

In the spontaneously broken phase with non-zero Higgs vev, the Lagrangian \mathcal{L}_Y can be written in the following simple form:

$$-\mathcal{L}_Y = \sum_f \frac{y_f v}{\sqrt{2}} \left(1 + \frac{h}{v}\right) \bar{f} f = \sum_f m_f \left(1 + \frac{h}{v}\right) \bar{f} f, \quad f = u, d, e, \quad (163)$$

where unitary gauge is utilized. One can see that SSB generates fermion masses m_f and, similarly to Eq.(155), *relates* them to the corresponding couplings of the Higgs boson h (see Fig.20a).

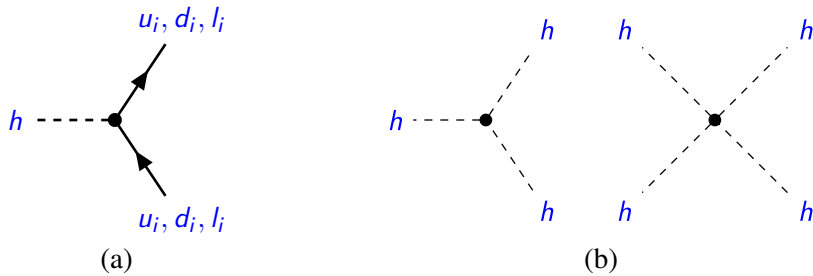


Fig. 20: Higgs–fermion couplings (a) and self-interactions of the Higgs boson (b).

It is worth noting that Eq.(162) is not the most general renormalizable Lagrangian involving the SM scalars and fermions. One can introduce *flavour* indices and non-diagonal *complex* Yukawa matrices y_f^{ij} to account for a possible mixing between the SM fermions, i.e.,

$$\mathcal{L}_{\text{Yukawa}} = -y_l^{ij} (\bar{L}_i \Phi) l_{jR} - y_d^{ij} (\bar{Q}_i \Phi) d_{jR} - y_u^{ij} (\bar{Q}_i \Phi^c) u_{jR} + \text{h.c.} \quad (164)$$

Substituting $\Phi \rightarrow \Phi_0$ we derive the expression for fermion mass matrices $m_f^{ij} = y_f^{ij} \frac{v}{\sqrt{2}}$, which can be diagonalized by suitable unitary rotations of left and right fields. In the SM the Yukawa matrices (164) are also diagonalized by the *same* transformations. This leads again (in the unitary gauge) to Eq.(163) but with the fields corresponding to the *mass* eigenstates. The latter *do not* coincide with *weak* states, which enter into \mathcal{L}_W (115). However, one can rewrite \mathcal{L}_W in terms of mass eigenstates. Due to large *flavour symmetry* of weak interactions²⁶, this introduces a single mixing matrix (the Cabibbo–Kobayashi–Maskawa matrix, or CKM), which manifests itself in the charged-current interactions \mathcal{L}_{CC} . A remarkable fact is that three generations are *required* to have \mathcal{CP} violation in the quark sector. Moreover, a single CKM with only four physical parameters (angles and one phase) proves to be very successful in accounting for plethora of phenomena involving transitions between different flavours. We will not discuss further details but refer to the dedicated lectures on Flavor Physics [26].

7 The SM: theory vs. experiment

Let us summarize and write down the full SM Lagrangian as

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{Gauge}}(g_s, g, g') + \mathcal{L}_{\text{Yukawa}}(y_u, y_d, y_l) + \mathcal{L}_{\text{Higgs}}(\lambda, m_\Phi^2) + \mathcal{L}_{\text{Gauge-fixing}} + \mathcal{L}_{\text{Ghosts}}. \quad (165)$$

The Yukawa part $\mathcal{L}_{\text{Yukawa}}$ is given in Eq.(164), while $\mathcal{L}_{\text{Higgs}} = -V(\Phi)$ is the Higgs potential from Eq.(152). After SSB the corresponding terms give rise to the Higgs couplings to the SM fermions (Fig.20a) and Higgs self-interactions (Fig.20b). The former are diagonal in the *mass* basis. The kinetic term for the Higgs field is included in

$$\mathcal{L}_{\text{Gauge}} = -\frac{1}{4} \underbrace{G_{\mu\nu}^a G_{\mu\nu}^a}_{SU(3)_c} - \frac{1}{4} \underbrace{W_{\mu\nu}^i W_{\mu\nu}^i}_{SU(2)_L} - \frac{1}{4} \underbrace{B_{\mu\nu} B_{\mu\nu}}_{U(1)_Y} + (D_\mu \Phi)^\dagger (D_\mu \Phi) \quad (166)$$

$$+ \underbrace{\bar{L}_i i \hat{D} L_i + \bar{Q}_i i \hat{D} Q_i}_{SU(2)_L \text{ doublets}} + \underbrace{\bar{l}_{Ri} i \hat{D} l_{Ri} + \bar{u}_{Ri} i \hat{D} u_{Ri} + \bar{d}_{Ri} i \hat{D} d_{Ri}}_{SU(2)_L \text{ singlets}}, \quad (167)$$

where for completeness we also add the colour group $SU(3)_c$ responsible for the strong force. The first three terms in Eq.(166) introduce gauge bosons for the SM gauge groups and in the non-Abelian case account for self-interactions of the latter (Fig. 15). The fourth term in (166) written in the form (155) accounts for gauge interactions of the Higgs field (Fig. 18). Finally, Eq.(167) gives rise to interactions between gauge bosons and the SM fermions (see, e.g., Fig. 12).

The SM Lagrangian Eq.(165) depends on 18 physical²⁷ parameters — 17 dimensionless couplings (gauge, Yukawa, and scalar self-interactions) and only 1 mass parameter m_Φ^2 (see Table. 3). It is worth emphasizing here that there is certain freedom in the definition of *input* parameters. In principle, one can write down the SM predictions for a set of 18 observables (e.g., physical particle masses or cross-sections at fixed kinematics) that can be measured in experiments. With the account of loop corrections the predictions become non-trivial functions of *all* the Lagrangian parameters. By means of PT it is possible to invert these relations and express these primary parameters in terms of the chosen measured quantities. This allows us to *predict other observables in terms of* a finite set of measured *observables*²⁸.

However, it is not always practical to strictly follow this procedure. For example, due to confinement we are not able to directly probe the strong coupling g_s and usually treat it as a scale-dependent parameter $(4\pi)\alpha_s = g_s^2$ defined in the modified minimal-subtraction ($\overline{\text{MS}}$) scheme. It is customary to use the value of $\alpha_s^{(5)}(M_Z) = 0.1181 \pm 0.011$ at the Z -mass scale as an input for theoretical predictions. A convenient choice of other input parameters is presented in Table.3. It is mostly dictated by the fact

²⁶In the SM the symmetry is $U(3)^5$ and corresponds to flavour rotations of left doublets, Q and L , and right singlets, u_R , d_R and l_R . Neutrinos are assumed to be massless.

²⁷We do not count unphysical gauge-fixing parameters entering into $\mathcal{L}_{\text{Gauge-fixing}}$ and $\mathcal{L}_{\text{Ghosts}}$.

²⁸One can even avoid the introduction of *renormalizable* parameters and use *bare* quantities at the intermediate step.

Table 3: Parameters of the SM.

18=	1	1	1	1	1	9	4
primary:	g_s	g	g'	λ	m_Φ^2	y_f	y_{ij}
practical:	α_s	M_Z^2	α	M_H^2	G_F	m_f	V_{CKM}

that the parameters from the ‘‘practical’’ set are measured with better precision than the others.

At the *tree* level one can write

$$\begin{aligned} \alpha_s &= \frac{g_s^2}{4\pi}, & (4\pi)\alpha &= g^2 g'^2 / (g^2 + g'^2), & M_Z^2 &= \frac{(g^2 + g'^2)v^2}{4}, \\ G_F &= \frac{1}{\sqrt{2}v^2}, & M_h^2 &= 2\lambda v^2 = 2|m_\Phi|^2, & m_f &= y_f v / \sqrt{2}. \end{aligned} \quad (168)$$

The relations are modified at higher orders in PT and perturbative corrections turn out to be mandatory if one wants to confront theory predictions [27–29] with high-precision experiments. A simple example to demonstrate this fact comes from the tree-level ‘‘prediction’’ for the W -mass M_W . From Eq.(158) and Eq.(168) we can derive

$$\frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2M_W^2(1 - M_W^2/M_Z^2)}. \quad (169)$$

Plugging recent PDG [21] values

$$\alpha^{-1} = 137.035999139(31), \quad M_Z = 91.1876(21) \text{ GeV}, \quad G_F = 1.1663787(6) \times 10^{-5} \text{ GeV}^{-2}, \quad (170)$$

in Eq.(169), one can predict

$$M_W^{tree} = 80.9387(25) \text{ GeV}, \quad (171)$$

where only uncertainties due to the input parameters (170) are taken into account. Comparing M_W^{tree}

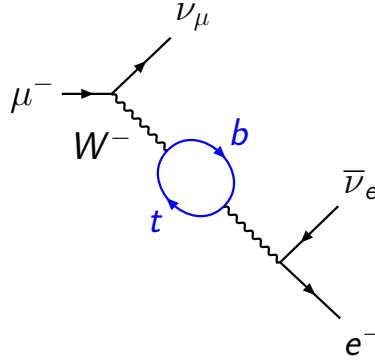


Fig. 21: An example of loop corrections to the muon decay, which give rise to the modification of the tree-level relation in Eq.(169).

with the measured value $M_W^{exp} = 80.379(12) \text{ GeV}$, one can see that our naive prediction is off by about 47σ ! Of course, this is not the reason to abandon the SM. We just need to take radiative corrections into account (see, e.g., Fig.21). Among other things the latter allows one to *connect* phenomena at different scales in the context of a single model.

A modern way to obtain the values of the SM parameters is to perform a global fit to confront state-of-the-art SM predictions with high-precision experimental data. Due to quantum effects, we can even probe New Physics that can contribute to the SM processes at low energies via virtual states. Indeed, LEP precision measurements interpreted in the context of the SM were used in a multidimensional

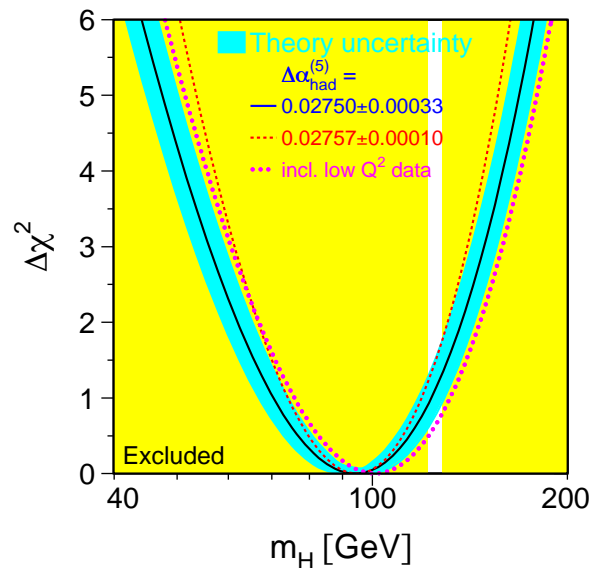


Fig. 22: The dependence of $\Delta\chi_{\min}^2(M_H^2) = \chi_{\min}^2(M_H^2) - \chi_{\min}^2$ on the value of M_H . The width of the shaded band around the curve shows the theoretical uncertainty. Exclusion regions due to LEP and LHC are also presented.

parameter fits to predict the mass of the top quark M_t (“New Physics”), prior to its discovery at the Tevatron. After M_t was measured it was included in the fit as an additional constraint, and the same approach led to the prediction of a *light* Higgs boson. In Fig. 22, the famous *blue-band* plot by the LEP Electroweak Working Group (LEPEWWG [30]) is presented. It was prepared a couple of months before the official announcement of the Higgs-boson discovery. One can see that the best-fit value corresponding to $\Delta\chi_{\min}^2 = 0$ lies just about 1σ below the region *not* excluded by LEP and LHC.

Obviously, at the moment the global EW fit is *overconstrained* and can be used to test overall consistency of the SM. In Fig. 23 we present the comparison between measurements of different (pseudo)observables O^{meas} and the SM predictions O^{fit} corresponding to the best-fit values of fitted parameters. Although there are several quantities where *pulls*, i.e., deviations between the theory and experiment, reach more than two standard deviations, the average situation should be considered as extremely good. A similar conclusion can be drawn from the recent Figs. 24 and 25, in which experimental results for various cross-sections measured by ATLAS and CMS are compared with the SM predictions. In case one is interested in the behavior of the SM at ultra-high energies, it is more convenient to get back to the primary parameters and use the renormalization group to estimate how they change with scale. In Fig. 26, the scale dependence of the SM parameters is presented. One can see that the gauge couplings tend to converge to a single value at about 10^{13-15} GeV, thus providing a hint for Grand Unification. Another important consequence of this kind of studies is related to the EW vacuum (meta)stability (see, e.g., [33]). In Fig. 26, it manifests itself at the scale $\mu \simeq 10^{10}$ GeV, at which the self-coupling λ becomes negative, making the tree-level potential unbounded from below.

8 Conclusions

Let us summarize and discuss briefly the pros and cons of the SM. The Standard Model has many nice features:

- it is based on Symmetry principles: Lorentz + $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry;
- it is renormalizable and unitary;

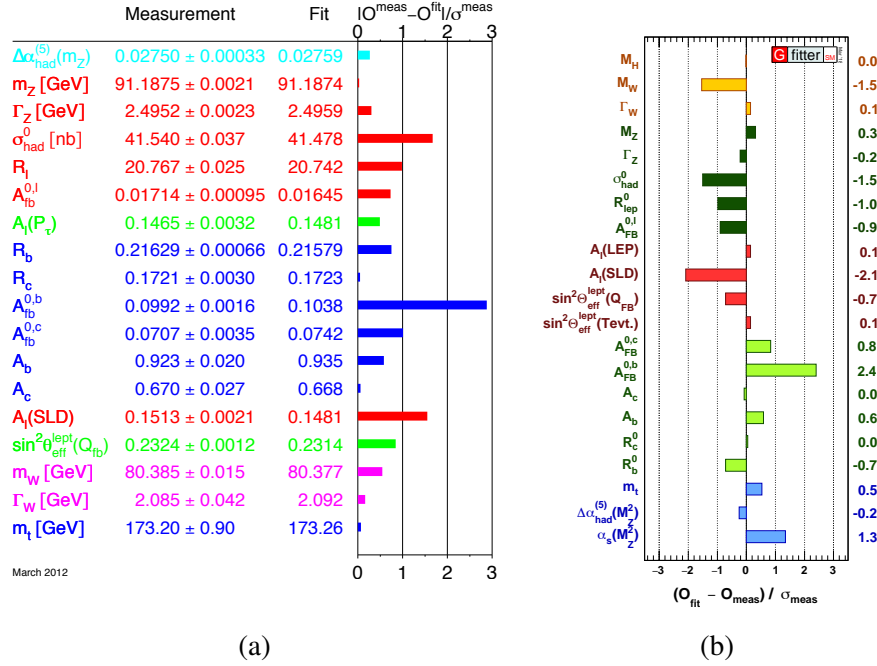


Fig. 23: Pulls of various (pseudo)observables due to (a) LEPEWWG [30] and (b) Gfitter [31].

- the structure of all interactions is fixed (but not all couplings are tested experimentally);
- it is an anomaly-free theory;
- it can account for rich Flavour Physics (see [26]);
- three generations allow \mathcal{CP} -violation (see [26]);
- it can be extended to incorporate neutrino masses and mixing (see [18]);
- it allows making systematic predictions for a wide range of phenomena at different scales;
- all predicted particles have been discovered experimentally;
- it survives stringent experimental tests.

Due to this, the SM is enormously successful (*Absolutely Amazing Theory of Almost Everything*). Since it works so well, *any* New Physics should reproduce it in the low-energy limit. Unfortunately, contrary to the Fermi-like non-renormalizable theories, the values of the SM parameters do not give us obvious *hints* for a New Physics scale. But why do we need New Physics if the model is so perfect? It turns out that we do not *understand*, why the SM works so well. For example, one needs to clarify the following:

- What explains the pattern behind Flavour Physics (hierarchy in masses and mixing, 3 generations)?
- Is there a symmetry behind the SM (electric) charge assignment?
- What is the origin of the Higgs potential?
- What is the origin of accidental Baryon and Lepton number symmetries?
- Why is there no \mathcal{CP} -violation in the strong interactions (strong \mathcal{CP} problem)²⁹?
- Why is the Higgs-boson mass so low? (Hierarchy/Naturalness problem, see [22])
- Is it possible to unify all the interactions, including gravity?

In addition, there are phenomenological problems that are waiting for solutions and probably require introduction of some New Physics:

²⁹The SM Gauge group allows such a term in the SM Lagrangian, $\mathcal{L} \ni \theta_{CP} \frac{1}{16\pi^2} F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a$. But it turns out that $\theta_{CP} = 0$.

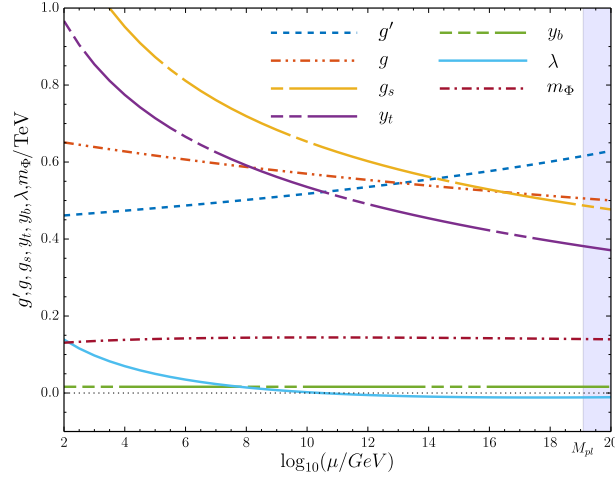


Fig. 26: Scale dependence of the SM parameters obtained by means of mr package [32].

To conclude, one of the most important *tasks* in modern high-energy physics is to find the scale at which the SM breaks down. There is a big chance that some new physical phenomena will eventually manifest themselves in the ongoing or future experiments, thus allowing us to single out viable model(s) in the enormous pool of existing NP scenarios.

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