6 Activity Reports

6.1 Beam-Beam Collisions with an Arbitrary Crossing Angle: Analytical tune shifts, tracking algorithm without Lorentz boost, Crab-Crossing

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6.1.1 Introduction

The beam-beam collision with a finite crossing angle has become a reality since the DAΦNE [1] and KEKB [2] colliders were put in operation. Nowadays, designs or/and upgrade options for a number of colliders and particle factories are based on lattices having a crossing angle (see, for example, [3, 4]). Recently new innovative schemes were proposed, such as crab-crossing for KEKB [5] and very large crossing angle for DAΦNE-II [6]. Hence it is clear that the beam physics related to the crossing angle has become critically important. In this paper we would like to continue discussing these questions, already raised in ICFA Beam Dynamics Letter 34 by Y.Cai [7]. Actually, the Introduction given in [7] can be applied to our paper as well, so we decided not to repeat it here. As compared to [7], we derived the final formulae for 3D Gaussian beams which can be directly used in estimates and simulations of beam-beam collisions with a crossing angle and that can be easily generalized on a Crab-Crossing scheme of collision. Initially we were focused on the analytical expressions for the beam-beam tune shifts with arbitrary crossing angles. These expressions were obtained in 2003 (see [8,9]) and partially checked with simulations. The full implementation of arbitrary crossing angle in the simulation code was performed by the end of 2003, that allowed us to complete the comparison for large crossing angles. Actually, this paper is a joining of [9] (that is a generalization of [8]) and a detailed explanation of the tracking algorithm without Lorentz boost. These two parts are closely connected, use the same notations, and crosscheck each other, so we decided to present them together as a whole.

6.1.2 Beam-beam tune shift formulae

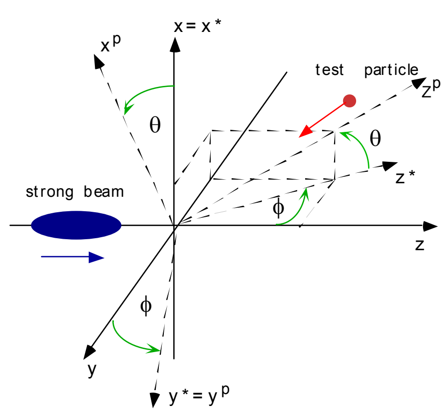


Figure 1: Scheme of beam-beam collision under a crossing angle.

Let us consider two ultra relativistic bunches colliding at an arbitrary angle, as shown in Fig. 1. The strong beam moves along z-axis of the right laboratory coordinate system. The coordinate system connected with the test particles of the weak beam is denoted with the index 'p' in Fig. 1. The coordinate transformations between the two systems are obtained, first, by a rotation of the strong bunch coordinate system by the angle ϕ around x-axis and, second, by a rotation of the resulting system x*-y*-z* around y*-axis by the angle θ.

The coordinate transformations from one system to the other are as follows:

$$
\begin{aligned}
& x = x^p \cos(\theta) + z^p \sin(\theta) \\
& y = y^p \cos(\phi) - \left(z^p \cos(\theta) - x^p \sin(\theta)\right)\sin(\phi) \\
& z = y^p \sin(\phi) + \left(z^p \cos(\theta) - x^p \sin(\theta)\right)\cos(\phi) \\
& \text{and} \\
& x^p = x\cos(\theta) - \left(z\cos(\phi) - y\sin(\phi)\right)\sin(\theta) \\
& y^p = y\cos(\phi) + z\sin(\phi) \\
& z^p = \left(z\cos(\phi) - y\sin(\phi)\right)\cos(\theta) + x\sin(\theta)
\end{aligned}
\tag{1}
$$

In the laboratory system components of the electromagnetic field, created by a 3D Gaussian bunch (strong bunch) moving with a velocity ~c is given by [10]:

$$
\begin{aligned}
& E_x = \frac{eN\gamma}{2\pi^{3/2}\varepsilon_0} x \int_0^\infty dw \frac{\exp\left\{-\frac{x^2}{\left(2\sigma_x^2 + w\right)} - \frac{y^2}{\left(2\sigma_y^2 + w\right)} - \frac{\gamma^2 (z - ct)^2}{\left(2\gamma^2\sigma_z^2 + w\right)}\right\}}{\left(2\sigma_x^2 + w\right)^{3/2}\sqrt{\left(2\sigma_y^2 + w\right)\left(2\gamma^2\sigma_z^2 + w\right)}} \\
& E_y = \frac{eN\gamma}{2\pi^{3/2}\varepsilon_0} y \int_0^\infty dw \frac{\exp\left\{-\frac{x^2}{\left(2\sigma_x^2 + w\right)} - \frac{y^2}{\left(2\sigma_y^2 + w\right)} - \frac{\gamma^2 (z - ct)^2}{\left(2\gamma^2\sigma_z^2 + w\right)}\right\}}{\left(2\sigma_y^2 + w\right)^{3/2}\sqrt{\left(2\sigma_x^2 + w\right)\left(2\gamma^2\sigma_z^2 + w\right)}} \\
& B_x = -\frac{E_y}{c} \\
& B_y = \frac{E_x}{c}
\end{aligned}
\tag{2}
$$

Equations of motion of a test particle belonging to the weak beam in this system are:

$$
\begin{aligned}
& x(t) = -c\sin(\theta)t + x_0 && v_x = -c\sin(\theta) \\
& y(t) = c\cos(\theta)\sin(\phi)t + y_0 && v_y = c\cos(\theta)\sin(\phi) \\
& z(t) = -c\cos(\theta)\cos(\phi)t + z_0 && v_z = -c\cos(\theta)\cos(\phi)
\end{aligned}
\tag{3}
$$

The Lorentz force acting on the test particle due to the electromagnetic fields produced by the strong beam:

$$\vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right) \quad with \quad \vec{v} \times \vec{B} = -v_z B_y \vec{i} + v_z B_x \vec{j} + \left(v_x B_y - v_y B_x\right)\vec{k} \tag{4}$$

has the following components:

$$\begin{aligned}
F_x &= e\left(E_x - v_z B_y\right) = e\left(E_x + c\cos(\theta)\cos(\phi)B_y\right) = eE_x\left(1+\cos(\theta)\cos(\phi)\right) \\
F_y &= e\left(E_y + v_z B_x\right) = e\left(E_x - c\cos(\theta)B_x\right) = eE_y\left(1+\cos(\theta)\cos(\phi)\right) \\
F_z &= e\left(v_x B_y - v_y B_x\right) = e\left(-c\sin(\theta)\frac{E_x}{c} + c\cos(\theta)\sin(\phi)\frac{E_y}{c}\right) = e\left(E_y\cos(\theta)\sin(\phi) - E_x\sin(\theta)\right)
\end{aligned} \tag{5}$$

The force projected onto the axes of the test particle coordinate system has:

$$\begin{aligned}
F_x^p &= F_x\cos(\theta) + F_y\sin(\phi)\sin(\theta) - F_z\cos(\phi)\sin(\theta) = eE_x\left(\cos(\phi)+\cos(\theta)\right) + eE_y\sin(\theta)\sin(\phi) \\
F_y^p &= F_y\cos(\phi) + F_z\sin(\phi) = eE_y\left(\cos(\phi)+\cos(\theta)\right) - eE_x\sin(\theta)\sin(\phi)
\end{aligned} \tag{6}$$

According to the tune shift definitions:

$$\begin{aligned}
\xi_{x^p} &= \Delta Q_{x^p} = \frac{1}{4\pi}\int_{-\infty}^{+\infty} dz^p \beta_x \left.\frac{\partial F_x^p\left(x\left(x^p,y^p,z^p\right),y\left(x^p,y^p,z^p\right),z\left(x^p,y^p,z^p\right)\right)}{\partial x^p}\right|_{x^p=y^p=0} \\
\xi_{y^p} &= \Delta Q_{y^p} = \frac{1}{4\pi}\int_{-\infty}^{+\infty} dz^p \beta_y \left.\frac{\partial F_y^p\left(x\left(x^p,y^p,z^p\right),y\left(x^p,y^p,z^p\right),z\left(x^p,y^p,z^p\right)\right)}{\partial y^p}\right|_{x^p=y^p=0}
\end{aligned} \tag{7}$$

Combining eqs. (1), (2) and (6), differentiating with respect to the transverse coordinate (eq. (7)) and integrating along z^p, one gets:

$$\begin{aligned}
\xi_{x^p} &= \frac{r_e N\beta_x}{2\pi\gamma}\times \\
&\int_0^{\infty} dw \frac{\left\{\begin{gathered}(\cos\theta+\cos\phi)^2(1+\cos\theta\cos\phi)(2\sigma_y^2+w)+\sin\phi^2(1+\cos\theta\cos\phi)(2\sigma_z^2+\frac{w}{\gamma^2}) \\ +\sin\theta^2\sin\phi^2\left(1-\cos\theta^2\cos\phi^2\right)(2\sigma_x^2+w)\end{gathered}\right\}}{\left((1+\cos\theta\cos\phi)^2(2\sigma_x^2+w)(2\sigma_y^2+w)+\cos\theta^2\sin\phi^2(2\sigma_x^2+w)(2\sigma_z^2+\frac{w}{\gamma^2})+\sin\theta^2(2\sigma_y^2+w)(2\sigma_z^2+\frac{w}{\gamma^2})\right)^{3/2}} \\
\xi_{y^p} &= \frac{r_e N\beta_y}{2\pi\gamma}\times \\
&\int_0^{\infty} dw \frac{\left\{\begin{gathered}(\cos\theta+\cos\phi)^2(1+\cos\theta\cos\phi)(2\sigma_x^2+w)+\sin\phi^2\sin\theta^2(1+\cos\theta\cos\phi)(2\sigma_y^2+w) \\ +\sin\theta^2\cos\phi\left(\cos\phi+\cos\theta-\cos\theta\sin\phi^2\right)(2\sigma_z^2+\frac{w}{\gamma^2})\end{gathered}\right\}}{\left((1+\cos\theta\cos\phi)^2(2\sigma_x^2+w)(2\sigma_y^2+w)+\cos\theta^2\sin\phi^2(2\sigma_x^2+w)(2\sigma_z^2+\frac{w}{\gamma^2})+\sin\theta^2(2\sigma_y^2+w)(2\sigma_z^2+\frac{w}{\gamma^2})\right)^{3/2}}
\end{aligned} \tag{8}$$

Note, that for combinations *($\theta = 0; \phi = 0$)* and *($\theta = \pi; \phi = \pi$)* the above expressions are reduced to the well know formulae for the head-on collision. Besides, for arbitrary θ and $\phi = 0$ eq. (8) reproduces the formulae (9) in [8] for the tune shifts with a horizontal crossing angle.

In case when $\gamma >> tg(\theta/2)$ we can neglect the term $w/(\gamma^2 ctg^2(\theta))$ and for the case of a horizontal crossing angle *($\phi = 0$)* we obtain:

$$
\begin{aligned}
\xi_{x^p} &= \frac{r_e N}{2\pi\gamma} \frac{\beta_x}{\sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)}\left(\sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)} + \sigma_y\right)} \\
\xi_{y^p} &= \frac{r_e N}{2\pi\gamma} \frac{\beta_y}{\sigma_y \left(\sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)} + \sigma_y\right)}
\end{aligned}
\tag{9}
$$

Similarly, for the vertical crossing angle *($\theta = 0$)* we get:

$$
\begin{aligned}
\xi_{x^p} &= \frac{r_e N}{2\pi\gamma} \frac{\beta_x}{\sigma_x \left(\sqrt{\left(\sigma_z^2 tg^2(\phi/2) + \sigma_y^2\right)} + \sigma_x\right)} \\
\xi_{y^p} &= \frac{r_e N}{2\pi\gamma} \frac{\beta_y}{\sqrt{\left(\sigma_z^2 tg^2(\phi/2) + \sigma_y^2\right)}\left(\sqrt{\left(\sigma_z^2 tg^2(\phi/2) + \sigma_y^2\right)} + \sigma_x\right)}
\end{aligned}
\tag{10}
$$

Considering the last expressions (8)-(9) and the luminosity formula given in [11]:

$$
L = \frac{N^2}{4\pi\sigma_y \sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)}} \tag{11}
$$

We can see that both eqs. (9) and (11) can be obtained from similar formulae for the head-on collision by simply substituting:

$$
\sigma_x \;{-}{-}{>}\; \sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)} \tag{12}
$$

in case of collisions with a horizontal crossing angle and:

$$
\sigma_y \;{-}{-}{>}\; \sqrt{\left(\sigma_z^2 tg^2(\phi/2) + \sigma_y^2\right)} \tag{13}
$$

for the collisions at a vertical angle.

6.1.3 Beam-beam interaction formulae for tracking

There is no need in sophisticated formulae in the case of small crossing angles. Indeed, after the test particle arrival to the Interaction Point (IP) we know its 6D coordinates. Then we simply add the crossing angles to the particle betatron angles ($X^/$ and $Y^/$) and can imagine that the particle belongs to the beam colliding with the strong bunch head-on. Now we can employ the well-known formulae for Beam-Beam Interaction [12, 13]. Despite the strong bunch has some longitudinal length and usually is represented as a number of slices, after these transformations we again will have the particle at the nominal IP. Finally, we have to subtract the crossing angles from the particle coordinates $X^{/}$ and $Y^{/}$ in order to get the particle 6D coordinates in its own frame, and continue tracking through the machine lattice. Nevertheless, some authors of Beam-Beam codes (K. Hirata for BBC code) prefer to perform the transformation through IP in the special frame where the collision is head-on, even for small crossing angles, that requires Lorentz transformations [14]. It was shown in [15] that for small crossing angles both approaches are in good agreement. In fact, the crossing angles used in the older versions of beam-beam code LIFETRAC [16] were $-\theta$ and ϕ. In this case, for small crossing angles, the particle coordinates $X^{/}$ and $Y^{/}$ in the laboratory frame are just equal to the coordinates $X^{`}$, $Y^{'}$ in the "p" frame plus the corresponding crossing angles.

In this paper we consider the case of arbitrary crossing angles. Besides, the bunches can be rotated in crab-cavities in order to make the longitudinal axes of both colliding bunches co-parallel. For that, the crab angle should be equal to the minus half crossing angle. However, in general case we shall consider they are independent.

It is well known that in the ultra relativistic case the charged particle creates an electro-magnetic field only in the plane perpendicular to its velocity. In order to track the test particle through the charged "strong" bunch, the following approximation is used. The strong bunch is represented as a number of thin slices (pancakes), the plane of a slice must be perpendicular to its velocity. The test particle experiences a beam-beam kick when it crosses the slice's plane. This is a "zero-time" interaction, which simplifies the transformation and makes it symplectic. The test particle interacts sequentially with all the slices, with a simple drift space transformation between them. Usually it is said that the strong bunch is divided by slices longitudinally. However, with a crab-crossing collision the slices are not perpendicular to the bunch's longitudinal axis since it is not parallel to the bunch velocity.

We do not consider here the transformations through the crab-cavities, if any. For the test particle it is implemented by constant 6×6 matrixes placed at the correct locations before and after IP. As for this paper, we assume that we know the 6D particle coordinates when it arrives to the nominal IP. Then, the beam-beam transformation is applied to the particle, after that it is tracked through the machine lattice, and so on. The only thing we shall consider concerning the crab-crossing – that is how it affects the strong bunch shape and distribution in the laboratory frame. It is important that the crab rotation is not equivalent to the normal rotation of the strong bunch as a solid body. Instead, it can be imagined as transverse shifts of the slices, where the shift value is proportional to the longitudinal slice coordinate. The particle transverse distributions within slices do not change after such transformation. It is convenient to define the crab angle by two angles in the polar coordinate system: θ_{cr} is the total deflection angle and ϕ_{cr} is the azimuth angle in XY plane. After the crab rotation the slice coordinates will be:

$$\begin{aligned} \Delta x &= z_0 \sin(\theta_{cr}) \cos(\phi_{cr}) \\ \Delta y &= z_0 \sin(\theta_{cr}) \sin(\phi_{cr}) \\ z_{cr} &= z_0 \cos(\theta_{cr}) \end{aligned} \tag{14}$$

where Z_0 and Z_{cr} are the longitudinal slice coordinates (with regard to the center of bunch) before and after the crab cavity, Δx and Δy are the transverse slice shifts.

6.1.3.1 *Kick from a single slice*

It is convenient to apply the beam-beam kick in the coordinate system connected with the strong bunch. So, first of all we have to find the test particle coordinates in this system using (1). However, the standard formulae for head-on collision cannot be directly used for large crossing angles. There are at least three factors, which must be considered. First, the contribution of magnetic field is not equal now to the electric one. Second, the "time-of-flight" factor, which can vary in a wide range depending on the crossing angle. And third, the particle energy dE/E cannot be more calculated in assumption that the transverse momentum is much smaller than the longitudinal one: for large crossing angles it can be comparable or even larger. Fortunately, for the transverse kicks the first two factors exactly compensate each other. Indeed, the Lorentz force acting on the test particle due to the electromagnetic fields produced by the strong slice has the following components (see eqs. 4,5):

$$
\begin{aligned}
F_x &= e\left(E_x - v_z B_y\right) = eE_x\left(1 - v_z/c\right) \\
F_y &= e\left(E_y + v_z B_x\right) = eE_y\left(1 - v_z/c\right) \\
F_z &= e\left(E_z + v_x B_y - v_y B_x\right) = e\left(E_z + E_x v_x/c + E_y v_y/c\right)
\end{aligned}
\tag{15}
$$

The kick (momentum change) gained by the test particle is proportional to the product of the force and the time of interaction, which is inversely proportional to the relative longitudinal speed of the slice and the particle. Thus, in the case of Gaussian beams, for the transverse kicks we can simply use the Bassetti-Erskine formulae [13] written in assumption that $|v_z| = c$.

$$
\frac{\Delta p_y}{p_0} = -\frac{Nr_e}{\gamma_0}\sqrt{\frac{2\pi}{\sigma_x^2-\sigma_y^2}}\cdot \mathrm{Re}\left(F(x,y)\right) \qquad \frac{\Delta p_x}{p_0} = -\frac{Nr_e}{\gamma_0}\sqrt{\frac{2\pi}{\sigma_x^2-\sigma_y^2}}\cdot \mathrm{Im}\left(F(x,y)\right) \tag{16}
$$

where p_0 is the total momentum of the test particle, N is the number of electrons in the strong slice, r_e is the classical electron radius, γ_0 is the relativistic factor of the test particle, σ_x and σ_y are the transverse sizes of the strong slice (it is assumed that $\sigma_x > \sigma_y$), and the function *F(x,y)* is represented through the complex error function *W(z)* as follows:

$$
F(x,y) = W\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{-\left(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right)}\cdot W\left(\frac{x\dfrac{\sigma_y}{\sigma_x}+iy\dfrac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) \tag{17}
$$

Now let us consider the longitudinal kick. The part connected with the magnetic field and transverse velocities (see the third row of (15), last two terms in the parentheses) can be rewritten as follows:

$$
\frac{\Delta p_{zB}}{p_0} = \frac{1}{1-p_z/p_0}\left(\frac{\Delta p_x}{p_0}\cdot\frac{p_x+\Delta p_x/2}{p_0}+\frac{\Delta p_y}{p_0}\cdot\frac{p_y+\Delta p_y/2}{p_0}\right) \tag{18}
$$

Since the change in the transverse momentum due to the interaction can be comparable with the initial one, we should use the "average transverse momentum during the interaction". The change in the longitudinal momentum also can be comparable with the initial one (for example, if the crossing angle is 90^0, so that $v_z \approx 0$), but in any case it must be much smaller than the total momentum p_0, and we may neglect the change of p_z in the right-hand part of (18). The longitudinal electric field E_z is created by the space charge of the strong slice if the derivatives $\partial\sigma/\partial s \neq 0$. For the Gaussian beams, using the corresponding formulae from [14] we obtain the following expression for the electric part of the longitudinal kick (it is called ***g*** in [14]):

$$\frac{\Delta p_{zE}}{p_0} = \frac{\left(x\cdot\frac{\Delta p_x}{p_0} + y\cdot\frac{\Delta p_y}{p_0} + \frac{2Nr_e}{\gamma_0}\right)\cdot\left(\frac{\partial\sigma_x^2}{\partial s} - \frac{\partial\sigma_y^2}{\partial s}\right) - \frac{2Nr_e}{\gamma_0} e^{-\left(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right)}\cdot\left(\frac{\sigma_x}{\sigma_y}\frac{\partial\sigma_x^2}{\partial s} - \frac{\sigma_y}{\sigma_x}\frac{\partial\sigma_y^2}{\partial s}\right)}{2\left(1 - p_z/p_0\right)\cdot\left(\sigma_x^2 - \sigma_y^2\right)} \tag{19}$$

The only difference with the head-on collision is the "time-of-flight" factor. Since the transverse momentum can be comparable or even larger than the longitudinal one, the energy change gained by the particle after the kick should be calculated by the formula:

$$\frac{\Delta E}{E_0} = \sqrt{\frac{\left(p_x + \Delta p_x\right)^2 + \left(p_y + \Delta p_y\right)^2 + \left(p_z + \Delta p_z\right)^2}{p_x^2 + p_y^2 + p_z^2}} - 1 \tag{20}$$

If the momentum change is much less than the initial one: $\Delta p << p_0$, that is usually the case, the energy change can be rewritten as:

$$\frac{\Delta E}{E_0} = \frac{\Delta p_x\cdot\left(p_x + \frac{\Delta p_x}{2}\right) + \Delta p_y\cdot\left(p_y + \frac{\Delta p_y}{2}\right) + \Delta p_z\cdot\left(p_z + \frac{\Delta p_z}{2}\right)}{p_0^2} \tag{21}$$

Let us consider, for example, the case without crossing angle: $p_z = -p_0$ and the change in longitudinal momentum is much less than the transverse kick: $\Delta p_z << \Delta p$. Then, using (18) and (21), we obtain:

$$\frac{\Delta E}{E_0} \approx \frac{\Delta p_x\cdot\left(p_x + \frac{\Delta p_x}{2}\right) + \Delta p_y\cdot\left(p_y + \frac{\Delta p_y}{2}\right)}{2p_0^2} - \frac{\Delta p_{zE}}{p_0} \tag{22}$$

That is in agreement with equation (38) from [14].

6.1.3.2 *Tracking algorithm*

After all, we describe the algorithm of Beam-Beam Interaction implemented in the tracking code LIFETRAC. We start from the test particle arrival to the nominal IP and mark this moment as τ_P, to distinguish it from τ_0 – the nominal collision time (when the centers of both colliding beams arrive to the IP). First of all, the transformation is made from the "p" to the laboratory coordinate system using (1), but with $z^P = 0$. It is important to note that z^P here is not the test particle's longitudinal coordinate with regard to the equilibrium particle of the weak beam, but the "p" azimuth of the point where the transformation is applied. The same transformation (1) is made for the particle velocities.

If there is a constant separation between the colliding beams (for example, at the Parasitic Crossings), the separation values are added to the particle coordinates X and Y. In the case of crab-crossing there is an additional transverse shift (14), which is individual for each slice. Besides, the strong bunch can be rotated in XY plane (a tilt) due to the betatron coupling. If any, the corresponding transformations are applied to the test particle 6D coordinates. Finally, we get the particle coordinates in the frame where the X- and Y-axes are parallel to the strong slice ellipse axes, and the Z-axis goes through the center of slice along its velocity.

The next step is to find the time interval between τ_P and the collision time τ_c (it can be negative!). The longitudinal slice coordinate at the moment τ_P is $z_S - s$, where z_S is the longitudinal slice shift with regard to the center of the strong bunch, and s is the longitudinal particle coordinate with regard to the center of the weak bunch – the 5th component of the standard 6D particle coordinates in the "p" frame. The test particle's longitudinal coordinate z at this moment is given by (1). So, we obtain:

$$\tau_c - \tau_p = \frac{z_s - s - z}{c \cdot (1 - p_z / p_0)} \tag{23}$$

where the numerator is the longitudinal distance between the slice and the particle at the time τ_P, and the denominator is the longitudinal velocity of rapprochement. Now we can find the particle coordinates at the moment of collision.

To calculate the kicks by the formulae (16–19), we need to know the transverse sizes of the slice and the derivatives $\partial\sigma / \partial s$. These can be obtained from the lattice functions of the strong beam at the nominal IP and the longitudinal coordinate of Collision Point (CP). The next CP will be shifted by the value of longitudinal distance between the two sequential slices, divided by $(1 – p_z/p_0)$. The transformation between CP's is a simple drift space, and then the kick from a slice is applied, and so on.

After the last slice passage, we may need to perform a transformation to the laboratory frame: back rotation in the XY plane (if there is a tilt of the strong bunch), and back transverse shifts (in the cases of crab-crossing and constant separation at the PC). Then the transformation from the laboratory to the particle frame is applied using (1). Finally, we need to shift the particle longitudinally to the nominal IP in the "p" frame – this is made as a simple drift space transformation.

6.1.4 Comparison of analytical tune shifts with numerical simulations

In order to check the formulae (8) describing the general case of collisions with an arbitrary crossing angle we use numerical simulations with beam-beam code LIFETRAC. The betatron tunes in the presence of beam-beam effects are calculated by tracking in the following way. First of all a test particle is tracked for one turn with the initial conditions:

$$X_i = \delta(i,j)\sigma_i q, \quad i = 1,2,..,6, \quad q << 1, \quad \delta(i,j) = \begin{cases} 0, & if \quad i \neq j \\ 1, & if \quad i = j \end{cases} \tag{24}$$

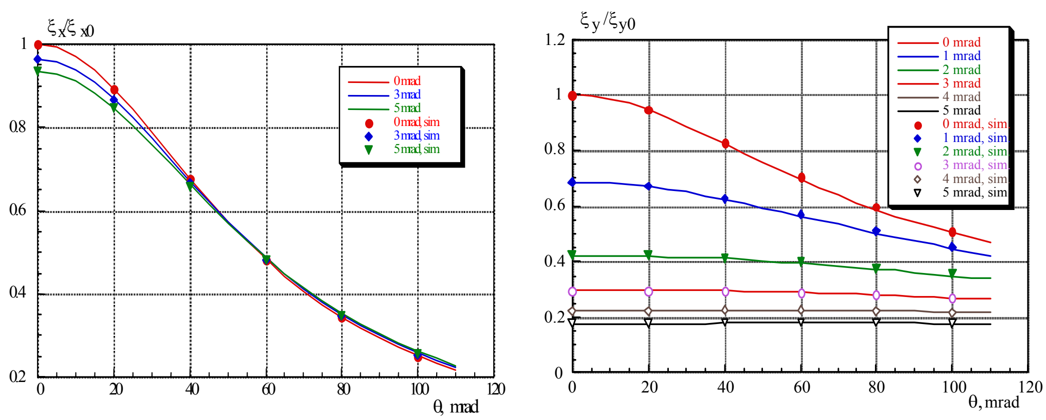
where X_i are the coordinates in the 6D phase space and σ_i are the respective rms sizes.

Doing this 6 times for j = 1,2,..,6 we obtain the 6×6 revolution matrix. Then, the matrix eigenvalues are calculated those give us the tunes. For these simulations we use a simple model of collider with linear transformations from IP to IP. In order to reproduce correctly the Gaussian longitudinal distribution we divide a strong bunch in much more longitudinal slices than in ordinary beam-beam simulations. Besides, the longitudinal rms bunch length is taken to be much shorter than the transverse beta functions at IP in order to satisfy approximations made to obtain eqs. (8). In these conditions the following equation is valid:

$$\cos(\mu) = \cos(\mu_0) - 2\pi\xi \sin(\mu_0) \qquad (25)$$

where μ_0 is the initial betatron tune (transverse or vertical, without beam-beam), and μ is the tune calculated by tracking. Thus, we can find the tune shift ξ.

First of all, we performed the comparison for rather small crossing angles, but sizeable Piwinski's angles. The following set of parameters was used: bunch length σ_L=3 cm, beta-functions at the IP β_x = 150 cm, β_y = 20 cm, emittances ε_x = $5\cdot10^{-5}$ cm·rad, ε_y=10^{-7} cm·rad. Figures 2 (a) and (b) show the normalised horizontal and vertical tune shifts calculated analytically and numerically for comparison. As it is seen, the agreement between the analytical formulae and the simulations is very much satisfactory.

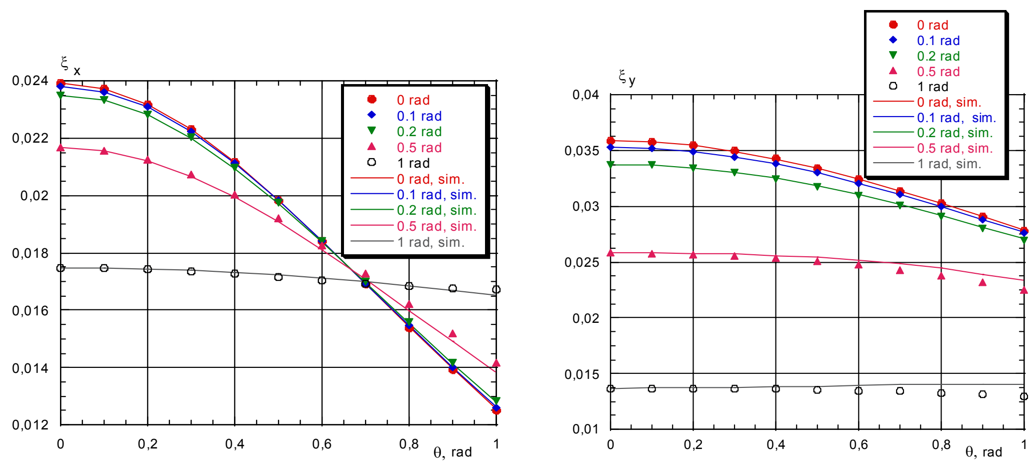


a) b)

Figure 2: The horizontal (a) and vertical (b) tune shifts as a function of angle θ for ϕ = 0,1,2,3,4 and 5 mrad (normalised by the value of the horizontal tune shift in head-on collisions). Solid lines – analytical results, dots – simulation results.

Then, we made the comparison for large crossing angles, in the range of 0.1 to 1.0 radians. The choice of parameters was based on the following statements. Bunch length must be much shorter than the beta-functions at the IP. Piwinski's angles of the order of 1-3 (at 1 rad) to have a reasonable and appreciable tune shift reduction in both planes. This also means that the transverse sizes are comparable (not order of magnitude different). Maximum tune shifts have to be reasonable, of the order of 0.02 - 0.05 or so. Finally, we came to the following set of parameters: bunch length σ_L = 0.3 cm, beta-functions at the IP $\beta_x = \beta_y$ = 100 cm, emittances ε_x = $2.25\cdot10^{-4}$ cm·rad, ε_y = 10^{-4} cm·rad.

The comparison results are presented in Figures 3 a) and b) for the horizontal and vertical tune shifts, correspondingly. Some important remarks must be applied to these results. First of all, since the vector of beam-beam kick (in the laboratory coordinate system) is perpendicular to the strong bunch's velocity, it has a component parallel to the weak bunch's velocity - that results in the energy change (18). The sign of the "energy kick" depends on the sign of the test particle longitudinal coordinate, so that IP behaves like a RF cavity (with rather specific potential shape, however). For large crossing angles, this even can result in the longitudinal instability. Moreover, due to the beam-beam interaction the system becomes 6D coupled, and the coupling depends not only on the ξ parameters and crossing angles, but on the initial tunes (both betatron and synchrotron) as well. So, the calculated by tracking tune shifts, since they are extracted from the eigenvalues, depend on the initial tunes. For example, in the case of the both crossing angles equal to 1 radian, but the initial tunes changed from (0.12, 0.15) to (0.12, 0.35), we have got the tune shifts changed from (0.01648638, 0.01396790) to (0.01758817, 0.01302020), that is rather big difference. We think, this coupling effect is the main source of discrepancy on large crossing angles, since it is not accounted at all in the analytical formulae (8). So, one can see that when the both angles are large, simulations give larger ξ_x and smaller ξ_y than the analytical formulae – it looks like a "redistribution" due to coupling. In these conditions we consider the agreement is quite acceptable.



a) b)

Figure 3: The horizontal (a) and vertical (b) tune shifts as a function of angle θ for $\phi = 0$, 0.1, 0.2, 0.5, and 1.0 rad. Solid lines – simulation results, dots – analytical results.

6.1.5 Conclusions

1. We have obtained the formulae for the beam-beam tune shifts in collisions with an arbitrary crossing angle. In particular, it has been shown that these formulae can be transformed from the similar formulae for head-on collisions by substituting the horizontal beam size by $\left(\sigma_x^2 + \sigma_z^2 tg^2(\theta/2)\right)^{1/2}$ in case of collisions with a horizontal crossing angle and the vertical beam size by $\left(\sigma_y^2 + \sigma_z^2 tg^2(\phi/2)\right)^{1/2}$ if bunches collide at a vertical crossing angle.

2. The formulae for symplectic 6D beam-beam kick have been derived in the laboratory coordinate system, without Lorentz boost. These are currently implemented in the beam-beam code LIFETRAC.

3. Analysing the tune shift formulae, we see that for flat beams:

   a) The luminosity and the tune shifts are reduced with the horizontal crossing angle. However, since

   $$L \sim \left(\sigma_x^2 + \sigma_z^2 tg^2(\theta/2)\right)^{-1/2}; \quad \xi_x \sim \left(\sigma_x^2 + \sigma_z^2 tg^2(\theta/2)\right)^{-1}; \quad \xi_y \sim \left(\sigma_x^2 + \sigma_z^2 tg^2(\theta/2)\right)^{-1/2}$$

   the horizontal tune shift drops faster than the luminosity does.

   b) In collisions with the vertical crossing angle the horizontal tune shift practically does not depend on the vertical angle if $\sigma_x >> \left(\sigma_y^2 + \sigma_z^2 tg^2(\phi/2)\right)^{1/2}$ while the vertical tune shift and the luminosity are reduces proportionally to $\left(\sigma_y^2 + \sigma_z^2 tg^2(\phi/2)\right)^{-1/2}$.

4. A comparison of analytical tune shifts calculations with eq. (8) and numerical simulations has shown a good agreement.

6.1.6 References

1. M. Zobov et al., PAC05, ROAA001; C. Milardi et al., EPAC04, pp. 233-235.
2. Y. Funakoshi et al., EPAC04, pp. 707-709.
3. C. Zhang et al., EPAC4, pp. 230-232.
4. J. Seeman et al., EPAC04, pp. 878-880.
5. H. Koiso, PAC05, TPPP003, to be published.
6. P. Raimondi, http://www.lnf.infn.it/conference/d2/TALKS/raimondi.pdf
7. Y. Cai, ICFA Beam Dyn.Newlett.34:11-19,2004.
8. P. Raimondi, M. Zobov, DAΦNE Tech. Note G-58, Frascati, April 30, 2003.
9. D. Shatilov, M. Zobov, DAΦNE Tech. Note G-59, Frascati, May 14, 2003.
10. A.A. Zholents, Preprint 91-18, Novosibirsk, 1991.
11. O. Napoly, Part.Accel.40:181-203,1993.
12. K. Hirata, H. Moshammer, F. Ruggiero, KEK Preprint 92-117.
13. M. Bassetti, G.Erskine, CERN ISR TH /80-06 (1980).
14. K. Hirata, Phys.Rev.Lett.74:2228-2231,1995.
15. D. Shatilov, KEK Report 96-14.
16. D. Shatilov, Part.Accel.52:65-93,1996.

6.2 A longitudinal coupled bunch feedback for HERA-p

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6.2.1 Introduction

A longitudinal broadband damper system to control coupled bunch instabilities is currently under construction to be installed in the 920 GeV proton accelerator HERA-p